A Survey of Graph Layout Problems

JOSEP DÍAZ, JORDI PETIT AND MARIA SERNA

Universitat Politècnica de Catalunya

Graph layout problems are a particular class of combinatorial optimization problems whose goal is to find a linear layout of an input graph in such way that a certain objective cost is optimized. This survey considers their motivation, complexity, approximation properties, upper and lower bounds, heuristics and probabilistic analysis on random graphs. The result is a complete view of the current state of the art with respect to layout problems from an algorithmic point of view.

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1. INTRODUCTION

Graph layout problems are a particular class of combinatorial optimization problems whose goal is to find a linear layout of an input graph in such a way that a certain objective function is optimized. A linear layout is a labelling of the vertices of a graph with distinct integers. A large number of relevant problems in different domains can be formulated as graph layout problems. These include optimization of networks for parallel computer architectures, VLSI circuit design, information retrieval, numerical analysis, computational biology, graph theory, scheduling and archaeology. Most interesting graph layout problems are NP-hard and their decisional versions NP-complete, but, for most of their applications, feasible solutions with an almost optimal cost are sufficient. As a consequence, approximation algorithms and effective heuristics are welcome in practice.

Because of their importance, there are many results associated with layout problems. Here we try to present a complete view of the current state of the art with respect to graph layout problems. Our focus is biased to algorithmic issues. Our purpose is two-fold: on one hand, to present a global view including the latest results; on the other to encourage the algorithmic community to work on this very exciting field of research. There are other surveys that deal

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Auhors' address: Departement de Llenguatges I Sistemes Informàtics, Universitat Politècnica de Catalunya, Mòdul C6, Campus Nord, 08034 Barcelona; email: {diaz,jpetit,mjserna}@lsi.upc.es.

with aspects of different graph layout problems, and this paper intersects with them; see for instance Chinn et al. [1982]; Chung [1988]; Monien and Sudborough [1990]; Diaz [1992]; Mohar and Poljak [1993]; Bezrukov [1999], Lai and Williams [1999]; Raspaud et al. [2000].

The road map for this survey is as follows: First of all, in Section 2, we formally define the layout problems we are interested in and state some basic results. A historical overview with motivations and applications for the study of layout problems is surveyed in Section 3. Section 4 presents NP-completeness results on layout problems and Section 5 presents some polynomial time solvable graph classes. In Section 6 we survey fixed parameterized complexity results. Afterwards, in Section 7 we survey approximation algorithms. In Section 8 we consider random graphs as inputs to layout problems and show the performance of some approximation algorithms on them. In Sections 9 and 10 we present, respectively, several techniques to obtain upper and lower bounds and several heuristic methods to obtain good feasible solutions. Finally, in Section 11 we present several extensions to the basic model of graph layout problems. We close the paper with some concluding remarks, mainly open problems.

2. DEFINITIONS AND BASIC OBSERVATIONS

We start with the definition of several graph layout problems and associated concepts. This enables us to treat different layout problems using a unique framework. Finally, we present some basic results regarding layout problems.

The graph theoretic definitions and notations that we use, for the most part, conform to the standard ones in computer science. Unless otherwise mentioned, graphs are finite, undirected and without loops. Given a graph G, its vertex set is denoted by $V(G)$ and its edge set by $E(G)$. The notation uv stands for the undirected edge $\{u, v\}$. The degree of a vertex u in a graph G is denoted as $\deg(u) = \deg_G(u)$ and the maximal degree of G as $\Delta(G)$. The set of neighbors of a vertex u in G is denoted $\Gamma(u) = \Gamma_G(u) = \{v \in V(G) : uv \in E(G)\}$.

A *linear layout*, or simply a *layout*, of an undirected graph $G = (V, E)$ with $n = |V|$ vertices is a bijective function $\varphi : V \to [n] = \{1, \ldots, n\}$. A layout has also been called a *linear ordering* [Adolphson and Hu 1973], a *linear arrangement* [Shiloach 1979], a *numbering* [Chinn et al. 1982] or a *labeling* [Juvan and Mohar 1992] of the vertices of a graph. We denote by $\Phi(G)$ the set of all layouts of a graph G.

Given a layout φ of a graph $G = (V, E)$ and an integer i, we define the set $L(i, \varphi, G) = \{u \in V : \varphi(u) \leq i\}$ and the set $R(i, \varphi, G) = \{u \in V : \varphi(u) > i\}$. The *edge cut* at position i of φ is defined as

$$\theta(i, \varphi, G) = |\{uv \in E : u \in L(i, \varphi, G) \wedge v \in R(i, \varphi, G)\}|$$

and the *modified edge cut* at position i of φ as

$$\zeta(i, \varphi, G) = |\{uv \in E : u \in L(i, \varphi, G) \wedge v \in R(i, \varphi, G) \wedge \varphi(u) \neq i\}|.$$

The *vertex cut* or *separation* at position i of φ is defined as

$$\delta(i, \varphi, G) = |\{u \in L(i, \varphi, G) : \exists v \in R(i, \varphi, G) : uv \in E\}|.$$

Given a layout φ of G and an edge $uv \in E$, the *length* of uv on φ is

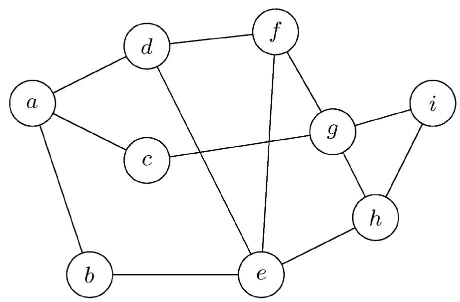
$$\lambda(uv, \varphi, G) = |\varphi(u) - \varphi(v)|.$$

These measures are summarized in Table I for ease of future reference.

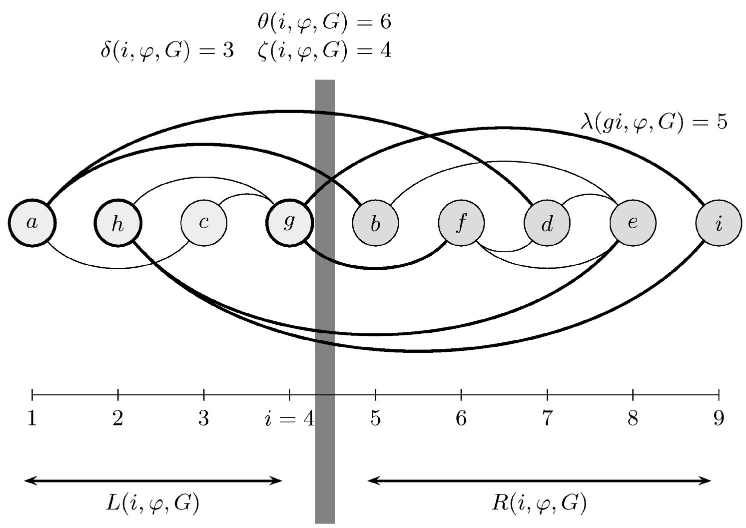
A common way to represent a layout φ of a graph G is to align its vertices on a horizontal line, mapping each vertex u to position $\varphi(u)$, as shown in Figure 1. This graphical representation gives an easy understanding of the previously defined measures: By drawing a vertical line just after position i and before position $i + 1$, the vertices at the left of the line belong to $L(i, \varphi, G)$ and the vertices at the right

Table I. Layout Measures for a Layout φ of a Graph $G = (V, E)$

$L(i, \varphi, G)$	$=$	$\{u \in V \ : \ \varphi(u) \leq i\}$.
$R(i, \varphi, G)$	$=$	$\{u \in V \ : \ \varphi(u) > i\}$.
$\theta(i, \varphi, G)$	$=$	$\lvert\{uv \in E \ : \ u \in L(i, \varphi, G) \wedge v \in R(i, \varphi, G)\}\rvert$.
$\zeta(i, \varphi, G)$	$=$	$\lvert\{uv \in E \ : \ u \in L(i, \varphi, G) \wedge v \in R(i, \varphi, G) \wedge \varphi(u) \neq i\}\rvert$.
$\delta(i, \varphi, G)$	$=$	$\lvert\{u \in L(i, \varphi, G) \ : \ \exists v \in R(i, \varphi, G) : uv \in E\}\rvert$.
$\lambda(uv, \varphi, G)$	$=$	$\lvert\varphi(u) - \varphi(v)\rvert, \qquad uv \in E$.



(a) $G = (V, E)$.



(b) Graphical representation of φ. Dividing the layout at $i = 4$, left vertices are shadowed light, while right vertices are shadowed dark, cut edges and separator vertices are bold.

Fig. 1. A graph G together with some layout measures and a graphical representation of the layout $\varphi = \{(a, 1), (b, 5), (c, 3), (d, 7), (e, 8), (f, 6), (g, 4), (j, 9), (h, 2)\}$.

of the line belong to $R(i, \varphi, G)$. It is easy to compute the cut $\theta(i, \varphi, G)$ by counting the number of edges that cross the vertical line. The modified cut $\zeta(i, \varphi, G)$ counts all the edges in $\theta(i, \varphi, G)$ except those that have vertex $\varphi^{-1}(i)$ as endpoint. It is also easy to compute the separation $\delta(i, \varphi, G)$ by counting the number of vertices at the left of the vertical line that are joined with some vertex at the right of the vertical line. Finally, the length $\lambda(uv, \varphi, G)$ of an edge uv corresponds to the natural distance between its endpoint images.

Given a layout φ of a graph $G = (V, E)$, its *reversed layout* is denoted φ^R and is defined by $\varphi^R(u) = |V| - \varphi(u) + 1$ for all $u \in V$.

A *layout cost* is a function F that associates to each layout φ of a graph G, an integer $F(\varphi, G)$. Let F be a layout cost; the optimization layout problem associated with F consists in determining some layout $\varphi^* \in \Phi(G)$ of an input graph G such that

$$F(\varphi^*, G) = \min_{\varphi \in \Phi(G)} F(\varphi, G).$$

For any F and G, we define $\text{MIN}F(G) = \min_{\varphi \in \Phi(G)} F(\varphi, G)$.

The particular costs we consider in this survey are listed below, together with the layout problems they give raise to:

—*Bandwidth* (BANDWIDTH): Given a graph $G = (V, E)$, find a layout $\varphi^* \in \Phi(G)$ such that $\text{BW}(\varphi^*, G) = \text{MINBW}(G)$ where

$$\text{BW}(\varphi, G) = \max_{uv \in E} \lambda(uv, \varphi, G).$$

—*Minimum Linear Arrangement* (MINLA): Given a graph $G = (V, E)$, find a layout $\varphi^* \in \Phi(G)$ such that $\text{LA}(\varphi^*, G) = \text{MINLA}(G)$ where

$$\text{LA}(\varphi, G) = \sum_{uv \in E} \lambda(uv, \varphi, G).$$

—*Cutwidth* (CUTWIDTH): Given a graph $G = (V, E)$, find a layout $\varphi^* \in \Phi(G)$ such that $\text{CW}(\varphi^*, G) = \text{MINCW}(G)$ where

$$\text{CW}(\varphi, G) = \max_{i \in [|V|]} \theta(i, \varphi, G).$$

—*Modified Cut* (MODCUT): Given a graph $G = (V, E)$, find a layout $\varphi^* \in \Phi(G)$ such that $\text{MC}(\varphi^*, G) = \text{MINMC}(G)$ where

$$\text{MC}(\varphi, G) = \sum_{i \in [|V|]} \zeta(i, \varphi, G).$$

—*Vertex Separation* (VERTSEP): Given a graph $G = (V, E)$, find a layout $\varphi^* \in \Phi(G)$ such that $\text{VS}(\varphi^*, G) = \text{MINVS}(G)$ where

$$\text{VS}(\varphi, G) = \max_{i \in [|V|]} \delta(i, \varphi, G).$$

—*Sum Cut* (SUMCUT): Given a graph $G = (V, E)$, find a layout $\varphi^* \in \Phi(G)$ such that $\text{SC}(\varphi^*, G) = \text{MINSC}(G)$ where

$$\text{SC}(\varphi, G) = \sum_{i \in [|V|]} \delta(i, \varphi, G).$$

—*Profile* (PROFILE): Given a graph $G = (V, E)$, find a layout $\varphi^* \in \Phi(G)$ such that $\text{PR}(\varphi^*, G) = \text{MINPR}(G)$ where

$$\text{PR}(\varphi, G) = \sum_{u \in V} \left(\varphi(u) - \min_{v \in \Gamma^*(u)} \varphi(v) \right)$$

and $\Gamma^*(u) = \{u\} \cup \{v \in V \ : \ uv \in E\}$.

—*Edge Bisection* (EDGEBIS): Given a graph $G = (V, E)$, find a layout $\varphi^* \in \Phi(G)$ such that $\text{EB}(\varphi^*, G) = \text{MINEB}(G)$ where

$$\text{EB}(\varphi, G) = \theta(\left\lfloor \frac{1}{2}|V| \right\rfloor, \varphi, G).$$

—*Vertex Bisection* (VERTBIS): Given a graph $G = (V, E)$, find a layout $\varphi^* \in \Phi(G)$ such that $\text{VB}(\varphi^*, G) = \text{MINVB}(G)$ where

$$\text{VB}(\varphi, G) = \delta\left(\left\lfloor \frac{1}{2}|V| \right\rfloor, \varphi, G \right).$$

Strictly speaking, the edge bisection and vertex bisection problems are not layout problems: Both problems ask for a partition of the set of vertices of the graph in two disjoint subsets of the same size (or differing by one if the number of vertices is odd) rather than for a permutation of the vertices. Nevertheless, bisection problems and layout problems are closely related, and bisection problems fit well in our framework for layout problems.

The definitions of the previous problems are summarized in Table II. In Section 3 we will reference the first appearances of these problems.

At this point it is relevant to point out some basic, but important, facts.

Observation 2.1 For any graph $G = (V, E)$ and any layout φ of G, the total edge length equals the sum of all edge cuts in the layout:

$$\sum_{uv \in E} \lambda(uv, \varphi, G) = \sum_{i \in [|V|]} \theta(i, \varphi, G).$$

This fact was first noticed by Harper [1966]. It follows from the observation that any edge $uv \in E$ with $\varphi(u) < \varphi(v)$ contributes $\varphi(v) - \varphi(u)$ to the left hand side and 1 to each one of the terms

Table II. Layout Problems and Costs for a Graph $G = (V, E)$ with $|V| = n$

Problem	Name	Cost
Bandwidth	BANDWIDTH	$\text{BW}(\varphi, G) = \max_{uv\in E} \lambda(uv, \varphi, G)$.
Min. Lin. Arrangement	MINLA	$\text{LA}(\varphi, G) = \begin{cases} \sum_{uv\in E} \lambda(uv, \varphi, G), \\ \sum_{i=1}^{n} \theta(i, \varphi, G). \end{cases}$
Cutwidth	CUTWIDTH	$\text{CW}(\varphi, G) = \max_{i=1}^{n} \theta(i, \varphi, G)$.
Modified Cut	MODCUT	$\text{MC}(\varphi, G) = \sum_{i=1}^{n} \zeta(i, \varphi, G)$.
Vertex Separation	VERTSEP	$\text{VS}(\varphi, G) = \max_{i=1}^{n} \delta(i, \varphi, G)$.
Sum Cut	SUMCUT	$\text{SC}(\varphi, G) = \sum_{i=1}^{n} \delta(i, \varphi, G)$.
Profile	PROFILE	$\text{PR}(\varphi, G) = \begin{cases} \sum_{u\in V} (\varphi(u) - \min_{v\in\Gamma^*(u)} \varphi(v)), \\ \text{SC}(\varphi^R, G). \end{cases}$
Edge Bisection	EDGEBIS	$\text{EB}(\varphi, G) = \theta(\lfloor n/2 \rfloor, \varphi, G)$.
Vertex Bisection	VERTBIS	$\text{VB}(\varphi, G) = \delta(\lfloor n/2 \rfloor, \varphi, G)$.

$\theta(\varphi(u), \varphi, G), \theta(\varphi(u) + 1, \varphi, G), \dots, \theta(\varphi(v), \varphi, G)$ in the right hand side.

Observation 2.2. For any graph $G = (V, E)$ and any layout φ of G,

$$\text{PR}(\varphi, G) = \text{SC}(\varphi^R, G).$$

This identity apparently has been well known for some time; a recent proof can be found in Golovach and Fomin [1998]. It is due to the fact that each vertex $u \in V$ contributes one unit $\varphi(u) - \min_{v\in\Gamma^*(u)} \varphi(v)$ times to the sum cut in the reversed layout. Notice that, as a consequence, PROFILE and SUMCUT are equivalent problems.

Observation 2.3. It is important to stress that the graph layout problems we have formulated explicitly ask for the construction of a layout with an optimal cost, rather than the cost of an optimal layout:

"Given a graph G, find *a layout $\varphi^* \in \Phi(G)$ such that $F(\varphi^*, G) = \text{MIN}F(G)$."*

All the problems can however be restated as decisional problems, where the task is to decide whether or not a graph admits a layout with cost not greater than an integer given as part of the input:

"Given a graph G and *an integer k,* is there *some layout $\varphi \in \Phi(G)$ such that $F(\varphi, G) \le k$?"*

We will not focus too much on whether we are considering the optimization or the decisional version of some problem, because they will be clearly differentiated by the context.

The following lemma gives useful relations between some layout costs. The proofs are straightforward.

LEMMA 2.4. *Let G be any graph with n vertices and m edges, and let φ be any layout of G. Then,*

$$\begin{array}{rclcl}
\text{LA}(\varphi, G) & \le & n \cdot \text{CW}(\varphi, G), & & \text{MINLA}(G) \le n \cdot \text{MINCW}(G), \\
\text{LA}(\varphi, G) & \le & m \cdot \text{BW}(\varphi, G), & & \text{MINLA}(G) \le m \cdot \text{MINBW}(G), \\
\text{MC}(\varphi, G) & \le & \text{LA}(\varphi, G), & & \text{MINMC}(G) \le \text{MINLA}(G), \\
\text{SC}(\varphi, G) & \le & n \cdot \text{VS}(\varphi, G), & & \text{MINSC}(G) \le n \cdot \text{MINVS}(G), \\
\text{VS}(\varphi, G) & \le & \text{BW}(\varphi, G), & & \text{MINVS}(G) \le \text{MINBW}(G), \\
\text{EB}(\varphi, G) & \le & \text{CW}(\varphi, G), & & \text{MINEB}(G) \le \text{MINCW}(G), \\
\text{VB}(\varphi, G) & \le & \text{VS}(\varphi, G), & & \text{MINVB}(G) \le \text{MINVS}(G), \\
\text{CW}(\varphi, G) & \le & \Delta(G) \cdot \text{BW}(\varphi, G), & & \text{MINCW}(G) \le \Delta(G) \cdot \text{MINBW}(G).
\end{array}$$

The next lemma relates some layout costs of a graph in terms of its connected components. A proof for bandwidth can be found in Chvátalová et al. [1975]; for the remaining problems the results can be proven in a similar way.

LEMMA 2.5. *Let G be a graph and $G_1, \ldots, G_k$ its connected components. Then,*

$$\begin{aligned} \text{MINBW}(G) &= \max_{i\in[k]} \text{MINBW}(G_i), & \text{MINMC}(G) &= \sum_{i\in[k]} \text{MINMC}(G_i), \\ \text{MINCW}(G) &= \max_{i\in[k]} \text{MINCW}(G_i), & \text{MINLA}(G) &= \sum_{i\in[k]} \text{MINLA}(G_i), \\ \text{MINVS}(G) &= \max_{i\in[k]} \text{MINVS}(G_i), & \text{MINSC}(G) &= \sum_{i\in[k]} \text{MINSC}(G_i). \end{aligned}$$

A useful consequence of the previous lemma is that, for any layout cost $F \in \{\text{BW}, \text{CW}, \text{VS}, \text{LA}, \text{MC}, \text{SC}\}$, it is possible to obtain an optimal layout for F of a graph just by computing the optimal layouts of its connected components. Observe, however, that the EB and VB bisection costs do not share this property: As a counterexample, consider a graph made of two components of the same size, each one being a clique.

The following lemma relates the layout costs of a graph and a subgraph of it. Given two graphs G and H, it is said that H is an edge induced graph of G if $V(H) = V(G)$ and $E(H) \subseteq E(G)$ and it said that H is a vertex induced graph of G if $V(H) \subseteq V(G)$ and $E(H) = \{uv \in E(G) \,:\, u, v \in V(H)\}$.

LEMMA 2.6. *Let H be an edge or vertex induced subgraph of a graph G. Then, for any layout cost $F \in \{\text{BW}, \text{CW}, \text{VS}, \text{LA}, \text{MC}, \text{SC}\}$, holds that $\text{MIN}F(H) \le \text{MIN}F(G)$. In the case that H is an edge induced subgraph of G, then $\text{MINEB}(H) \le \text{MINEB}(G)$ and $\text{MINVB}(H) \le \text{MINVB}(G)$.*

3. HISTORICAL PERSPECTIVE AND APPLICATIONS

In its simpler form, a layout is an embedding into the natural line. As we will mention, the terms *layout* and *layout problem* are due to the early application of these problems to the optimal layout of circuits. We present here some background that motivates research on layout problems, as well as some of their applications. We start with a historical overview.

The Minimum Linear Arrangement problem (MINLA) was first stated in Harper [1964]. Harper's aim was to design error-correcting codes with minimal average absolute errors on certain classes of graphs. Latter, this problem was considered in Mitchison and Durbin [1986] as an over-simplified model of some nervous activity in the cortex. MINLA also has applications in single machine job scheduling [Adolphson 1977; Ravi et al. 1991] and in graph drawing [Shahrokhi et al. 2001]. The Minimum Linear Arrangement problem has received some alternative names, such as the *Optimal Linear Ordering*, the *Edge Sum*, the *Minimum-1-sum*, the *bandwidth sum* or the *wirelength* problem.

Bandwidth had received much attention during the fifties in order to speed up several computations on sparse matrices. According to Dewdney [1976], the introduction of the bandwidth problem for graphs (BANDWIDTH) was first stated in Harary [1967], however the problem was formally defined in Harper [1966].

The cutwidth problem (CUTWIDTH) was first used in the seventies as a theoretical model for the number of channels in an optimal layout of a circuit [Adolphson and Hu 1973]; see also the Introduction in Makedon and Sudborough [1989]. More recent applications of this problem include network reliability [Karger 1999], automatic graph drawing [Mutzel 1995] and information retrieval [Botafogo 1993].

The vertex separation problem (VERTSEP) was originally motivated by the general problem of finding good separators for graphs [Lipton and Tarjan 1979], and has applications in algorithms for VLSI design [Leiserson 1980]. As we shall see, this problem is also equivalent to some other well known problems.

The sum cut and profile problems on graphs (SUMCUT and PROFILE) were independently defined in Diàz et al. [1991] and in Lin and Yuan [1994b]. The sum cut problem was originally proposed as a simplified version of the δ-operator problem [Díaz 1979]. The profile problem was proposed as a way to reduce the amount of storage of sparse matrices [Tewarson 1973; Lepin 1986; Lin and Yuan 1994a]. Both problems turn out to be equivalent to the interval graph completion problem [Ravi et al. 1991], which has applications in archaeology [Kendall 1969] and clone fingerprinting [Karp 1993]. The problem was rediscovered in Golovach [1997] under the name *total vertex separation number*.

The edge bisection problem (EDGEBIS) has a wide range of applications, notably in the area of parallel computing and VLSI [Bhatt and Leighton 1984; Shing and Hu 1986; Hromkovič and Monien 1992; Leighton 1993; Diekmann et al. 1994]. The vertex bisection problem (VERTBIS) is relevant to fault-tolerance, and is related to the complexity of sending messages to processors in interconnection networks via vertex-disjoint paths [Klasing 1998].

In the remaining of this section, we present several further applications related to layout problems.

3.1. Layout Problems in Numerical Analysis

In the area of numerical analysis, it is desirable for many engineering applications to reorder the rows and columns of very large sparse symmetric matrices in such a way that their non-zero entries lie as close as possible to the diagonal. Recall that a matrix is sparse when it has very few non-zero entries. Specifically, the *bandwidth* of a symmetric matrix M is the largest integer b for which there is a non-zero entry at $M[j, j+b]$ and the *profile* of M is $\sum_{i\in[n]}(i - p_i)$ where p_j is the index of the first non-zero entry of row j. Reducing the bandwidth and/or the profile of a matrix leads to a reduction of the amount of space needed for some storage schemes and to an improvement of the performance of several common operations such as Choleski factorization of non-singular systems of equations [Saad 1996]. The problem of reducing the bandwidth or the profile of a matrix M consists in finding a permutation matrix P such that $M' = P \cdot M \cdot P^T$ has minimal bandwidth or minimal profile. Recall that a permutation matrix P is an identity matrix with the same size of M whose columns have been permuted.

Observe that if we identify the non-zero entries of a symmetric matrix with the edges of a graph and the permutations of rows and columns with flips of the vertex labels, then the bandwidth of the graph equals the bandwidth of the matrix, and the profile of the graph equals the profile of the matrix.

The problem of reducing the bandwidth or the profile of a sparse symmetric matrix has a long history since it originated in the fifties; see for instance the references given in Gibbs et al. [1976]. Nowadays, there exist general sparse methods that are more efficient than these "envelope schemes." However, many commercial packages still offer functions to reduce the bandwidth or the profile of sparse matrices as a preprocessing step. Thus, improvements in these methods can be ported to this software without a complete reorganization of their architecture Barnard et al. [1995]. Efficient algorithms to perform several operations on matrices with small bandwidth can be found, for instance, in Saad [1996]. Information retrieval to browse hypertext is a recent area where bandwidth and profile reduction techniques are also used [Botafogo 1993; Simon and Teng 1997].

3.2. Layout Problems in VLSI

Many layout problems are originally motivated as simplified mathematical models of VLSI layout. Given a set of modules, the VLSI layout problem consists in placing the modules on a board in a nonoverlapping manner and wiring together the terminals on the different modules according to a given wiring specification and in such a way that the wires do not interfere among them. There are two stages in VLSI layout: placement and routing. The *placement problem* consists in placing the modules on a board; the *routing problem* consists in wiring together the terminals on different modules that should be connected. A VLSI circuit can be modeled by the means of a graph, where the edges represent the wires and the vertices represent modules. Of course, this graph is an oversimplified model of the circuit, but understanding and solving problems in this simple model can help to obtain better solutions for the real-world model.

A first approach to solve the placement phase was to use the minimum linear arrangement problem in order to minimize the total wire length [Harper 1970; Adolphson and Hu 1973]. Recently an alternative approach to solve this problem has been considered; it consists in finding recursively minimal cuts with minimal capacity among all cuts that separates the graph into two components of equal size. The edge bisection problem aims at this approach [Simon and Teng 1997]. Nowadays, integrated circuit technology has changed substantially and some of the early applications of layouts are obsolete.

The cutwidth of a graph times the order of the graph gives a measure of the area needed to represent the graph in a VLSI layout when vertices are laid out in a row [Lengauer 1982]. In fact, Raspaud et al. [1995] prove a new relation between the cutwidth and the area of the VLSI layout of a graph: the minimal area of a VLSI layout of a graph is not less than the square of its cutwidth. A similar relation between area and edge bisection was well known for graphs with maximum degree 4 [Thompson 1979].

3.3. Layout Problems in Graph Drawing

Perhaps one of the most important goals in graph drawing is to produce aestethic representations of graphs. Reducing the number of crossing edges is a way to improve the readability and comprehension of a graph. A bipartite drawing or 2-layer drawing is a graph representation where the vertices of a bipartite graph are placed in two parallel lines and the edges are drawn with straight lines between them. The bipartite crossing number of a bipartite graph is the minimal number of edge crossings over all bipartite drawings. In Shahrokhi et al. [2001] it is proved that for a large class of bipartite graphs, reducing the bipartite crossing number is equivalent to reducing the total edge length, that is, to the minimum linear arrangement problem. Moreover, an approximate solution of MinLA can be used to generate an approximate solution to the bipartite crossing number problem. See Shahrokhi et al. [1997] for a bound and applications of crossing numbers and Vrťo [2002] for an extensive bibliography.

3.4. Layout Problems as Embedding Problems

Linear arrangements are a particular case of embedding graphs in d-dimensional grids or other graphs. In its most general form, the embedding of a graph G into a host graph H consists in defining an injective function mapping the vertices of G to the vertices of H, and associating a path in H for each edge of G. Three parameters are fundamental to assess the quality of an embedding: the dilation, the congestion, and the load. The *dilation* of an embedding is the length of the largest associated path. The *congestion* of an embedding is the maximal number of paths that share an edge of H. The *load* of an embedding is the maximum number of vertices of G that are mapped to a vertex of H. Making use of good embeddings is essential in certain contexts, as in parallel computing where embeddings can be used to simulate an algorithm designed for one type of network on a parallel machine with a different type of network; see Monien and Sudborough [1990] for a nice survey.

The case in which a graph with n vertices must be embedded into a path graph P_n of n vertices with load 1 is perhaps the simplest nontrivial embedding problem and has been intensively studied in the literature [Kendall 1969; Adolphson and Hu 1973; Adolphson 1977; Harper 1977; Robertson and Seymour 1985; Makedon and Sudborough 1989; Ravi et al. 1991; Botafogo 1993; Karp 1993; Leighton 1993; Saad 1996]. In this particular case, some layout problems and embedding problems are closely related. Specifically, the bandwidth of a graph corresponds to the minimal dilation, and the cutwidth to the minimal congestion.

There exists other interesting embeddings on more general graphs than the path. For instance, Raspaud et al. [2000] present a survey on cyclic cutwidth and cyclic bandwidth, that is, when the graph is embedded into a cycle, rather than a path. Few results are known for other cyclic width parameters, Ching Guu solved the cyclic MINLA problem for the d-cube (H. L. Harper, Personal communication). Of practical importance are also the embeddings into grids [Miller [1991]; Lin 1994; Bezrukov et al. 2000a]. Several variants of embeddings into trees were studied in Hu [1974]; Johnson et al. [1978]; Seymour and Thomas [1994]; Ding and Oporowski [1995]; Wu et al. [1999]; Àlvarez et al. [2000].

3.5. Layout Problems in Parallel and Distributed Processing

Many parallel computers are made up of a set of processors with their own private memory that exchange messages by the way of a communication network. In order to get good speedups when using such a system, it is important to distribute the total amount of work among the processors as evenly as possible to minimize idle times. It is also important to reduce the amount of communication among the processors, because communicating through the network is much slower than the speed of the processors. Well established mapping and load balancing techniques have been developed to address these situations. In certain cases, these techniques lead to graph partitioning problems; see Diekmann et al. [1994]; Leighton [1993]; Hromkovič and Monien [1992].

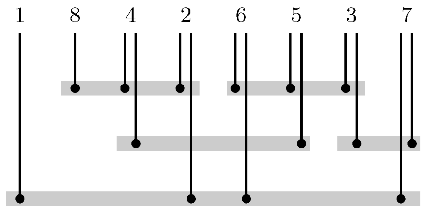
The graph partitioning problem consists in partitioning the vertices of a given graph in k sets of nearly same size in such a way that the number of cutting edges between the k sets is minimal. The edge bisection problem is a particular case of graph partitioning where $k = 2$. Recursive bisection is a popular technique to obtain partitions when k is a power of 2. See Simon and Teng [1997] for an analysis of recursive bisection and see the references in Bezrukov [1999] for more information on graph partitioning.

Edge bisection can be of use when solving partial differential equations and using finite elements methods in parallel systems. Simplifying, in these problems a particular iterative computational task has to be carried out in every vertex of a mesh (or the graph defined by the particular topology of the system) and its computation involves data from this vertex and in its neighbors. A way to distribute the total amount of computation between two processors is to assign to each one half of the vertices in the grid. But as border vertices need to communicate in order to get their operands, it is necessary to reduce the cut of the bisection.

3.6. Some Equivalent Problems

The vertex separation problem is strongly connected with several other important NP-complete problems: gate matrix layout, pathwidth, and vertex search number. The bandwidth problem is also related to proper pathwidth.

The gate matrix layout is a well studied problem with application in CMOS circuit design [Deo et al. 1987]. An instance of the gate matrix layout problem consists in a collection of nets $\{N_1, \ldots, N_n\}$ and their respective connection to a set of gates $\{G_1, \ldots, G_m\}$. Nets are identified with rows and gates are identified with



	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8
N_1	1	1	0	0	0	1	1	0
N_2	0	0	0	1	1	0	0	0
N_3	0	0	1	0	0	0	1	0
N_4	0	1	0	1	0	0	0	1
N_5	0	0	1	0	1	1	0	0

Fig. 2. Example of a gate matrix layout with three tracks (figure after Bodlaender).

columns. The goal of the problem is to seek a permutation of the columns that minimizing the number of tracks required to lay out the chip, which is equivalent to minimizing its area. Figure 2 shows an example of gate matrix layout. We denote by $\text{MINGML}(G)$, the minimal number of tracks needed by a graph G.

The pathwidth problem has received a great interest in recent years due to its relation with the graph minors theory [Robertson and Seymour 1985]. A *path-decomposition* of a graph $G = (V, E)$ is a sequence of subsets of vertices $(X_1, \ldots, X_r)$ such that

—$\bigcup_{i=1}^{r} X_i = V$;
—both endpoints of any edge $e \in E$ belong to some X_i for $1 \leq i \leq r$; and
—for all $i \leq j \leq k$, it is the case that $X_i \cap X_k \subseteq X_j$.

The pathwidth of a path decomposition $(X_1, \ldots, X_r)$ is

$$\text{MINPW}(G, (X_1, \ldots, X_r)) = \max_{i \in [r]} |X_i| - 1.$$

The pathwidth of G, denoted $\text{MINPW}(G)$, is the minimal pathwidth over all possible path decompositions of G. The pathwidth problem (PATHWIDTH) consists in determining a path decomposition with minimal pathwidth. Figure 3 shows a graph and one of its path decompositions. See Bodlaender [1993] and Downey and Fellows [1999] for more information on the pathwidth problem.

The vertex search number problem was introduced in Kirousis and Papadimitriou [1986]. Briefly stated, this problem asks how many searchers are needed to capture an unlimited number of intruders moving around the edges of a given graph. We denote by $\text{MINSN}(G)$ the minimal vertex search number of graph G.

The equivalence between the gate matrix layout, search number, pathwidth and vertex separation problems is a consequence of the results in Kirousis and Papadimitriou [1986]; Kinnersley [1992]; Fellows and Langston [1994]:

THEOREM 3.1 *For any graph G,*

$$\text{MINVS}(G) = \text{MINPW}(G) = \text{MINSN}(G) - 1 = \text{MINGML}(G) + 1.$$

Let $I_v = \{i : v \in X_i\}$. A *proper path decomposition* is a path decomposition that also satisfies $I_u \not\subset I_v$ for all $u, v \in V$. The proper pathwidth of G, denoted $\text{MINPPW}(G)$, is the minimal pathwidth over all possible proper path decompositions of G. The proper pathwidth problem consists in determining a proper path decomposition with minimal pathwidth. The proper pathwidth of a graph G can also be defined as the minimum cardinality of a maximum clique of a proper interval subgraph of G decreased by one.

The following equivalence between bandwidth and proper pathwidth is due to Kaplan and Shamir [1996]:

THEOREM 3.2 *For any graph G,* $\text{MINPPW}(G) = \text{MINBW}(G)$.

4. NP-COMPLETENESS RESULTS

It is widely believed that showing that a problem is NP-complete is equivalent to prove its computational intractability [Garey and Johnson 1979]. The following theorem indicates the difficulty of the considered layout problems on arbitrary graphs.

THEOREM 4.1 *The decisional versions of the layout problems* BANDWIDTH, MINLA*,*

Table III. Review of NP-Completeness Results for Decisional Graph Layout Problems

Problem	NP-complete	Ref.
Bandwidth	in general	[Papadimitriou 1976]
	for trees with maximum degree 3	[Garey et al. 1978]
	for caterpillars with hair-length ≤ 3	[Monien 1986]
	for caterpillars with ≤ 1 hair per backbone vertex	[Monien 1986]
	for cyclic caterpillars with hair-length 1	[Muradyan 1999]
	for grid graphs and unit disk graphs	[Díaz et al. 2001a]
MinLA	in general	[Garey et al. 1976]
	for bipartite graphs	[Even and Shiloach 1975]
Cutwidth	in general	[Gavril 1977]
	for graphs with maximum degree 3	[Makedon et al. 1985]
	for planar graphs with maximum degree 3	[Monien and Sudborough 1988]
	for grid graphs and unit disk graphs	[Díaz et al. 2001a]
ModCut	for planar graphs with maximum degree 3	[Monien and Sudborough 1988]
VertSep	in general	[Lengauer 1981]
	for planar graphs with maximum degree 3	[Monien and Sudborough 1988]
	for chordal graphs	[Gustedt 1993]
	for bipartite graphs	[Goldberg et al. 1995]
	for grid graphs and unit disk graphs	[Díaz et al. 2001a]
SumCut	in general	[Díaz et al. 1991]
		[Lin and Yuan 1994b]
		[Golovach 1997]
	for cobipartite graphs	[Yuan et al. 1998]
EdgeBis	in general	[Garey et al. 1976]
	for graphs with maximum degree 3	[MacGregor 1978]
	for graphs with maximum degree bounded	[MacGregor 1978]
	for d-regular graphs	[Bui et al. 1987]

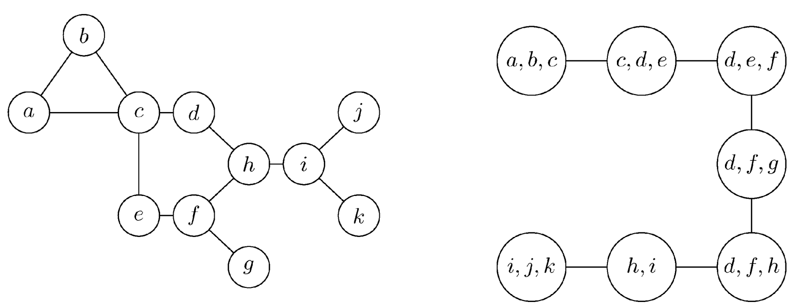


Fig. 3. Example of a graph and one of its path decompositions with pathwidth 2.

Cutwidth , ModCut , VertSep , SumCut , EdgeBis *are NP-complete.*

The reduction for Bandwidthis given in Papadimitriou [1976], the proofs for MinLA and EdgeBis are given in Garey et al. [1976], for Cutwidth the completeness is due to Gavril [1977], for ModCut is given in Monien and Sudborough [1988], for VertSep the proof is in Lengauer [1981], and for SumCut it can be found in Diàz et al. [1991]; Lin and Yuan [1994b]; Golovach [1997].

Many layout problems remain NP-complete even for certain restricted classes of graph; Table III draws a synthetic overview on these results. For instance, Bandwidthis NP-complete even when restricting its inputs to trees with maximum degree three [Garey et al. 1978]. This result was improved in Monien [1986] proving that Bandwidthremains

NP-complete for caterpillars with hairs of length at most three, and for caterpillars with at most one unbounded hair attached to the backbone. The problem is also NP-complete for cyclic caterpillars as reported in Muradyan [1999]. Recall that a caterpillar is a particular class of tree made of a set of paths, called the hairs, attached by one of their leaves to the vertices of another path, called the backbone. A cyclic caterpillar has as backbone, a cycle instead of a path.

The CUTWIDTH problem is NP-complete even for graphs with maximum degree three [Makedon et al. 1985]. This result was strengthened by showing that CUTWIDTH , VERTSEP and MODCUT are NP-complete even for planar graphs with maximum degree three [Monien and Sudborough 1988]. EDGEBIS is also NP-complete for d-regular graphs [Bui et al. 1987] and for graphs with maximal degree 3 [MacGregor 1978]. It is also known that MINLA is NP-complete when restricted to bipartite graphs [Even and Shiloach 1975]. The NP-completeness of VERTSEP for chordal graphs was proved in Gustedt [1993], and for bipartite graphs in Goldberg, et al [1995]. In Diàz et al. [2001a] it is proved that BANDWIDTH, CUTWIDTH and VERTSEP remain NP-complete even when restricted to grid graphs or unit disk graphs. Finally, it is also known that SUMCUT is NP-complete when restricted to cobipartite graphs [Yuan et al. 1998].

The complexity of most of the layout problems for many families remains open. For instance, it is unknown if MINLA or SUMCUT are NP-complete for sparse graphs. Also the NP-completeness of MINLA and SUMCUT remain open for planar and/or for series-parallel graphs. The complexity of VERTBIS for the general case is unknown.

5. CLASSES OF GRAPHS WITH POLYNOMIAL TIME ALGORITHMS

NP-completeness results do not rule out the existence of efficient algorithms to get optimal solutions on particular classes of graphs. We now review this type of result for layout problems.

In the case of the MINLA problem, the optimal value of MINLA for hypercubes is known [Harper 1964] and there exists a closed formula for the value of the de Bruijn graph of order four [Harper 1970]. The motivation for the former case was to minimize the total edge length of the wires needed to connect a Viterbi decoder and the motivation for the latter case was to design error-correcting codes with minimal errors. In the case of a d-dimensional hypercube Q_d, $\text{MINLA}(Q_d) = 2^{d-1}(2^d - 1)$.

According to Chung [1988], Goldberg and Klipker [1976] were the first to give an $O(n^3)$ algorithm to solve the minimum linear arrangement problem for trees. Adolphson and Hu [1973] provide an $O(n \log n)$ algorithm to compute the MINLA of a rooted tree with n vertices. Shiloach [1979] improves the result by presenting an algorithm to solve the MINLA on trees of n vertices in $O(n^{2.2})$ time. This was further improved by Chung [1988], who gave an $O(n^{\log 3/\log 2})$ algorithm. The optimal value for the MINLA problem on a complete binary tree with k levels $T_{2,k}$ has an explicit expression discovered by Chung: For all $k \geq 2$,

$$\text{MINLA}(T_{2,k}) = 2^k \left(\frac{1}{3}k + \frac{5}{18}\right) + \frac{2}{9}(-1)^k - 2.$$

A recursive expression was also presented by Chung for the case of complete ternary trees.

With respect to parallel algorithms, Diàz et al. [1997a] proved that MINLA for trees is in NC, as it can be solved in $O(\log^2 n)$ time using a CREW PRAM with $O(n^{3.6})$ processors; see Greenlaw et al. [1995] for concepts on parallel complexity.

The MINLA problem on square or rectangular grids has a peculiar history. Let $L_{m,n}$ denote a $m \times n$ rectangular grid graph and let $L_n = L_{n,n}$ denote a $n \times n$ square grid graph. The problem was first solved by Muradyan and Piliposyan [1980], in a paper written in Russian, for the general case of rectangular grids. Latter, in 1986, Mitchison and Durbin [1986] presented

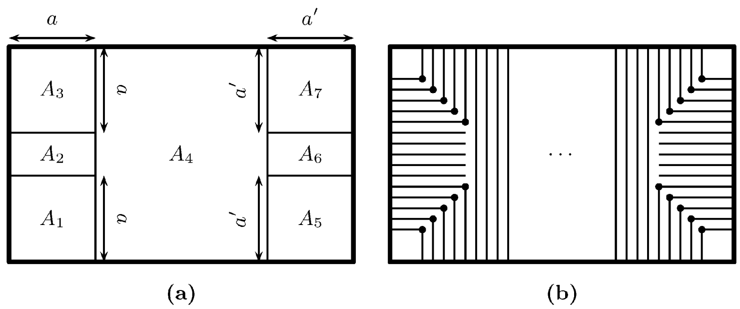


Fig. 4. Schematic representation of the optimal layout of MINLA($L_{m,m'}$) for rectangular $m \times m'$ grids (figure reproduced from Muradyan and Piliposyan [1980]).

the solution for square grids only. In 1981, Niepel and Tomasta [1981] incorrectly conjectured that the lexicographic layout is optimal for MINLA(L_m). In a paper published in 1994, Nakano [1994] again referenced this conjecture. Still in 2000, Fishburn et al. [2000] presented a solution for MINLA($L_{m,m'}$).

Indeed, the optimal layout for MINLA($L_{m,m'}$) on $m \times m'$ rectangular grids has an interesting solution, which is well described in Bezrukov [1999]. The optimal numbering is shown schematically in Figure 4. The numbering starts with the left lower corner of the grid and then fills the areas $A_1, A_2, \ldots, A_7$, where A_1, A_3 are $a \times a$ squares and A_5, A_7 are $a' \times a'$ squares; see Figure 4(a). The values of a and a' must satisfy

$$a, a' \in \left\{ \left\lceil m - \frac{1}{2} - \sqrt{\frac{1}{2}m^2 - \frac{1}{2}m + \frac{1}{4}} \right\rceil, \right. \times \left. \left\lfloor m + \frac{1}{2} - \sqrt{\frac{1}{2}m^2 - \frac{1}{2}m + \frac{1}{4}}. \right\rfloor \right.$$

The way to number the areas is shown in Figure 4(b). Each square must be numbered sequentially. The square A_1 must be filled row after column. The square A_2 must be filled by consecutive rows, from bottom to top and from left to right. The square A_3 must be enumerated with the reverse order with respect to A_1. The square A_4 must be numbered by columns from bottom to top and from left to right. Finally, the squares A_5, A_6 and A_7 are filled in the same way. Using this solution, we get

$$\begin{aligned} \text{MINLA}(L_{m,m'}) &= -\frac{2}{3}a^3 + 2ma^2 \\ &\quad - \left(m^2 + m - \frac{2}{3}\right)a + m' \\ &\quad \times (m^2 + m - 1) - m \end{aligned}$$

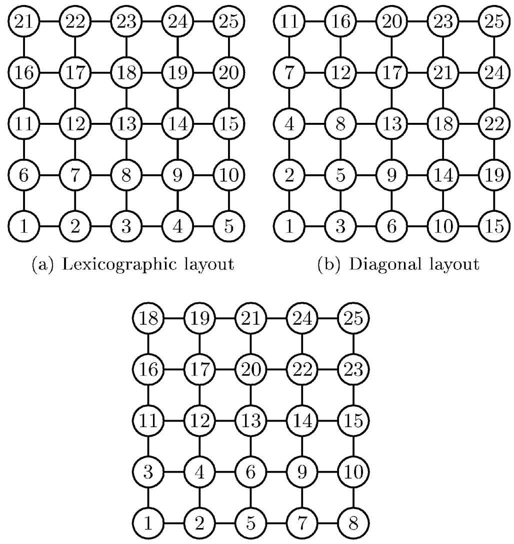
and

$$\text{MINLA}(L_m) = \frac{4 - \sqrt{2}}{3}m^3 + O(m^2).$$

The following theorem due to Muradyan and Piliposyan [1980]; Mitchison and Durbin [1986]; Bollobás and Leader [1991]; Leighton [1993]; Diàz [2000]; Fishburn et al. [2000] gathers the optimal values of layout problems for square grids:

THEOREM 5.1 *Let L_m be an square grid of side m. Then,*

$$\begin{aligned} \text{MINVS}(L_m) &= m, \text{MINCW}(L_m) = m \\ &\quad + (\text{odd } m), \\ \text{MINSC}(L_m) &= \frac{2}{3}m^3 + \frac{1}{2}m^2 \\ &\quad - \frac{7}{6}m, \text{MINVB}(L_m) = m, \\ \text{MINEB}(L_m) &= m + (\text{odd } m), \text{MINLA}(L_m) \\ &= \frac{1}{3}(4 - \sqrt{2})m^3 + O(m^2). \end{aligned}$$



(a) Lexicographic layout (b) Diagonal layout

(c) Muradyan–Piliposjan layout

Fig. 5. Optimal layouts for 5×5 square grids.

Optimal layouts for 5×5 square grids are shown in Figure 5. The lexicographic layout is optimal for VERTSEP , BANDWIDTH, EDGEBIS and CUTWIDTH; the Muradyan–Piliposjan layout is optimal for MINLA; the diagonal layout is optimal for VERTSEP, VERTBIS, SUMCUT and BANDWIDTH.

The exact results on VERTBIS, SUMCUT and BANDWIDTHare also known for other multi-dimensional grids and tori; see Bezrukov [1999]. The VERTSEP problem for n-dimensional grids was solved in Bollobás and Leader [1991]. Notice that nothing is known about the optimal solutions of grid graphs with holes, except for the edge bisection problem, for which Papadimitriou and Sideri [1996] gave an $O(n^5)$ algorithm. Finding a faster algorithm is interesting due to its practical relevance.

A d-dimensional c-ary clique is a graph with vertices labeled by integers from 0 to $c^d - 1$ and edges connecting vertices whose c-ary representation differ in one and only one digit. Lindsey [1964] solved the MINLA problem on this kind of graph, another proof can be found in Nakano [1994]. More exact MINLA results for several other particular classes of graphs have been identified; see Bezrukov [1999] and references therein.

The cutwidth problem has a very similar trajectory. Harper [1966] seems to be the one who first solved it for the case of hypercubes. Chung et al. [1982] presented an $O(n \log^{d-2} n)$ time algorithm for the cutwidth of trees with n vertices and with maximum degree d. Yannakakis [1985] improved that result by giving an algorithm to determine the cutwidth of a tree of n vertices in $O(n \log n)$ time. In the case of a k-level t-ary tree $T_{t,k}$, it holds that

$$\text{MINCW}(T_{t,k}) = \left\lceil \frac{1}{2}(k-1)(t-1) \right\rceil, \quad \forall k \geq 3.$$

Exact cutwidths of 2-dimensional grids and 3-dimensional grids have been found by Rolim et al. [1995], who also present results for cylindrical and toroidal meshes. In particular, for $m, n \geq 2$, they prove that

$$\text{MINCW}(L_{m,n}) = \begin{cases} 2, & \text{if } m = n = 2, \\ \min\{m+1, n+1\}, & \text{otherwise.} \end{cases}$$

On the other hand, Thilikos et al. [2001] present an algorithm to compute the cutwidth of bounded degree graphs with small treewidth in polynomial time.

With respect to parallel algorithms, Díaz et al. [1997a] proved that an optimal layout for the cutwidth of a tree with n vertices and degree Δ can be computed in $O(\Delta \log^2 n)$ time using a CREW PRAM with $O(n^{3.6})$ processors. The parallel complexity of the CUTWIDTH problem for trees with unbounded degree is an open problem.

For the SUMCUT or PROFILE problems, Lepin [1986] gave the first polynomial exact algorithm for trees. This result was improved by the linear time algorithm presented in Díaz et al. [1991]. These authors also gave a parallel algorithm for computing the optimal sum cut layout of a tree with n vertices in $O(\log n)$ time using a CREW PRAM with $O(n^2 \log n)$ processors. Kuo and Chang [1994] also gave a sequential polynomial time algorithm for the same problem on trees.

In the case of the VERTSEP problem, Ellis et al. [1979] gave a linear algorithm to compute the optimal vertex separation of a tree, and an $O(n \log n)$ algorithm to find the optimal layout. Recently, an linear time algorithm to find the optimal layout has been presented in Skodinis [2000]. Further polynomial time algorithms to compute the vertex separation of permutation graphs and cographs were given in Bodlaender et al. [1995b] and Bodlaender and Möhring et al. [1993] respectively.

In the case of the edge bisection problem, Leighton [1993] showed how to minimize the bisection width of Cartesian products of paths of the same length, provided the length is even. Nakano [1994] solved the problem for odd lengths. Rolim et al. [1995] have determined the optimal edge bisection of ordinary, cylindrical and toroidal 2-dimensional grids and of ordinary and toroidal 3-dimensional grids. The following theorem states their results for $L_{m,n}$:

THEOREM 5.2 *Let $L_{m,n}$ be a rectangular grid. For $2 \le m \le n$,*

$$\text{MINEB}(L_{m,n}) = m + (\text{odd } n)$$

and, for $m \ge 2$, $n \ge 3$,

$$\text{MINEB}(L_{m,n}) = \begin{cases} \min\{2m, n\}, & \text{if } m \text{ and } n \text{ are even}, \\ \min\{2m, n+2\}, & \text{if } m \text{ is odd and } n \text{ is even}, \\ \min\{2m+1, n\}, & \text{if } m \text{ is even and } n \text{ is odd}, \\ \min\{2m+1, n+2\}, & \text{if } m \text{ and } n \text{ are odd}. \end{cases}$$

A related result is considered in Azizoğlu and Eğecioğlu [2002] for the bisection of d-dimensional generalized cylinders (products of d graphs, each of them being a path or a cycle). The bisection of the hypercube seems to have been solved by many people concurrently (see e.g., [Nakano 1994]). A closed formula for the edge bisection of cube-connected cycles graphs is given in Manabe et al. [1984]. For the case of the trees, MacGregor [1978] gives an $O(n^3)$ algorithm. A parallel algorithm running on $O(\log^2 n \log \log n)$ time on a CRCW PRAM with $O(n^2)$ processors for the bisection of trees, based on MacGregor's algorithm is presented On Goldberg and Miller [1988]. An $O(n^2)$ algorithm to compute optimal bisection of partial k-trees is given in Soumyanath and Deogun [1990]. Recall that a partial k-tree is a graph with bounded treewidth.

In Muradyan and Piliposyan [1980], the MINLA and CUTWIDTH problems are solved for complete p-partite graphs. The complete p-partite graph $K(N_1, \dots, N_p)$ is defined as a graph whose vertex set can be partitioned into p, so that two vertices are adjacent if and only if they belong to different partitions. If the number of vertices in two different partitions differ in at most one, a p-partite is called balanced. It should be noted also that $\text{MINLA}(K(N_1, N_2, \dots, N_p))$ has a nested solution.

As previously mentioned, the Bandwidth problem is NP-complete when restricted to trees. However, in the case of a k-level t-ary tree $T_{t,k}$, it holds that

$$\text{MINBW}(T_{t,k}) = \left\lceil \frac{t(t^{k-1} - 1)}{2(k-1)(t-1)} \right\rceil.$$

The embedding of complete binary trees with optimal bandwidth was presented in Heckmann et al. [1998]. There also exists an $O(n \log n)$ algorithm to determine the bandwidth of caterpillars with hairs of length at most two [Assman et al. 1981].

Other classes of graphs whose bandwidth can be computed efficiently are interval graphs [Muradyan 1986; Mahesh et al. 1991; Sprague 1994], butterflies [Lai 1997] and chain graphs [Kloks et al. 1998]. Recall that interval graphs are intersection graphs of a set of intervals over the real line, and that chain graphs are bipartite graphs $G = (X, Y, E)$ where there is an ordering $x_1, x_2, \dots, x_{|X|}$ of X such that $\Gamma(x_1) \subseteq \Gamma(x_2) \subseteq \cdots \subseteq \Gamma(x_{|X|})$. The first paper containing a polynomial time algorithm for BANDWIDTHfor interval graphs was Muradyan [1986]. The second paper claiming the result was Kratsch [1987].

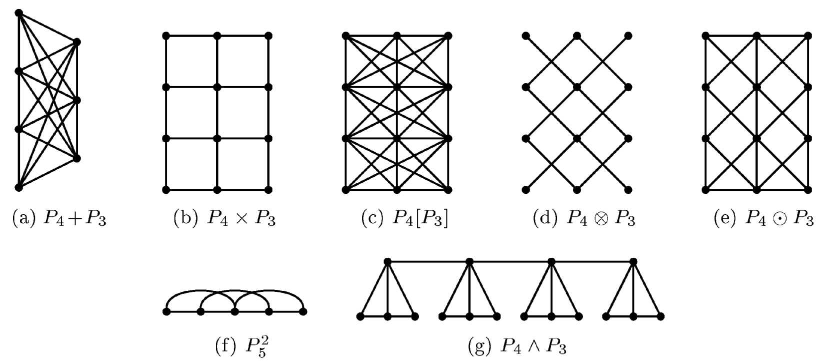


Fig. 6. Examples of graph composition: sum (a), cartesian product (b), composition (c), tensor product (d), strong product (e), power (f) and corona (e).

The third algorithm was proposed in Kleitman and Vohra [1990] and takes time $O(n\text{BW}(G))$. Flaws in the proof of Kratsch's algorithm were discovered independently, in two papers that propose a correct algorithm Mahesh et al. [1991] and [Sprague 1994]. The algorithm in Mahesh et al. [1991], takes time $O(n^2)$ and is a corrected version of Kratsch's algorithm. The algorithm in Sprague [1994] takes time $O(n \log n)$ and extends the result in Kleitman and Vohra [1990]. SumCut is the other layout problem that has a polynomial time algorithm for interval graphs Lin and Yuan [1994a].

Table IV summarizes which classes of graphs are known to be optimally solvable in polynomial time for graph layout problems.

Other than grids, several results are known for families of graphs that can be described applying basic composition operations on paths (P_n), cycles (C_n), trees (T_n), complete graphs (K_n), or complete bipartite graphs ($K_{n,m}$). The definitions of the operations are taken from Lai and Williams [1999].

—The r-th power of a graph G is the graph G^r, with the same vertex set as G and an edge uv if $d(u, v) \leq r$ in G.

—The *sum* of $k \geq 2$ pairwise disjoint graphs $G_1, \ldots, G_k$, denoted as $G_1 + \cdots + G_k$, is the graph G with vertex set $\cup_{1 \leq i \leq k} V(G_i)$ and an edge uv if for some $i \neq j$ $u \in V(G_i)$ and $v \in V(G_j)$ or if for some i $u, v \in V(G_i)$ and $uv \in E(G_i)$.

—The *cartesian product* of two graphs G and H, denoted as $G \times H$, is the graph with vertex set $V(G) \times V(H)$ where (u_1, v_1) is adjacent to (u_2, v_2) if either $u_1u_2 \in E(G)$ and $v_1 = v_2$ or $v_1v_2 \in E(H)$ and $u_1 = u_2$.

—The *composition* of two graphs G and H, denoted as $G[H]$, is the graph with vertex set $V(G) \times V(H)$ where (u_1, v_1) is adjacent to (u_2, v_2) if either $u_1u_2 \in E(G)$ or $v_1v_2 \in E(H)$ and $u_1 = u_2$.

—The *tensor product* of two graphs G and H, denoted as $G \otimes H$, is the graph with vertex set $V(G) \times V(H)$ where (u_1, v_1) is adjacent to (u_2, v_2) if $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$.

—The *strong product* of two graphs G and H, denoted as $G \odot H$, is the graph with vertex set $V(G) \times V(H)$ where (u_1, v_1) is adjacent to (u_2, v_2) if one of the following holds: (a) $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$, (b) $u_1 = u_2$ and $v_1v_2 \in E(H)$, or (c) $v_1 = v_2$ and $u_1u_2 \in E(G)$.

—The *corona* of two graphs G and H is denoted as $G \wedge H$ and contains a copy of G and a copy of H for each vertex of G. Each vertex of G is connected to every vertex in the corresponding copy of H.

Figure 6 illustrates these definitions and Table V summarizes the known results.

The problem of computing the bandwidth of a *Hamming graph* is an interesting open problem, because it gives a measure of the effects of noise in the

Table IV. Review of Classes of Graphs Optimally Solvable in Polynomial Time (n denotes the number of vertices in the graph, m its number of edges and Δ its maximal degree)

Problem	Class of graph	Complexity	Ref.
BANDWIDTH	caterpillars with hair-length ≤ 2	$O(n \log n)$	[Assman et al. 1981]
	hypercubes	$O(n \log n)$	[Harper 1966]
	butterflies	$O(n \log n)$	[Lai 1997]
	interval graphs	$O(n\Delta^2 \log \Delta)$	[Muradyan 1986]
	interval graphs	$O(n \log n)$	[Mahesh et al. 1991]
	interval graphs	$O(n \log n)$	[Sprague 1994]
	chain graphs	$O(n^2 \log n)$	[Kloks et al. 1998]
	complete k-level t-ary tree	$O(n)$	[Heckmann et al. 1998]
	square grids	$O(n)$	[Mai and Luo 1984]
			[Díaz et al. 2000]
MINLA	trees	$O(n^3)$	[Goldberg and Klipker 1976]
	rooted trees	$O(n \log n)$	[Adolphson and Hu 1973]
	trees	$O(n^{2.2})$	[Shiloach 1979]
	trees	$O(n^{\log 3/\log 2})$	[Chung 1988]
	rectangular grids	$O(n)$	[Muradyan and Piliposyan 1980]
	square grids	$O(n)$	[Mitchison and Durbin 1986]
	2-dimensional cylinder	$O(n)$	[Muradyan 1982]
	hypercubes	$O(n)$	[Harper 1964]
	de Bruijn graph of order 4	$O(n)$	[Harper 1970]
	d-dimensional c-ary cliques	$O(n)$	[Lindsey 1964]
	complete p-partite graphs	$O(n + p\log(p))$	[Muradyan and Piliposyan 1988]
CUTWIDTH	trees	$O(n \log^{\Delta-2} n)$	[Chung et al. 1982]
	trees	$O(n \log n)$	[Yannakakis 1985]
	hypercubes	$O(n)$	[Harper 1964]
	d-dimensional c-ary cliques	$O(n)$	[Nakano 1994]
	max degree $\leq \Delta$ and treewidth $\leq k$	$O(n^{\Delta k^2})$	[Thilikos et al. 2001]
	ordinary 2- and 3-dim. meshes	$O(n^2)$	[Rolim et al. 1995]
	toroidal and cylindrical 2-dim. meshes	$O(n^2)$	[Rolim et al. 1995]
	toroidal 3-dim. meshes	$O(n^2)$	[Rolim et al. 1995]
	complete p-partite graphs	$O(n + p\log(p))$	[Muradyan and Piliposyan 1988]
VERTSEP	trees	$O(n \log n)$	[Ellis et al. 1979]
	trees	$O(n)$	[Skodinis 2000]
	cographs	$O(n)$	[Bodlaender and Möhring 1993]
	permutation graphs	$O(n^2)$	[Bodlaender et al. 1995b]
	n-dimensional grids	$O(n^2)$	[Bollobás and Leader 1991]
SUMCUT	trees	$O(n^{2.3})$	[Lepin 1986]
	trees	$O(n)$	[Díaz et al. 1991]
	trees	$O(n^{1.722})$	[Kuo and Chang 1994]
	square grids	$O(n)$	[Díaz et al. 2000]
	interval graphs		[Lin and Yuan 1994a]
EDGEBIS	trees	$O(n^3)$	[MacGregor 1978]
	hypercubes	$O(n)$	[Nakano 1994]
	d-dimensional c-ary arrays	$O(n)$	[Nakano 1994]
	d-dimensional c-ary cliques	$O(n)$	[Nakano 1994]
	ordinary 2- and 3-dim. meshes	$O(n^2)$	[Rolim et al. 1995]
	toroidal and cylindrical 2-dim. meshes	$O(n^2)$	[Rolim et al. 1995]
	toroidal 3-dim. meshes	$O(n^2)$	[Rolim et al. 1995]
	grid graphs	$O(n^5)$	[Papadimitriou and Sideri 1996]
	treewidth $\leq k$	$O(n^2)$	[Soumyanath and Deogun 1990]
	cube-connected cycles graphs	$O(n)$	[Manabe et al. 1984]

Table V. Review of Families of Composite Graphs Optimally Solvable in Polynomial Time or with a Closed Formula. Graph names and operators are defined in the text

Problem	Class of graph	Ref.
Bandwidth	$G_1 + \cdots + G_k$	[Lai et al. 1994]
	$P_n \times \overset{d}{\ldots} \times P_n$	[Chvátalová 1975]
	$P_n \times C_m$	[Muradyan 1982]
	$C_n \times C_m$	[Lai and Williams 1995]
	$K_n[G]$, $P_n[G]$, $C_n[G]$	[Hendrich and Stiebitz 1992]
	$K_{1,n}[G]$, $K_n[G]$ for $G \in \{P_m, T_m, C_m\}$ or $G = C_{n_1} \times \ldots \times C_{n_k}$ with $n_i \leq 5$ or $G = P_{n_1} \times \ldots \times P_{n_k}$ with $n_i > 1$	[Liu and Williams 1995]
	$P_n^r[G]$, $C_n^r[G]$	[Chinn et al. 1995]
	$P_n \times P_m[G]$, $P_n \times C_m[G]$ with $2n \neq m$ and $C_n \times C_m[G]$ $6 \leq 2n \leq 2s$	[Zhou and Yuan 1998]
	$K_n \odot P_m$	[Hendrich and Stiebitz 1992]
	$P_n \odot P_m$, $C_n \odot P_m$, $P_n \odot C_m$, $C_n \odot C_m$	[Lai and Williams 1995]
	$P_n \otimes P_m$, $C_n \otimes P_m$, $C_n \otimes C_m$	[Lai and Williams 1997]
	$P_k \otimes K_{nm}$	[Williams 1994]
	$P_k \otimes K_n$, $C_k \otimes K_n$	[Williams 1996]
	$K_n \wedge K_m$, $C_n \wedge K_1$, $P_n \wedge K_1$, $C_n \wedge (K_1 \cup \ldots \cup K_1)$	[Chinn et al. 1992]
MinLA	$G_1 + \cdots + G_k$ all G_i are sum deterministic	[Lai and Williams 1994]
	$P_n[P_m]$, $P_n[C_m]$	[Liu 1992]
	$K_n[P_m]$, $K_n[C_m]$	[Liu and Williams 1995]
	$P_k \otimes K_{nm}$	[Williams 1994]
	$P_n \wedge P_m$, $K_n \wedge K_1$	[Williams 1993]
Cutwidth	C_n^r, $P_n \times P_m$, $P_n \times C_m$, $C_n \times C_m$, $K_n \times P_m$ and $K_n \times C_m$, $C_n^s \times C_m^r$, $K_n \times K_m$, $P_n \odot C_m$	[Liu and Yuan 1995]
SumCut	$G_1 + G_2$	[Lin and Yuan 1994b]
	$P_n \times K_m$, $C_n \times K_m$, $C_n \times C_m$	[Mai 1996]
	$K_n[G]$, $P_n[G]$, $C_n[G]$	[Lai 1997]
	$P_k \times K_{nm}$	[Lai 2001]
	$P_n \wedge G$, $C_n \wedge G$, $K_n \wedge G$, $K_{nm} \wedge G$	[Lai 1997]
	$G \wedge H$ where H is a 1-caterpillar, K_n or C_n	[Chang and Lai 2001]

multi-channel transmission of data with that numbering [Berger-Wolf and Reingold 2000]. The best known result, due to Harper [2001], shows that the bandwidth of the Hamming graph is asymptotic to $\sqrt{\frac{2}{\pi d}} n^d$ as $d \to \infty$.

6. FIXED PARAMETER RESULTS

In defining a parameterization, the issue is not whether a problem is hard, but what makes the problem hard or easy to compute. To study the *structural hardness* of a difficult problem, the approach is to split the input into two parts: a *difficult* part (the non-parameterized) and an *easy* part (the parameterized), where we impose some restrictions. For several problems, it is known that the parameterization of the input does not break the NP-completeness barrier. A classical example is the *coloring problem* parameterized by the number k of colors that can be used. It is well known that the problem is NP-complete, for $k \geq 3$. On the other hand, graph problems like the *maximum independent set* or the *minimum vertex cover* become polynomially solvable, for every k, when we parameterize them by the size k of the maximum independent set or the minimum vertex cover.

Even in the cases where a parameterization of an NP-complete problem leads to a polynomial time algorithm, there are different types of upper bounds of the running time for the best known algorithm. For instance if the problem has k as a parameter, the running time could be $O(n^{f(k)})$, or it could be $O(f(k)n^\alpha)$, or it could be $O(f(k) + n^\alpha)$, where $f(k)$ is a function of k and $\alpha \in \mathbb{N}$ and the big-oh notations hide a constant independent of k. A parameterized problem is said to be *fixed parameter tractable* if there exists an $O(f(k)n^\alpha)$-algorithm that solves the problem. The class FPT is the class of all fixed parameter tractable problems. Downey and Fellows defined a parameterized complexity hierarchy, the W*-hierarchy*. Similarly to the theory developed in structural complexity, the W-hierarchy consists of classes of parameterized problems, with different levels of parameterized complexity, between FPT and W [P]: FPT $\subseteq$ W [1] $\subseteq$ W [2] $\subseteq \cdots \subseteq$ W [P], along with suitable notions of reducibility and completeness. The interested reader can have a look at Downey and Fellows [1999] for a nice exposition of parameterized complexity. It is an open problem whether the inclusion in the hierarchy is strict. Moreover, there is evidence that if a problem is complete for some level of the W-hierarchy, then it is not expected to have an $O(f(k)n^\alpha)$-algorithm. For example, the parameterized *vertex cover problem* belongs to FPT [Balasubramanian et al. 1998], while the the parameterized *independent set problem* is known to be W[1]-complete [Downey and Fellows 1995].

Several parameterized complexity results are known for some layout problems. In these results the problems are parameterized by the minimum value of the layout measure. For the case of BANDWIDTH, we will consider the following parameterization for the BANDWIDTH problem:

BANDWIDTH (k): Given a graph G, determine whether $\text{MINBW}(G) \leq k$.

We will use the same notation and the same parameterization for the remaining layout problems.

The earliest result proved that it is possible to decide the BANDWIDTH (2) problem and the CUTWIDTH (2) problem in linear time [Garey et al. 1978]. Regarding the BANDWIDTH problem, Saxe [1980] presented an $O(n^{k+1})$ algorithm to decide BANDWIDTH (k) for any constant k. Gurari and Sudborough [1984] improved this result thanks to an $O(n^k)$ algorithm. This is essentially the best that can be done, as Bodlaender et al. [1994] proved that, for any k, the BANDWIDTH (k) problem is W[k]-hard. Recent work addresses the problem of simplifying the algorithm for the BANDWIDTH (2) problem, see Makedon et al. [1993] for biconnected graphs and and Caprara et al. [2002] for general graphs.

In the case of CUTWIDTH , Gurari and Sudborough [1984] presented a $O(n^k)$ algorithm to decide the CUTWIDTH (k) problem for any input graph with n vertices and any constant k. This result was improved in Makedon and Sudborough [1989] with an $O(n^{k-1})$ algorithm. Latter, Fellows and Langston [1988] obtained an $O(n^2)$ algorithm. The result for CUTWIDTH (k) has recently been improved in Thilikos et al. [2000], where a linear time algorithm is presented. Recall that the treewidth notion is similar to the pathwidth notion, but for a tree decomposition rather than a path decomposition.

On the other hand, Fellows and Langston [1988] have proved that VERTSEP and MODCUT are fixed-parameter tractable. In particular, Bodlaender [1996] proved that VERTSEP (k) can be decided in linear time.

Furthermore, in Bui and Peck [1992] it is shown that the EDGEBIS ($k \log n$) can be solved in polynomial time for planar graphs.

Table VI summarizes these results.

The fixed parameter complexity of the layout problems not included in Table VI remains an open problem. Furthermore, when considering the efficiency of the algorithms for parameterized problems, bear in mind that even if k is a constant, the multiplicative factor hidden in the big-oh notation is often exponential in k. This rises as an open problem the quest of practicable linear time algorithms.

7. APPROXIMATION ALGORITHMS

One of the approaches to dealing with intractable problems is to design an approximation algorithm that, in polynomial time will give a feasible solution "close" to the optimal one. In this section we state a formal definition of what it

Table VI. Fixed Parameterized Complexity Results for Layout Problems (n denotes the size of the graph and k the parameter)

Problem	Complexity	Ref.
Bandwidth (2)	$O(n)$	[Garey et al. 1978]
Bandwidth (k)	$O(n^{k+1})$	[Saxe 1980]
Bandwidth (k)	$O(n^k)$	[Gurari and Sudborough 1984]
Bandwidth (k)	W[k]	[Bodlaender et al. 1994]
Cutwidth (2)	$O(n)$	[Garey et al. 1978]
Cutwidth (k)	$O(n^k)$	[Gurari and Sudborough 1984]
Cutwidth (k)	$O(n^{k-1})$	[Makedon and Sudborough 1989]
Cutwidth (k)	$O(n^2)$	[Fellows and Langston 1992]
Cutwidth (k)	$O(n)$	[Thilikos et al. 2000]
ModCut (k)	$O(n^2)$	[Fellows and Langston 1992]
VertSep (k)	$O(n^2)$	[Fellows and Langston 1988]
VertSep (k)	$O(n)$	[Bodlaender 1996]

means to have a close solution and present approximability results for layout problems. For a complete text on the theory of approximability the reader can look at Garey and Johnson [1979]; Ausiello et al. [1999]; Vazirani [2001].

Recall that, given a minimization problem Π, an $r(n)$-approximation algorithm is an algorithm that, for any input x of size n, finds a solution to Π whose cost is at most $r(n)$ times the cost of an optimal solution to the problem's instance. When a problem Π has some $r(n)$-approximation algorithm, it is said to be $r(n)$-*approximable*. When there exists an algorithm for Π such that, for all $\epsilon < 1$, A_ϵ returns a feasible solution σ such that the ratio between the obtained value and the optimal value is less than $1 + \epsilon$, and it runs in polynomial time with respect to $|x|$, A_ϵ is said to be a *polynomial time approximation scheme*. Moreover, when A_ϵ runs in polynomial time with respect to $|x|$ and $1/\epsilon$, A_ϵ is said to be a *fully polynomial time approximation scheme*. A combinatorial optimization problem belongs to the class APX if it is ϵ-approximable for some constant $\epsilon > 1$, to the class PTAS if it admits an approximation scheme, and to the class FPTAS if it admits a fully approximation scheme. It is known that FPTAS $\subseteq$ PTAS $\subseteq$ APX, where the inclusions are strict if and only if P $\neq$ NP. There are other parallel analogues to sequential approximation classes; see Díaz et al. [1997b] for an introduction to parallel approximation.

For the graph layout problems under consideration, the set of instances I corresponds to the set of all undirected graphs, an instance $x \in I$ corresponds to a particular undirected graph G, the set of feasible solutions $S(G)$ corresponds to $\Phi(G)$, and the objective function is a layout cost $f \in \{$LA, BW, SC, VS, CW, MC, EB, VB$\}$.

Many of the more recent positive results put the emphasis on improving the approximation bounds, rather than improving the time complexity. Two of the techniques used are relaxations of semidefinite programs and spreading metrics. A *spreading metric* on a graph is an assignment of lengths to its edges or its vertices, so that nontrivial subgraphs are spread apart in the associated metric space [Seymour 1995]. The volume of a spreading metric, defined as the sum of the lengths of all edges or vertices, provides a lower bound of solving the problem that guides a divide and conquer strategy. The drawback of the algorithms based on spreading metrics is that they require solving a linear program with an exponential number of constraints, which can make them impractical. In fact, their running time is dominated by the complexity of finding a spreading metric, which can be computed by general linear programming algorithms, as the Ellipsoid method. In general, most of them take $\tilde{O}(m^2 n)$ time, where $\tilde{O}$ ignores poly-logarithmic factors; see Even et al. [1999]. *Flow metrics* are a new approach to approximate some layout problems [Bornstein and Vempala 2002]. Even if flow metrics do not improve previous results, they represent a much simpler framework where the exponential number of constraints is reduced to a polynomial number.

For the BANDWIDTH problem, some particular kinds of graphs have approximation algorithms. In the case of γ-dense graphs, there exists a polynomial time 3-approximation, which appeared in Karpinski et al. [1997]. Recall that a graph with n vertices is γ-dense if its minimum degree is at least γn. The proof of this result uses the construction of perfect matchings in bipartite graphs as a building block, together with a simplest 4-approximation algorithm. There is a 2-approximation algorithm to bandwidth for asteroidal triple free graphs, permutation graphs and trapezoidal graphs [Kloks et al. 1999]. There are also polynomial time $O(\log n)$-approximation algorithms for caterpillars [Haralambides et al. 1991], and for a larger class of trees, denoted as GHB-trees, which are characterized as trees such that for any node v, the depth difference of any two non-empty subtrees rooted at v is bounded by a constant [Haralambides and Makedon 1997]. For the case of general trees and chordal graphs, Gupta [2001] presents a randomized $O(\log^{2.5} n)$-approximation algorithm. For general graphs, bandwidth has several polylogarithmic approximation algorithms running in polynomial randomized time, due to Blum et al. [2000], using semidefinite-relaxation; to Feige [2000], using spreading metrics and volume respecting embeddings; and to Dunagan and Vempala [2001], using semidefinite-relaxation and Euclidean embeddings. On the negative side, in Blache et al. [1998] it is shown that it is NP-complete to find a $\frac{3}{2}$-approximation for general graphs, and it is also NP-complete to find a $\frac{4}{3}$-approximation for trees. As a consequence BANDWIDTH does not belong to PTAS. In fact, Unger [1998] proves that, for any constant k, it is NP-complete to find any k-approximation, even for caterpillars. Therefore, BANDWIDTH does not belong to APX. The approximability of BANDWIDTH between a constant and a polylogarithmic factor remains open.

Recall that graphs with n vertices and m edges are dense if $m = \Theta(n^2)$. There exist fully polynomial time approximation schemes for dense graphs, Arora et al. [1996, 1999], for MINLA and CUTWIDTH, and Frieze and Kannan [1996] for EDGEBIS. The regularity lemma of Szemerèdi is the base technique to prove these results.

Several approximation algorithms have been proposed for MINLA, CUTWIDTH and SUMCUT problems. The first nontrivial approximation algorithm for MINLA and CUTWIDTH on general graphs was a direct application of an $O(\log n)$ approximation algorithm for finding balanced partitions in a graph [Leighton and Rao 1999], which produced an $O(\log^2 n)$ approximation algorithm for MINLA and CUTWIDTH. The original paper appeared as a conference paper in 1989. Using ideas from the previous work, Hansen [1989] gave an $O(\log^2 n)$-approximation algorithm for MINLA. Using the spreading metrics technique Even et al. [2000] gave an $O(\log n \log\log n)$-approximation algorithms for MINLA and SUMCUT. Up to date, the best polynomial time approximation algorithms for MINLA and SUMCUT are $O(\log n)$-approximations for general graphs, and $O(\log\log n)$-approximations for planar graphs. Both results appear in Rao and Richa [1998], who also use the spreading metric technique. Their technique works for the general case of graphs with weighted edges.

In the case of the VERTSEP problem, Bodlaender et al. [1995a] present a polynomial time $O(\log^2 n)$-approximation algorithm for general graphs, and show how to use results from Seymour and Thomas [1994] to get an $O(\log n)$-approximation algorithm for planar graphs.

The first approximation algorithm for the EDGEBIS problem on general graphs with sublinear approximation ratio, was given in Feige et al. [2000], this result was further improved in Feige and Krauthgamer [2002], where an $O(\log^2 n)$ approximation ratio is obtained. For graphs, excluding any fixed graph as a minor (e.g. planar graphs), an improved ratio of $O(\log n)$ is obtained. Albeit its importance in many contexts, no better approximability results are currently known for this problem.

Table VII summarizes these approximability results.

8. RANDOM GRAPHS

Worst-case analysis does not always catch the real difficulty of a problem. A standard way of evaluating the *real* efficiency of an algorithm or heuristic, from a practical point of view, is to evaluate its performance on random instances.

Recall that a sequence of events $(E_n)_{n\geq 1}$, occurs *with high probability* if $\Pr[E_n] \to 1$, and *with overwhelming probability* if $\Pr[E_n] \geq 1 - 2^{-\Omega(n)}$ for all n. Also, if $(X_n)_{n\geq 1}$ is a sequence of random variables, and X is a random variable, it is said that X_n converges *in probability* to X ($X_n \xrightarrow{\Pr} X$) if $\Pr[|X_n - X| > \epsilon] \to 0$, for all $\epsilon > 0$.

8.1. Binomial Random Graphs

Given a positive integer n and a probability p_n, the class of *binomial random graphs*, $\mathcal{G}_{n,p_n}$, is a probability space over the set of undirected graphs $G = (V, E)$ on the node set $V = [n]$ determined by $\Pr[uv \in E] = p_n$, with these events mutually independent [Janson et al. 2000]. This well studied model offers a natural starting point from which to analyze approximability properties of layout problems in a probabilistic setting, see Figure 7(a).

BANDWIDTH can be approximated within a constant on binomial random graphs; see Bollobás [1985 page 374] and Turner [1986]. The same result for EDGEBIS was obtained in Boppana [1987]. These results can be extended to all layout problems considered in this survey, using a unique framework involving "mixing graphs", which is a concept close to expander graphs developed in Crescenzi et al. [2001]. The following result appears in Díaz et al. [2001c].

THEOREM 8.1 *Let $\epsilon \in (0, \frac{1}{6})$, $\gamma \in (0,1)$ and define $C_{\epsilon,\gamma} = 3(1+\ln 3)(\epsilon\gamma)^{-2}$. Consider a sequence $(c_n)_{n\geq 1}$ such that $C_{\epsilon,\gamma} \leq c_n \leq n$ for all $n \geq n_0$ for some natural n_0. Then, with overwhelming probability,* BANDWIDTH, MINLA, CUTWIDTH, MODCUT, SUMCUT, VERTSEP, EDGEBIS *and* VERTBIS, *can be approximated within an $O(\epsilon + \gamma)$ factor on binomial random graphs $\mathcal{G}_{n,p_n}$ where $p_n = c_n/n$.*

Actually, the proof of the above theorem shows that *any* algorithm computing a feasible layout, no matter how good or bad, will perform rather well on random graphs, pointing out that evaluating heuristics on binomial random graphs may be unworthy for layout problems.

Theorem 8.1 has been improved for EDGEBIS : Luczak and McDiarmid [2001] show that if $c > \ln 4$, then $\text{MINEB}(\mathcal{G}_{n,c/n}) = \Theta(n)$ with high probability; while if $c < \ln 4$, then $\text{MINEB}(\mathcal{G}_{n,c/n}) = 0$ with high probability.

8.2. Random Grid Graphs

Given an integer m and a probability p, the class of *random grid graphs*, $\mathcal{L}_{m,p}$, is a probability space over the set of grid graphs $G = (V, E)$ on the node set $V \subseteq [m]^2$ determined by $\Pr[u \in V] = p$ with these events mutually independent and $E = \{uv : u, v \in V \wedge \|u - v\|_1 = 1\}$.

The study of random grid graphs is based on *percolation* theory [Grimmett 1999]. Consider a *site percolation* process, where nodes from the infinite grid ($\mathbb{Z}^2$) are selected with some probability p. Let C_0 be the connected component where the origin belongs. A basic question in percolation theory is whether or not C_0 can be infinite. Let $\vartheta(p)$ denote the probability that $|C_0| = \infty$ and set $p_c = \inf\{p : \vartheta(p) > 0\}$, the *critical value* of p. In the *subcritical* limiting regimes $p \in (0, p_c)$, all components are almost surely finite.

The following theorem from Díaz et al. [2000] states the behavior of some layout problems on subcritical random grid graphs:

THEOREM 8.2 *Let $p \in (0, p_c)$; then there exist constants $0 < c_1 < c_2$ such that*

$$\lim_{m\to\infty} Pr\left[c_1 \leq \frac{\text{VS}(\mathcal{L}_{m,p})}{\sqrt{\log m}} \leq \frac{\text{CW}(\mathcal{L}_{m,p})}{\sqrt{\log m}} \leq c_2\right] = 1$$

Table VII. Review of Approximability Results for Layout Problems (n is the number of vertices of the input graph, m is its number of edges, and ϵ is the parameter of the approximation scheme). Refer to the text for more information on the polynomial running times

Problem	Approximability	Time	Ref.
BANDWIDTH	3-approximable for δ-dense graphs	$n^{O(1/\delta)}$	[Karpinski et al. 1997]
	2-approximable for AT-free graphs	$O(n^3)$	[Kloks et al. 1999]
	$O(\log n)$-approximable for caterpillars	$O(n^2)$	[Haralambides et al. 1991]
	$O(\log n)$-approximable for GHB-trees	$O(n^2)$	[Haralambides and Makedon 1997]
	random $O(\log^{4.5} n)$-approximable	$O(m(\log n)^4 \log\log n)$	[Feige 2000]
	random $O(\log^3 n \sqrt{\log\log n})$-approximable	$\mathrm{pol}(n)$	[Dunagan and Vempala 2001]
	random $O(\sqrt{n/\mathrm{BW}(G)} \log n)$-approximable	$\mathrm{pol}(n)$	[Blum et al. 2000]
	random $O(\sqrt{\log^{2.5} n})$-approx. for trees and chordal graphs	$\mathrm{pol}(n)$	[Gupta 2001]
	no PTAS	—	[Blache et al. 1998]
	no PTAS for trees	—	[Blache et al. 1998]
	no APX	—	[Unger 1998]
VERTSEP	$O(\log^2 n)$-approximable	$\mathrm{pol}(n)$	[Bodlaender et al. 1995a]
	$O(\log n)$-approximable for planar graph	$\mathrm{pol}(n)$	[Bodlaender et al. 1995a]
MINLA	PTAS for dense graphs	$n^{O(1/\epsilon^2)}$	[Arora et al. 1996]
	$O(\log^2 n)$-approximable	$\mathrm{pol}(n)$	[Hansen 1989; Leighton and Rao 1999]
	$O(\log n \log\log n)$-approximable	$\mathrm{pol}(n)$	[Even et al. 2000]
	$O(\log n)$-approximable	$\mathrm{pol}(n)$	[Rao and Richa 1998]
	$O(\log\log n)$-approximable for planar graphs	$\mathrm{pol}(n)$	[Rao and Richa 1998]
CUTWIDTH	PTAS for dense graphs	$n^{O(1/\epsilon^2)}$	[Arora et al. 1996]
	$O(\log^2 n)$-approximable	$\mathrm{pol}(n)$	[Leighton and Rao 1999]
SUMCUT	$O(\log n \log\log n)$-approximable	$\mathrm{pol}(n)$	[Even et al. 2000]
	$O(\log n)$-approximable	$\mathrm{pol}(n)$	[Rao and Richa 1998]
	$O(\log\log n)$-approximable for planar graphs	$\mathrm{pol}(n)$	[Rao and Richa 1998]
EDGEBIS	PTAS for dense graphs	$\tilde{O}(1/\epsilon^2)n + 2^{\tilde{O}(1/\epsilon^2)}$	[Frieze and Kannan 1996]
	$O(\log^2 n)$-approximable	$\mathrm{pol}(n)$	[Feige and Krauthgamer 2002]
	$O(\log n)$-approximable for planar graph	$\mathrm{pol}(n)$	[Feige and Krauthgamer 2002]

and there exist two constants $\beta_{\mathrm{LA}}(p) > 0$ and $\beta_{\mathrm{SC}}(p) > 0$ such that, as $m \to \infty$,

$$\mathrm{LA}(\mathcal{L}_{m,p})/m^2 \xrightarrow{\Pr} \beta_{\mathrm{LA}}(p) \quad \text{and} \quad \mathrm{SC}(\mathcal{L}_{m,p})/m^2 \xrightarrow{\Pr} \beta_{\mathrm{SC}}(p).$$

Results for the supercritical case $p \in (p_c, 1)$ are derived in Penrose [2000]:

THEOREM 8.3 *Let $p \in (p_c, 1)$. Then, there exists a constant $c > 0$ such that, with overwhelming probability,*

$$\begin{aligned} cm^2 &\le \mathrm{MINSC}(\mathcal{L}_{m,p}) \le m^2, & cm^2 &\le \mathrm{MINLA}(\mathcal{L}_{m,p}) \le 4m^2,\\ cm &\le \mathrm{MINBW}(\mathcal{L}_{m,p}) \le m, & cm &\le \mathrm{MINVS}(\mathcal{L}_{m,p}) \le m,\\ cm &\le \mathrm{MINCW}(\mathcal{L}_{m,p}) \le 4m. \end{aligned}$$

Moreover, for all p such that $\vartheta(p) > \frac{1}{2}$, with overwhelming probability,

$$cm \le \mathrm{MINEB}(\mathcal{L}_{m,p}) \le 4m.$$

8.3. Random Geometric Graphs

Random geometric graphs $\mathcal{G}(n; r_n)$ are graphs whose n nodes are n points uniformly distributed in the unit square, and whose edges between any pair of nodes exist when their distance is smaller than some parameter r_n, see Figure 7(c). Random geometric graphs have been proposed as a possible model to take into account the structural characteristics of instances that appear in many practical applications [Johnson et al. 1989; Berry and Goldberg 1999; Lang and Rao 1993].

As random grid graphs, random geometric graphs also exhibit a phase transition [Penrose 1995]: When $nr_n^2 \to \lambda$ there exists a critical parameter λ_c such that if $\lambda < \lambda_c$, graphs $\mathcal{G}(n; r_n)$ are likely to have $O(\log n)$ points in each connected component, while if $\lambda > \lambda_c$, there is likely to be a component with $\Theta(n)$ nodes. The next two theorems, proven in Díaz et al. [2000] and Penrose [2000] respectively, characterize the behavior of several layout costs in the subcritical and supercrital phases.

THEOREM 8.4 *Suppose $nr_n^2 \to \lambda \in (0, \lambda_c)$. Then, there exist constants $0 < c_1 < c_2$, $0 < c_3 < c_4$, $\tilde\beta_{\mathrm{LA}}(\lambda) > 0$ and $\tilde\beta_{\mathrm{SC}}(\lambda) > 0$ such that*

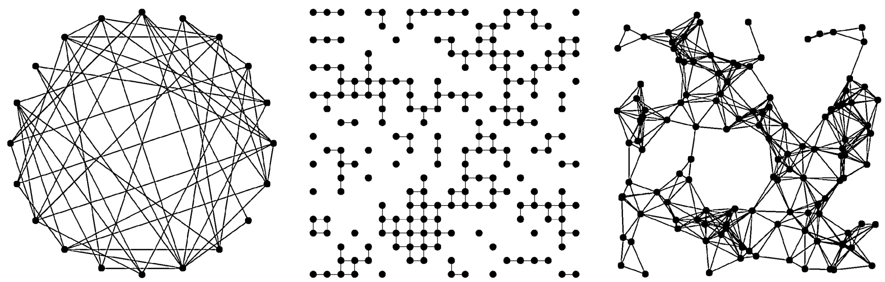
$$\mathrm{MINEB}(\mathcal{G}(n; r_n)) \xrightarrow{\Pr} 0, \qquad \mathrm{MINVB}(\mathcal{G}(n; r_n)) \xrightarrow{\Pr} 0,$$

$$\mathrm{MINLA}(\mathcal{G}(n; r_n))/n \xrightarrow{\Pr} \tilde\beta_{\mathrm{LA}}(\lambda), \qquad \mathrm{MINSC}(\mathcal{G}(n; r_n))/n \xrightarrow{\Pr} \tilde\beta_{\mathrm{SC}}(\lambda),$$

$$Pr\left[c_1 \le \frac{\mathrm{VS}(\mathcal{G}(n; r_n))}{\log n/\log\log n} \le c_2\right] \to 1, \qquad Pr\left[c_3 \le \frac{\mathrm{CW}(\mathcal{G}(n; r_n))}{(\log n/\log\log n)^2} \le c_4\right] \to 1.$$

THEOREM 8.5 *Suppose $nr_n^2 \to \lambda \in (\lambda_c, \infty)$. Then,*

$$\begin{aligned} \mathrm{MINCW}(\mathcal{G}(n; r_n)) &= \Theta\big(n^2 r_n^3\big), & \mathrm{MINEB}(\mathcal{G}(n; r_n)) &= \Theta\big(n^2 r_n^3\big),\\ \mathrm{MINSC}(\mathcal{G}(n; r_n)) &= \Theta\big(n^2 r_n\big), & \mathrm{MINLA}(\mathcal{G}(n; r_n)) &= \Theta\big(n^3 r_n^3\big),\\ \mathrm{MINVS}(\mathcal{G}(n; r_n)) &= \Theta(n r_n), & \mathrm{MINBW}(\mathcal{G}(n; r_n)) &= \Theta(n r_n). \end{aligned}$$



(a) Binomial random graph (b) Random grid graph (c) Random geometric graph

Fig. 7. Different models of random graphs.

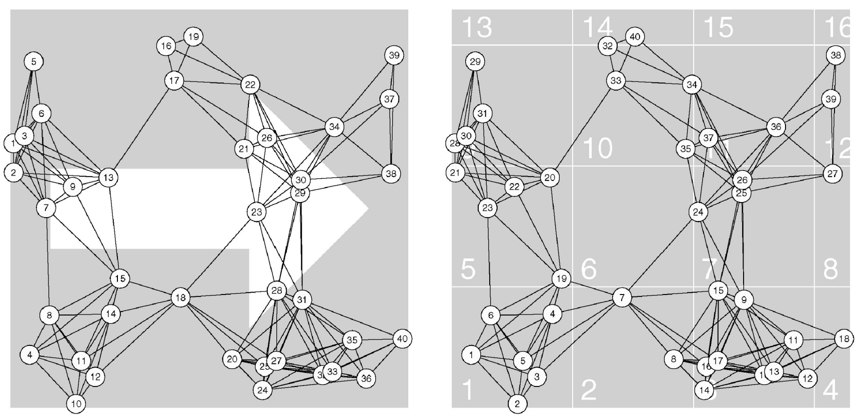


Fig. 8. Illustration of the projection (left) and dissection (right) algorithms. The projection algorithm creates a layout by ordering the vertices according to their projection onto the x-axis. The dissection algorithm creates a layout dissecting the unit square in boxes of the appropriate size and enumerating the points, following the order of the boxes in lexicographic order (points in the same box are enumerated arbitrarily).

In the case that the radius is slightly greater, Díaz et al. [2001a] show that it is possible to obtain nice approximations:

THEOREM 8.6 *Let* $(r_i)_{i\geq 1}$ *be a sequence of positive numbers with* $r_n \to 0$ *and* $nr_n^2/\log n \to \infty$. *Then, with high probability,* BANDWIDTH*,* MINLA *,* CUTWIDTH *,* MODCUT *,* SUMCUT *,* VERTSEP *,* EDGEBIS *and* VERTBIS *can be approximated within a constant on random geometric graphs* $\mathcal{G}(n; r_n)$. *Moreover, for* BANDWIDTH *and* VERTSEP*, the approximation factor can be arbitrarily close to 1.*

The proofs of the above results involve the use of isoperimetric inequalities to get lower bounds, and the analysis of two simple heuristics using geometrical information to get matching upper bounds; see Figure 8. In the case of the bandwidth and vertex separation problems, the solutions returned by either of the heuristics are asymptotically optimal.

These results on random geometric graphs can be applied to give empirical evidence of the goodness of several well-known heuristics for layout and partitioning problems [Díaz et al. 2001c]. These heuristics include global methods, such as spectral, multilevel and greedy methods; and local methods, such as simulated; annealing or Kernighan-Lin.

Geometric graphs have also been considered as possible models for wireless communication, where two transmitters are in contact whenever their distance is at most r. Therefore it seems natural to investigate the behavior of layout problems with respect to unreliable random geometric graphs, that is, random geometric graphs whose edges or nodes may fail at random independently. In Díaz et al. [2001b] it is shown that random geometric networks can tolerate a constant edge or node failure probability maintaining the order of magnitude of the MINEB, MINLA and MINCW measures.

9. UPPER AND LOWER BOUNDS

As we already indicated, all the layout problems presented, are hard to solve for general graphs. Therefore, it is important to have algorithms to obtain good lower bounds. On the other hand, it is also interesting to have upper bounds for the layout costs restricted to general classes of graphs. Formally, given a layout cost F, we say that an algorithm L computes a lower bound of the cost of a graph G for F if $F(\varphi, G) \geq L(G)$ for all $\varphi \in \Phi(G)$. Also, we say that U is an upper bound of the cost of a graph G for F if $F(\varphi, G) \leq U(G)$ for all $\varphi \in \Phi(G)$.

For some layout costs and some classes of graphs, even if no exact solutions are known, there exist lower bounds that asymptotically match upper bounds. This happens, for instance, for the edge and vertex bisection of the de Bruijn and Kautz graphs [Rolim et al. 1998], the cutwidth of the de Bruijn graph [Raspaud et al. 1995], the edge bisection of transposition graphs [Stacho and Vrťo 1998], the cutwidth of the d-dimensional mesh of r-ary trees [Vrťo 2000], and for the minimum linear arrangement and cutwidth on graphs with bounded maximal degree and genus [Sýkora and Vrťo 1993]. In Barth et al. [1995], bounds on bandwidth and cutwidth of a given graph are obtained from bounds for its quotient graphs, which is a useful result because some techniques involve clustering of vertices. In Diks et al. [1993], edge separation problems are investigated for planar graphs, outerplanar graphs, and trees of given maximum degree. The results lead to estimations on the edge bisection and the cutwidth of these graphs. Specifically, Diks et al. [1993] prove that the MINEB(G) and MINCW(G) are $O(\sqrt{kn})$ for graphs G with n vertices and maximal degree k, and that there exist graphs whose edge bisection and the cutwidth is $\Omega(\sqrt{kn})$.

The remainder of this section presents several approaches to get lower bounds for some of the layout problems under consideration.

9.1. The Path Method

The path method was introduced in Juvan and Mohar [1992] to compute a lower bound for the MINLA and BANDWIDTH problems. Let $P_n^k = (V_n, E_n^k)$ denote the k-th power graph of the path graph: $V_n = [n]$ and $E_n^k = \{ij \ : \ 0 < |i - j| \ \leq \ k\}$. A direct calculation gives

$$\text{MINLA}(P_n^k) = \frac{1}{6}k(k+1)(3n-2k-1).$$

Let $c(n, m)$ be the largest k for which $|E(P_n^k)| \leq m$. Then,

$$c(n, m) = n - \frac{1}{2}\sqrt{(2n-1)^2 - 8m} - \frac{1}{2}.$$

The use of these expressions to get a lower bound to MINLA and MINBW is given by the following theorem due to Juvan and Mohar [1992]:

THEOREM 9.1 *Let G be a graph with n nodes and m edges and let $k = \lfloor c(n, m) \rfloor$. Then,* MINLA$(G) \geq$ MINLA(P_n^k) *and* MINBW$(G) \geq k$.

9.2. Bounds Based on Spectral Properties

Let $G = ([n], E)$ be a graph and let L_G be its Laplacian matrix, defined by

$$L_G[u, v] = \begin{cases} -1 & \text{if } uv \in E, \\ 0 & \text{if} uv \notin E, \\ \deg(u) & \text{if } u = v. \end{cases}$$

By construction, L_G is positive semidefinite. Therefore it has n nonnegative real eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. The sequence $\lambda_1, \lambda_2, \ldots, \lambda_n$ is known as the *spectrum* of the graph G. It is know that the multiplicity of the value 0 as an eigenvalue of L_G is equal to the number of connected components of G; in particular, if G is connected $0 = \lambda_1 < \lambda_2$ [Mohar and Poljak 1993].

The following theorem gathers results from Juvan and Mohar [1992] and Mohar and Poljak [1993].

THEOREM 9.2 *Let G be a connected graph with n vertices and let λ_2 be the second smallest eigenvalue of the Laplacian matrix of G. Then,*

$$\begin{aligned} \text{MINLA}(G) &\geq \lambda_2(n^2 - 1)/6, \\ \text{MINCW}(G) &\geq \lambda_2 \left\lfloor \frac{1}{2}n \right\rfloor \left\lceil \frac{1}{2}n \right\rceil /n, \\ \text{MINEB}(G) &\geq \begin{cases} \lambda_2 n/4 & \textit{if } n \textit{ is even,} \\ \lambda_2(n^2 - 1)/4n & \textit{if } n \textit{ is odd.} \end{cases} \end{aligned}$$

Newer bounds for the EDGEBIS problem related to the level structure of a graph can be found in Bezrukov et al. [2000b].

The following result, presented in Helmberg et al. [1995], bounds the bandwidth of a graph using the ratio between the two extremal eigenvalues of its spectrum:

THEOREM 9.3 *Let G be a graph with n vertices and at least one edge. Let λ_2 and λ_n denote the second smallest and the largest eigenvalue of the Laplacian of G, respectively. Let α be the largest integer smaller than $n\lambda_2/\lambda_n$. Then,*

$$\text{MINBW}(G) \geq \begin{cases} n - 1 & \textit{if } \alpha \geq n - 2, \\ \alpha + 1 & \textit{if } \alpha \leq n - 2 \textit{ and } n \textit{ is even,} \\ \alpha & \textit{otherwise.} \end{cases}$$

The same reference also contains lower bounds for MINLA that involve more complex eigenvalues. Lower bounds for VERTSEP and BANDWIDTH can be obtained using the following theorem [Mohar and Poljak 1993]:

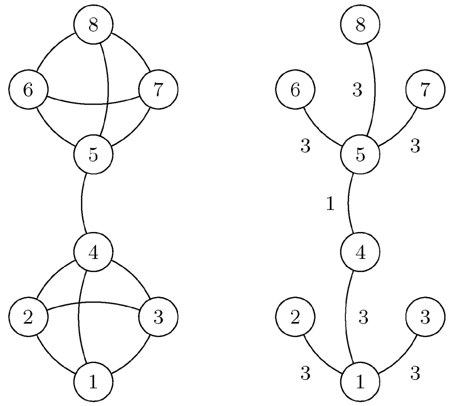
THEOREM 9.4 *Let $G = (V, E)$ be a connected graph with n vertices and maximal degree Δ. Let λ_2 be the second smallest eigenvalue of the Laplacian matrix of G. If $S \subset V$ separates vertices sets A and B then*

$$|C| \geq \frac{4\lambda_2|A||B|}{\Delta n - \lambda_2|A \cup B|}.$$

A slightly weaker version of this result was obtained by Alon and Milman [1985].

9.3. Bounds Based on Fundamental Cuts

Let $G = (V, E)$ be a graph with n vertices and let s and t be two distinguished vertices of G, which we call the source and the sink, respectively. The well-known max-flow min-cut theorem states that the maximal flow value from s to t is equal to the minimal edge cut separating s and t [Ford and Fulkerson 1962]. As there exist efficient algorithms to compute such a minimal cut, and there are $\frac{1}{2}n(n-1)$ possible choices for s and t, it is possible to build a symmetric $n \times n$ matrix f where $f[i, j]$ stores the maximal flow value between two distinct vertices i and j. In Gomory and Hu [1975] it is shown that matrix f can simply be represented by a weighted spanning tree of G where each edge represents a *fundamental cut* of G and has weight equal to the corresponding minimal cut. The maximum flow $f[i, j]$ between any pair of vertices i and j can be obtained by finding the unique path between i and j in the weighted spanning



f	1	2	3	4	5	6	7	8
1		3	3	3	1	1	1	1
2	3		3	3	1	1	1	1
3	3	3		3	1	1	1	1
4	3	3	3		1	1	1	1
5	1	1	1	1		3	3	3
6	1	1	1	1	3		3	3
7	1	1	1	1	3	3		3
8	1	1	1	1	3	3	3	

Fig. 9. A graph G, its Gomory–Hu tree and its matrix f of max-flows min-cuts. Applying Theorem 9.5, $\text{MINLA}(G) \geq 19$, $\text{MINCW}(G) \geq 3$ and $\text{MINBW}(G) \geq 3$. In fact, $\text{MINLA}(G) = 21$, $\text{MINCW}(G) = 3$ and $\text{MINBW}(G) = 3$.

tree and finding the minimal weight over all the edges in this path. Figure 9 shows a graph, its corresponding Gomory–Hu tree and its corresponding matrix f. An algorithm to construct the Gomory–Hu tree can be found in the same paper.

By construction, it is easy to see that the maximal fundamental cut for a graph is a lower bound of the cutwidth problem. Moreover, Adolphson and Hu [1973] proved that the total cut capacity of the $n-1$ fundamental cuts is a lower bound on the cost of the minimum linear arrangement problem. Using Lemma 2.4 we conclude that $\text{MINBW}(G) \geq \lceil \text{MINLA}(G)/n \rceil$. Therefore, we have the following lower bounds:

THEOREM 9.5 *Let $G = (V, E)$ be a graph and $T = (V, E', w)$ its weighted Gomory–Hu tree. Then,*

$$\text{MINCW}(G) \geq \max_{e \in E'} w(e),$$
$$\text{MINLA}(G) \geq \sum_{e \in E'} w(e),$$
$$\text{MINBW}(G) \geq \left\lceil \frac{1}{|V|} \sum_{e \in E'} w(e) \right\rceil.$$

Adolphson and Hu [1973] also proved that if the Gomory–Hu tree is a path, then that path is the optimal layout for the MINLA problem.

10. HEURISTICS

Resorting to heuristics is an alternative method to obtain solutions for optimization problems. In general, a *heuristic* is a rule of thumb, simplification or guess that reduces or limits the search for solutions in domains that are difficult and poorly understood. In the context of layout problems, a heuristic is a procedure that, given an input graph G, returns a feasible layout $\varphi \in \Phi(G)$. Unlike approximation algorithms, heuristics do not provide a theoretical guarantee on their cost of the returned layout nor on their running time. In spite of that, heuristics are often used in practice, but the assessment of their effectiveness and efficiency is inherently empirical: "It works well with my inputs." In this section, we review several heuristics for some layout problems.

10.1. Heuristics for Bandwidth

Due to their importance in engineering applications, many heuristics have been developed to reduce the bandwidth and/or profile of sparse matrices. Chinn et al. [1982] references a paper citing 49 different heuristics! The most well-known heuristics for bandwidth/profile reduction are the CutHill–McKee algorithm [Cuthill and Mckee 1969], King's algorithm [King 1970], and the Gibbs–Poole–Stockmeyer algorithms [Gibbs et al. 1976]. Most of them belong to a family of heuristics called *level algorithms*. Level algorithms are based on a level structure of the graph, which partitions its vertex set into levels $L_0, \ldots, L_s$ such that the endpoints of every edge in the graph are either in the same level L_i or in two consecutive levels L_i and L_{i+1}. The assessment of the goodness of level algorithms for BANDWIDTH has been considered by Turner [1986], who has analyzed their behavior on a particular distribution of random graphs with bandwidth no larger than an integer B. Specifically, he considers graphs $\mathcal{G}_{n,p,B}$, resulting from the following experiment: The vertex set is $[n]$, and the edge set is made by connecting, with probability $p \in (0, 1)$, any pair of vertices $u, v \in [n]$ such that $0 < |u - v| \leq B$. Turner first proved that for all $\epsilon > 0$, almost all $G \in \mathcal{G}_{n,p,B}$ satisfy $1 \leq B/\text{MINBW}(G) \leq 1 + \epsilon$, and then proved that for any level algorithm A, it holds that $A(G)/\text{MINBW}(G) \leq (1 + \epsilon)(3 - p)$ for almost all $G \in \mathcal{G}_{n,p,k}$, provided that $B = \omega(\log n)$ and p is fixed. The research in Feige and Krauthgamer [1998] improve on Turner's results, by allowing smaller edges probabilities.

10.2. Heuristics for Edge Bisection

On the other hand, the areas of VLSI and of parallel computing have given rise to many heuristics for the edge bisection problem; see Elsner [1997] for a nice survey. One of the first proposed heuristics to bisect a graph was the Kernighan–Lin heuristic [Kernighan and Lin 1970]. This heuristic, originally developed to minimize the number of connections in electronic circuits, belongs to a more general family of local search heuristics [Aarts and Lenstra 1997].

Helpful sets is a more recent local search heuristic developed in Diekmann et al. [1995]. The basic idea of this heuristic, appears in the proof of a result we have already mentioned to get upper bounds on the bisection of 4-regular graphs [Hromkovič and Monien 1992]. Consider a bisection (V_1, V_2) of a graph $G = (V, E)$. For all $u \in V$, let $\text{ext}(u) = \{uv \in E \ : \ u \in V_i \wedge v \in V_j,\ i \neq j\}$ and let $\text{int}(u) = \{uv \in E \ : \ u, v \in V_i,\ i \in \{1, 2\}\}$. Let S be a subset of vertices in V_1; then, for all $u \in S$, let $\text{int}(u, S) = |\{v \in S \ : \ uv \in E\}|$. The *helpfulness* of S is $h(S) = \sum_{u \in S}(\text{ext}(u) - \text{int}(u) + \text{int}(u, S))$. Any S with helpfulness h is said to be h-helpful, as moving S to the other side of the bisection decreases the edge bisection by h edges. On the other hand, given an h-helpful set S in V_1, a subset of vertices S' of $V_2 \cup S$ is a h-balancing set of S if $|S'| = |S|$ and S' is at least $(1 - k)$-helpful. Local search algorithms based on helpful sets find h-helpful sets S and swap their vertices with the ones in some k-balancing set S'. For instance, helpful sets can be combined with simulated annealing to obtain a nice heuristic for edge bisection [Diekmann et al. 1996].

Spectral bisection is another popular heuristic for EDGEBIS based on spectral properties (see Section 9.2). Its basic principle is to compute the Fiedler vector of the Laplacian matrix of an input graph $G = (V, E)$, compute the median M of the components of the Fiedler vector, and return a bisection $(A, V \setminus A)$ where A is made of vertices $v \in V$ satisfying $x_v^{(2)} \leq M$. A related algorithm based on eigenvalues is presented and analyzed in Boppana [1987]. Also, Spielman and Teng [1996] show that spectral bisection methods work well on bounded-degree planar graphs and finite element grids.

There exist several papers that compare the effectiveness of several heuristics for EDGEBIS from an experimental point of view; see Johnson et al. [1989]; Lelandand Hendrickson [1994]; Diekmann et al. [1995]; Battiti and Bertossi [1999]. Unfortunately, these studies do not seem to give an indication of the reason why these heuristics work. Other heuristics for EDGEBIS include multilevel algorithms [Barnard and Simon 1994], tabu search [Glover and Laguna 1997, Section 8.6.2] or genetic algorithms [Kadluczka and Wala 1995].

Simpler local search heuristics for the edge bisection problem have been theoretically analyzed on particular random graphs M. Jerrum and Sorkin [1998] analyzed the metropolis algorithm for the edge bisection problem on graphs with a *planted bisection*. Their results involve an algorithm that, starting from a random bisection, iteratively takes a pair of vertices in different sides of the current bisection and interchanges them with probability $1/(1 + \exp(\delta/t))$, where δ is the increase of the cut size and t is a parameter called temperature. Their analysis considers planted bisection $\mathcal{G}_{4n,p,r}$ random graphs, generated as follows: A $\mathcal{G}_{4n,p,r}$ graph has $4n$ vertices, half of them black and half of them white; edges between vertices of the same color are included independently with probability p, whereas edges between vertices of different colors are included independently with probability $r < p$. They first prove that the planted bisection, defined by the original coloring, is, with high probability, the unique optimal bisection on $\mathcal{G}_{4n,p,r}$ random graphs. Then, they prove that, with overwhelming probability, for a certain choice of t, the metropolis algorithm can find the planted bisection of $\mathcal{G}_{4n,p,r}$ random graphs in time $O(n^2)$, provided $p - r = \Omega(n^{-1/6})$. In his thesis Juels [1996] improves these results. To cope with the simpler hill-climbing algorithm: interchanges are accepted only if they decrement the size of the bisection. His results state that hillclimbing can find the planted bisection of $\mathcal{G}_{4n,p,r}$ random graphs in expected time $O(n^2)$ with probability $c > 0$, provided $p - r$ is constant. Repeated executions can boost the probability of success. Juels also reports experimental results that show that hillclimbing may be much more effective than what his theoretical results indicate on the random graphs he considers. The previous results of Jerrum, Sorkin and Juels have been recently improved in Carson and Impagliazzo [2001], where it is proved that with a constant non-zero probability, the hill climbing heuristic is also able to find the minimal bisection of random graphs in the $\mathcal{G}_{4n,p,r}$ model with $p - r = \Omega(n^{-1/4} \log^4 n)$.

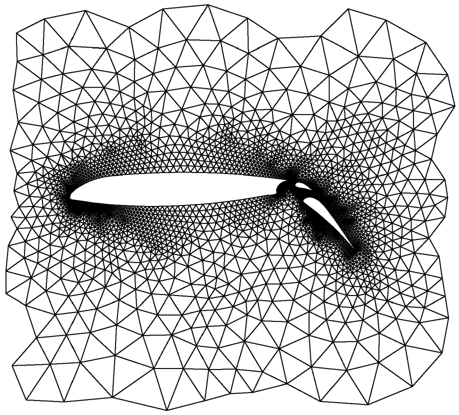
Similar results for a generalized version of $\mathcal{G}_{4n,p,r}$ graphs, but with a successive augmentation heuristic rather than with local search were proved in Condon and Karp [2001]. Their heuristic is based on a greedy algorithm that repeatedly selects a new pair of vertices and adds one to each side of the bisection. They show that their linear time heuristic hits the planted bisection with overwhelming probability, provided $p - r \geq n^{-1/2}$.

10.3. Heuristics for MINLA

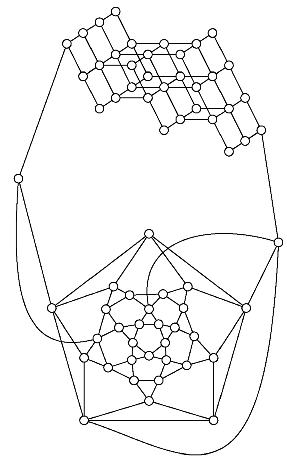
For the MINLA problem, a heuristic that we call *spectral sequencing* was proposed in Juvan and Mohar [1992]. Spectral sequencing first computes the eigenvector $x^{(2)}$ corresponding to the second smallest eigenvalue λ_2 of the Laplacian matrix $L(G)$ of the input graph G and then ranks each position of $x^{(2)}$. Thus, the heuristic returns an arrangement φ satisfying $\varphi(u) \leq \varphi(v)$ whenever $x^{(2)}_u \leq x^{(2)}_v$.

Another algorithm to get upper bounds on $\text{MINLA}(G)$ for arbitrary G is described in Muradyan [1985]. This algorithm first creates an initial arrangement of vertices $u_1, u_2, \ldots, u_n$, so that u_1 is a vertex with the minimal degree in G, u_2 is a vertex with the minimal degree in $G \setminus \{u_1\}$, and so on and working out this arrangement, creates a new arrangement for which $\text{MINLA}(G)$ is estimated.

An experimental study of heuristics for the MINLA problem was undertaken in Petit [1998, 2001a, 2001b]. The empirical results are based on two random models (binomial random graphs and random geometric graphs), "real life" graphs (including graphs arising from finite element discretizations, VLSI design and graph-drawing) and graphs with known optimal solutions (trees, hypercubes, grids). The first goal was to obtain an evaluation on the behavior of different families of heuristics through experimental results. The second goal was to use, in a systematic way, this experimental knowledge as a fundamental method to guide the design of new heuristics. Furthermore, approximation heuristics and methods to find lower bounds for MINLA were presented, evaluated and compared experimentally when applied to sparse graphs. The methods belong to the families of successive



(a) airfoil1



(b) gd96c

Fig. 10. Two graphs used in the comparison of heuristic for the MINLA problem.

augmentation heuristics, local search heuristics, and spectral sequencing. The conclusion is that the best approximations are obtained using simulated annealing, which involves a large amount of computation time. However, solutions found by Spectral Sequencing are also good and can be found in radically less time (see Figure 11).

Motivated by the long running time of simulated annealing compared to spectral sequencing, Petit [2000] presents and analyzes sequential and parallel versions of a new heuristic to approximate the MINLA problem on graphs embedded and clustered in some geometry. The heuristics consist in obtaining a first global solution using spectral sequencing and improving it locally through simulated annealing. The measurements obtained on a commodity cluster of nine work stations show that the new parallel, heuristic maintains solution quality, decreases the running time and offers an excellent speedup when run in parallel, without sacrificing solution quality.

Figure 11 gives a sample of the obtained measurements for the two graphs depicted in Figure 10.

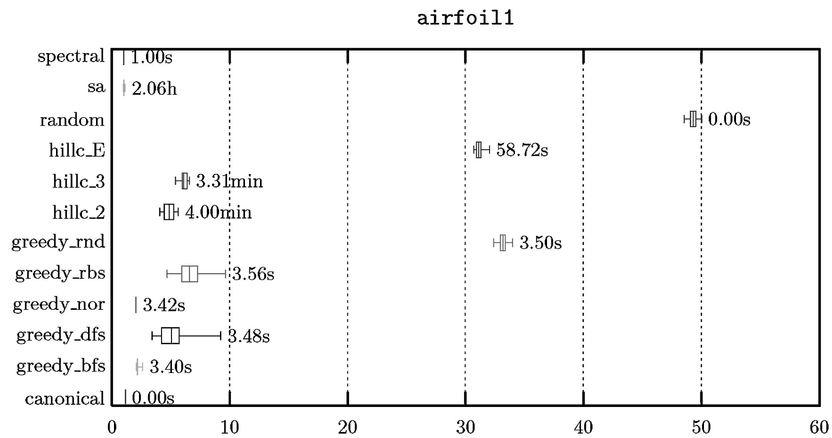
It must be remarked, however, that the experimental results for MINLA evidence a big gap between the best known upper and lower bounds. Therefore it remains as open problem to devise better techniques to obtain better lower bounds.

10.4. Heuristics for SUMCUT

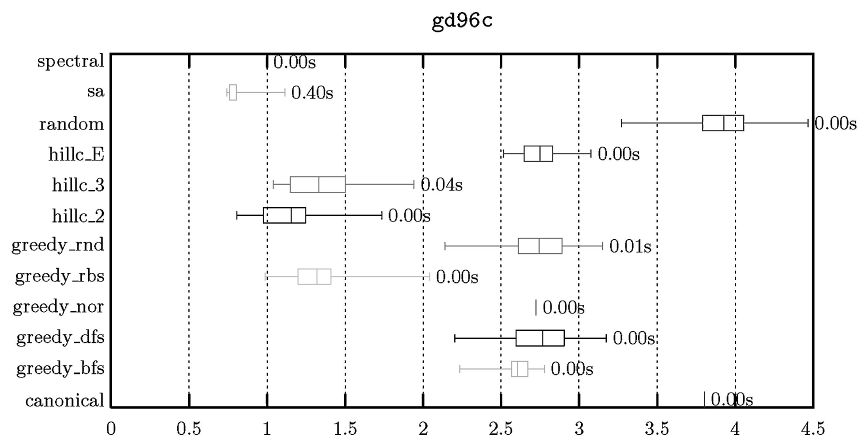
Several heuristics have also successfully been applied for the SUMCUT problem in order to reduce the profile of symmetric matrices. The early heuristics of Cuthill and Mckee [1969]; King [1970]; Gibbs et al. [1976], have now been improved in order to use simulated annealing, spectral methods, multilevel algorithm and hybrid schemes; see for example Everstine [1979]; Lewis [1994]; Barnard et al. [1995]; Kumfert and Pothen [1997]; Hager [2002].

10.5. Implementations

There exist several software libraries that implement many of the above mentioned heuristics. The Party library [Preis and Diekmann 1996] and the Chaco library [Hendrickson and Leland 1997] are packages that include a variety of different methods to partition or bisect graphs. In particular, Chaco enables a fast execution of the spectral sequencing heuristic for MINLA . Also, Metis is a library of programs for partitioning graphs and computing profile reducing orderings of sparse



(a) Results for **airfoil1**



(b) Results for **gd96c**

Fig. 11. Comparison of the relative effectiveness and efficiency of several heuristics for the MinLA problem for the graphs in Figure 10. The results are shown using boxplots, which summarize the distribution of the values of a group of samples. The box in a boxplot shows the median value as a line and the first (25th percentile) and third quartile (75th percentile) of the value distribution as the lower and upper parts of the box. The bars shown at the sides of the boxes represent the largest and smallest observed values. The average time elapsed to compute an individual of the sample is displayed at the right of the corresponding boxplot.

matrices [Karypis 2001]. The `llsh` toolkit offers several heuristics to approximate the MinLA problem [Petit 2001a].

11. VARIATIONS ON THE BASIC MODEL

Several variations on the basic layout model have been considered; there are modifications in the type of layout or in the fact that the graph carries additional information that must be preserved in some particular way in the layout. We survey here, some results on stack and queue layouts, as well as associated problems for weighted, directed and colored input graphs.

11.1. Stack and Queue Layouts

Stack and queue layouts were motivated from different contexts as VLSI design, fault-tolerant processing, parallel processing, sorting networks and parallel matrix computations [Chung et al. 1987; Heath et al. 1992; Heath and Rosenberg 1992; Heath et al. 1993].

A *k-queue layout* of an undirected graph $G = (V, E)$ consists of a linear order φ of the vertex set V and an assignment ψ of each edge in E into exactly one of k-queues. Observe that the ordering assigns a direction to each edge in E. The queue policy operates as follows: the vertices of V are scanned in left-to-right order. When vertex i is encountered, any edge assigned to a queue that has i as right end-point must be at the front of the queue, and it is dequeued. All the edges that has i as left end-point are enqueued in the assigned queues. The *queue-number* of G, MINQN(G), is the smallest k such that G has a k-queue layout. The *queue layout* problem is to find a queue layout that uses minimum number of queues, for a given graph.

In the same manner, a *k-stack layout* of an undirected graph $G = (V, E)$ consists of a linear order of the vertex set V and an assignment of each edge in E to exactly one of the k-stacks. The queue policy operates as before, scanning in left-to-right order V. When vertex i is encountered, any edges assigned to a stack that have i as right end-point must be at the top of the stack, and are popped. All the edges that have i as left end-point are pushed in the assigned stacks. Similarly, we can define the *stack-number* of a given graph. The *stack layout* problem is to find, for a given graph, a stack layout that uses minimum number of stacks.

Stack layouts were introduced in Bernhart and Kainen [1979] under the name *embedding graphs in books*; the stack-number is also referred as the *page-number* or the *page thickness*. In Chung et al. [1987] and in Bilski [1993], optimal stack layouts are constructed for a variety of graph classes. The queue layout problem is introduced Heath and Rosenberg [1992]. In the same paper, the authors prove that recognizing graphs with queue-number 1 is an NP-complete problem. On the other hand, recognizing graphs with stack-number 1 can be done in linear time as this class of graphs equals the class of outerplanar graphs [Bernhart and Kainen 1979]. Notice that, when the vertex ordering of the layout is fixed, the minimum number of queues required for the layout can be obtained in polynomial time, while computing the minimum number of stacks is NP-hard [Even and Itai 1971; Garey et al. 1980]. A comparative study of the power of queues and stack layouts was performed in Heath et al. [1992].

11.2. Directed Acyclic Graphs

A linear layout φ of a *directed* graph is a layout that provides a topological sorting of the vertices, that is, for any arc (u, v) we have that $\varphi(u) < \varphi(v)$. This restricted type of layout can only be possible for directed acyclic graphs, and several layout problems have also been considered for this class of directed graphs.

The BANDWIDTH problem remains NP-complete for directed trees with maximum in-degree 1 and maximum out-degree 2 [Garey et al. 1978].

The CUTWIDTH and MINLA problems for directed acyclic graphs are NP-complete [Even and Shiloach 1975], but they can be solved in polynomial time for rooted trees [Adolphson and Hu 1973]. A polylogarithmic factor approximation algorithm for the MINLA problem on directed acyclic graphs is given in Ravi et al. [1991].

In some applications of stack and queue layouts, the input graph is a directed acyclic graph: Heath and Pemmaraju [1999] provide several results for different classes of directed acyclic graphs and Heath et al. [1999] analyze both problems from an algorithmic point of view. In this last paper it is shown that recognizing 1-queue directed acyclic graphs can be done in polynomial time, but recognizing 4-queue or 6-stack directed acyclic graphs are NP-complete problems. Extensions of these results to directed acyclic graphs obtained from posets are reported in Heath and Pemmaraju [1997] and Heath et al. [1999].

11.3. Colored Graphs

There is a close relationship between intervalizing graph problems and graph layout problems on colored graphs. Interval graphs have been studied intensively because of their wide applicability to practical problems [Golumbic 1980]. In the last years quite a lot of effort has been devoted to the study of problems where the goal is to complete a graph into an interval graph. This kind of problem is used to model ambiguity in physical mapping or consistency in temporal reasoning [Golumbic et al. 1995]. The problem of *physical mapping* consists in determining the relative position of several fragments of DNA, which can be used by biologists to characterize individual genes. As the DNA fragments are obtained scrambled, in order to get a map of the original sequence, it is necessary to re-assemble them using information of their pairwise overlaps [Karp 1993].

A *k-coloring* of a graph $G = (V, E)$ is a mapping $\kappa : V \longrightarrow \{1, \ldots, k\}$ such that no two adjacent vertices have the same color. A *k-colored* graph is a graph together with a k-coloring. A *colored layout* of a given k-colored graph (G, κ) is a layout φ of G such that for any $u, x, v \in V$, if $(u, v) \in E$ and $\varphi(u) < \varphi(x) < \varphi(v)$ then $\kappa(u) \neq \kappa(x)$. A *proper colored layout* of (G, κ) is a colored layout φ of G such that for all $u, v \in V$ and $x \in V$ with degree at least 2, if $(u, v) \in E$ and $\varphi(u) < \varphi(x) < \varphi(v)$ then there exists a vertex y such that $\varphi(v) < \varphi(y)$ and $(x, y) \in E$. We consider the following layout problems for colored graphs:

—*Colored Layout Problem* (CLP): Given a k-colored graph (G, κ), decide whether there is a colored layout φ of (G, κ).

—*Proper Colored Layout Problem* (PCLP): Given a k-colored graph (G, κ), decide whether there is a proper colored layout φ of (G, κ).

Àlvarez et al. [1998] showed that the CLP is NP-complete for caterpillars with hairs of length at most 2 and that the problem can be solved in NC for caterpillars with hairs of length at most 1, independently of the number of colors. In Àlvarez et al. [2001] it was shown that the problem is still NP-complete when the input is restricted to be four colored caterpillar with unbounded hair length. The PCLP is also hard for caterpillars of hair length 2 see Àlvarez and Serna [1999]. These two problems are closely related to intervalizing problems:

—*Interval Colored Graph* (ICG): Given a k-colored graph $(G = (V, E), \kappa)$, decide whether there is an edge superset E' such that the graph $G' = (V, E')$ is an interval graph and κ is still a proper coloring of G'.

—*Proper Interval Colored Graph* (PICG): Given a k-colored graph $(G = (V, E), \kappa)$, decide whether there is an edge superset E' such that the graph $G' = (V, E')$ is a proper interval graph and κ is still a proper coloring of G'.

The ICG problem is a special case of the *Interval Sandwich Problem* [Golumbic and Shamir 1993], and has received a lot of attention, as a simplified model for reconstructing the ordering physical DNA mapping problems [Fellows et al. 1993]. Moreover, the problem was shown to be NP-complete in Fellows et al. [1993] (see also Golumbic et al. [1994]) and it is polynomial time equivalent to the CLP [Dinneen 1995; Àlvarez et al. 1998], therefore it is also hard for caterpillars. When the problem was given parameterized on the number of colors, hardness results were obtained in Bodlaender et al. [1994]. Later it was shown that the problem is NP-complete for the case of 4 or more colors, while if we only have 2 and 3 colors the problem is solvable in linear and quadratic time respectively [Bodlaender and de Fluiter 1996]. A different approach to study the problem is to fix the number of colors and look at the complexity of the problem considering the degree of vertices. For fixed number of colors k and degree bounded by a constant d, there is a $O(n^{k-1})$ algorithm for the ICG [Kaplan and Shamir 1999].

In Kaplan and Shamir [1996] and Kaplan et al. [1999] it was shown that the parameterized version of the PICG problem, with parameter the number of colors, is $W[1]$-hard, and this implies the NP-completeness of PICG. They also gave a polynomial time algorithm for constant number of colors. For another simpler NP-completeness proof for this problem see Goldberg et al. [1995]. When the input graph is a colored tree, the problem is hard even for caterpillars of hair length 2. The proof of this last result exploits the relationship with the PICG problem [Àlvarez and Serna 1999].

11.4. Weighted Graphs

When we consider edge weighted graphs the cost of a layout measure, involving edges, is easily extended by multiplying the contribution of an edge by its weight. Some of the hardness results for layout problems are stronger, for example the CUTWIDTH is NP-complete for weighted trees [Monien and Sudborough 1988].

The MINLA is solvable in polynomial time for directed weighted trees Adolphson and Hu [1973]. Interestingly, the approximability results for several graph layout problems of Even et al. [2000] and Rao and Richa [1998] also apply to weighted graphs.

The research reported in Chirravuri et al. [1996] has shown that a formulation of the physical mapping problem is closely related to the MINLA problem on edge weighted graphs.

11.5. Graph Labelings

A vertex *labeling* of a graph G is an assignment ϕ of labels to the vertices of G that induces for each edge $uv \in E(G)$ an edge label depending on $\phi(u)$ and $\phi(v)$. A function ϕ is called a *graceful labeling* of a graph with m edges if ϕ is an injection from $V(G)$ to $\{0, 1, \ldots, m\}$, such that, for all $uv \in E(G)$, each edge label $|\phi(u) - \phi(v)|$ is distinct. A function ϕ is called a *harmonious labeling* of a graph with m edges if ϕ is an injection from $V(G)$ to the group of integers modulo m, such that, for all $uv \in E(G)$, each edge label $\phi(u) + \phi(v) \bmod m$ is distinct. See Gallian [1998] for a survey on graph labeling.

12. CONCLUDING REMARKS

In this paper we have presented a current view on the main known results about graph layout problems. The reader has surely observed that plenty of problems remain open: In general, everything that in this paper is not referred as done might turn into an interesting research problem.

For instance, with regard to complexity, little is known about MINLA for sparse graphs. It is not known if it is NP-complete or it is in the class P. Also, as noted in Bui and Peck [1992] and Papadimitriou and Sideri [1996] the NP-completeness of EDGEBIS for planar graphs remains open. Also, it would be interesting to know whether problems different from BANDWIDTH do or not belong to APX for general graphs or even for planar graphs.

Another task to carry out would be the study of other probabilistic classes of graphs for which good estimations of their optimal values could be found easily (binomial random graphs and random geometric graphs are an example of such a class). These classes could serve as a source of new graph generators to help in the benchmark and analysis of heuristics as well as in the design of networks with some specified properties. A concrete example of this class of graphs could be non uniform random geometric graphs, as they can be used as a model for mobile computing networks.

Moreover, the design of efficient hybrid heuristics is another interesting area, where plenty of research can be done.

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