Common Developments of Three Different Orthogonal Boxes

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Abstract

We investigate common developments that can fold into plural incongruent orthogonal boxes. It was shown that there are infinitely many orthogonal polygons that fold into two incongruent orthogonal boxes in 2008. In 2011, it was shown that there exists an orthogonal polygon that folds into three boxes of size $1 \times 1 \times 5$, $1 \times 2 \times 3$, and $0 \times 1 \times 11$. It remained open whether there exists an orthogonal polygon that folds into three boxes of positive volume. We give an affirmative answer to this open problem: there exists an orthogonal polygon that folds into three boxes of size $7 \times 8 \times 56$, $7 \times 14 \times 38$, and $2 \times 13 \times 58$. The construction idea can be generalized, and hence there exists an infinite number of orthogonal polygons that fold into three incongruent orthogonal boxes.

1 Introduction

Since Lubiw and O'Rourke posed the problem in 1996 [5], polygons that can fold into a (convex) polyhedron have been investigated. In the book on geometric folding algorithms by Demaine and O'Rourke in 2007, many results about such polygons are given [4, Chapter 25]. Such polygons have many applications including toys and puzzles. For example, the puzzle "cubigami" (Figure 1) is developed by Miller and Knuth, and it is a common development of all tetracubes except one (since the last one has surface area 16, while the others have surface area 18). One of the many interesting problems in this area is whether there exists a polygon that folds into plural incongruent orthogonal boxes. Biedl et al. gave two polygons that fold into two incongruent orthogonal boxes [3] (see also [4, Figure 25.53]). Later, Mitani and Uehara constructed infinite families of orthogonal polygons that fold into two incongruent orthogonal boxes [6]. Last year, Abel et al. showed an orthogonal polygon that folds into three boxes of size $1 \times 1 \times 5$, $1 \times 2 \times 3$, and $0 \times 1 \times 11$ [1]. However, the last "box" has volume zero; this is a so called "doubly covered rectangle" (e.g., [2]). Therefore, it remains open to show whether there is a polygon that can fold into three or more boxes of positive volume.

We give an affirmative answer to this open problem; there exists an orthogonal polygon that can fold into three incongruent orthogonal boxes of size $7 \times 8 \times 56$, $7 \times 14 \times 38$, and $2 \times 13 \times 58$ (Figure 2)[1].

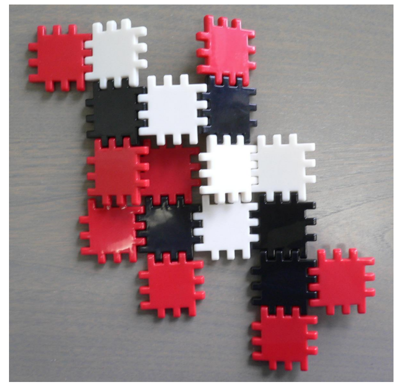

Figure 1: Cubigami.

The construction idea can be generalized. Therefore, we conclude that there exist infinitely many orthogonal polygons that can fold into three incongruent orthogonal boxes.

2 Construction of the common development

The definition of the *development* of a solid can be found in [4, Chap. 21]. Roughly, the development is the unfolding obtained by slicing the surface of the solid, and it forms a single connected simple polygon without self-overlap. The *common development* of two (or more) solids is the development that can fold into the solids. In this paper, as developments, we only consider orthogonal polygons that consist of unit squares.

Intuitively, the basic construction idea is simple. We first choose a common development of two different boxes of size $a \times b \times c$ and $a' \times b' \times c'$. We select one of these two boxes; let it have size $a \times b \times c$. We cut the two rectangles of size $a \times b$ (one at the *top*, and another at the *bottom* of the box) into two pieces of size $a \times b/2$ each. Then we squash the box and make these two rectangles of size $a \times b$ into two rectangles of size $(a + b/2) \times b/2 = 2a \times b/2$ (Figure 3). However, this simple idea immediately comes to a dead end; this operation can be done properly if and only if $a = b/2$, and hence we only change the rectangle of size 1×2 into the

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[1]This figure is also available at http://www.jaist.ac.jp/~uehara/etc/puzzle/nets/3box.pdf for ease to cut and fold.

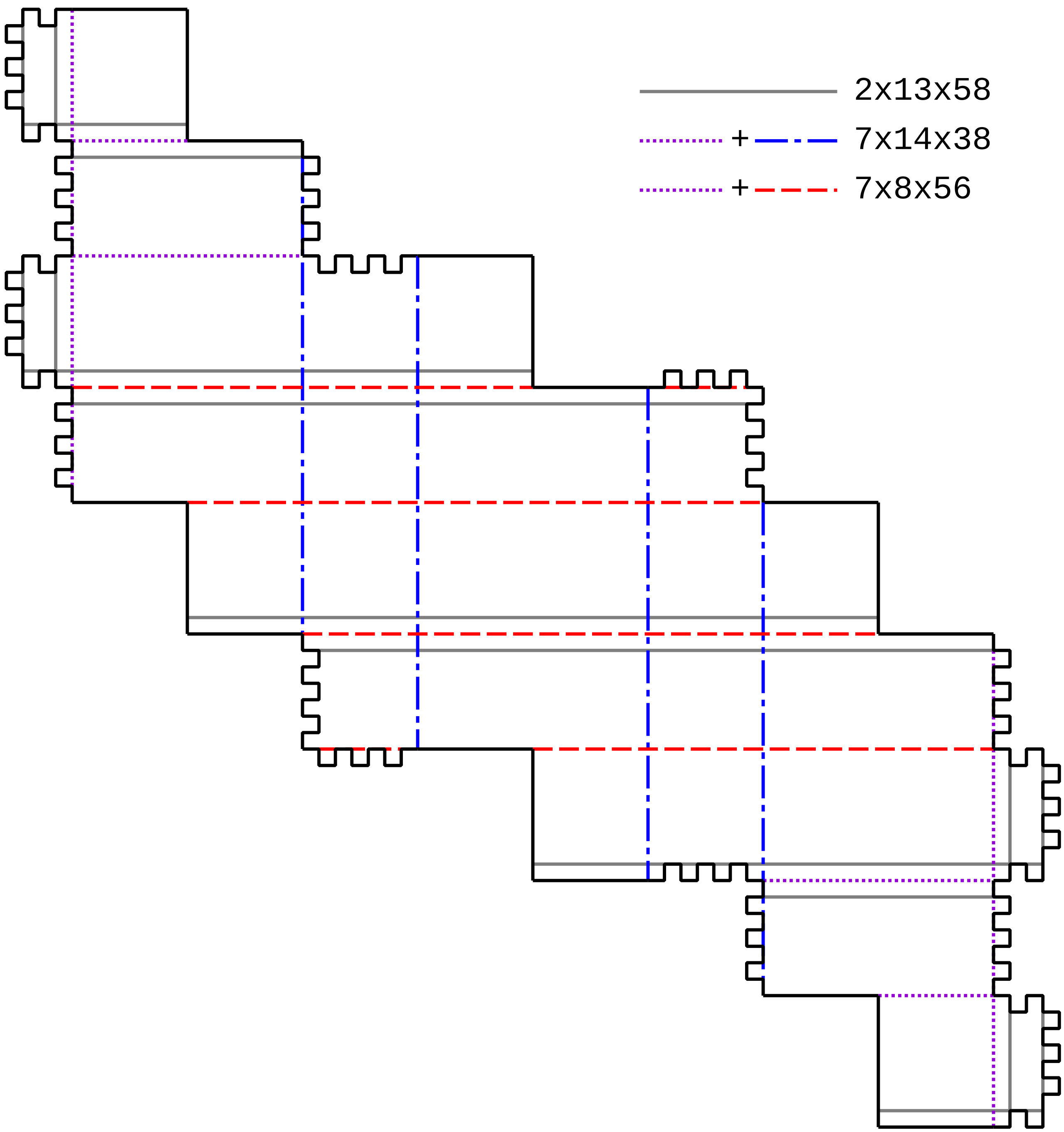

Figure 2: A common development of three different boxes of size $7 \times 8 \times 56$, $7 \times 14 \times 38$, and $2 \times 13 \times 58$.

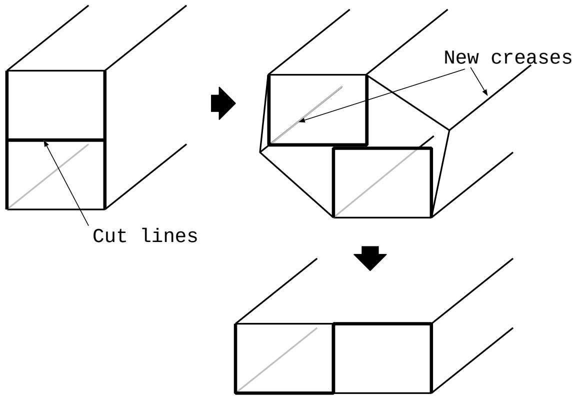

Figure 3: Basic idea: squash the box.

other rectangle of size 2×1, which are congruent.

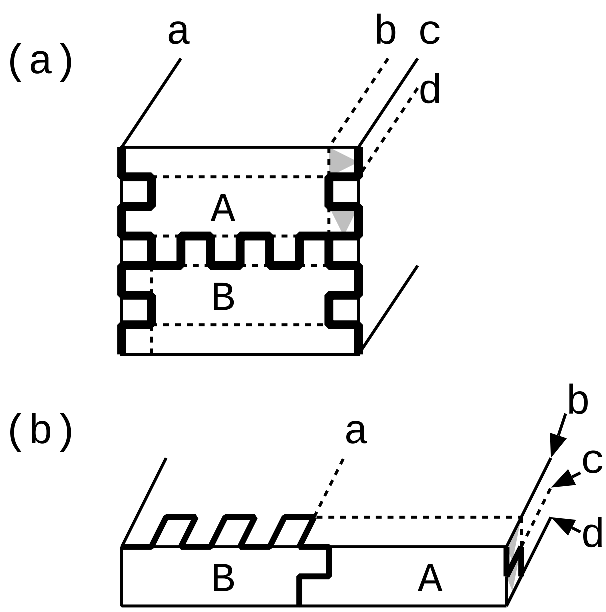

Figure 4: Squash the box: cut and fold.

The main trick to avoid this problem is to move pieces of the rectangles of size $a \times b$ of the box to the side rectangles of size $b \times c$ and $a \times c$. That is, after the squash operation above, the surface areas of the resultant top and bottom rectangles decrease, and the side rectangles grow a little. A specific example is given in Figure 4; in this example, the rectangle of size 8×7 is split into two congruent pieces by a mid zig-zag line; each piece in turn is divided into one central piece (labeled A, B in Figure 4). The result is a rectangle of size 13×2. (In Figure 4(a), the bold lines are cut lines, and dotted lines are folding lines to obtain (b). The lines a, b, c, and d are corresponding, and the gray triangles indicate how two squares are arranged by the operation.) Among the 56 squares, $56 - 26 = 30$ squares are moved to the four sides. We note that the perimeter of these two rectangles is not changed since $7+8+7+8 = 2+13+2+13 = 30$.

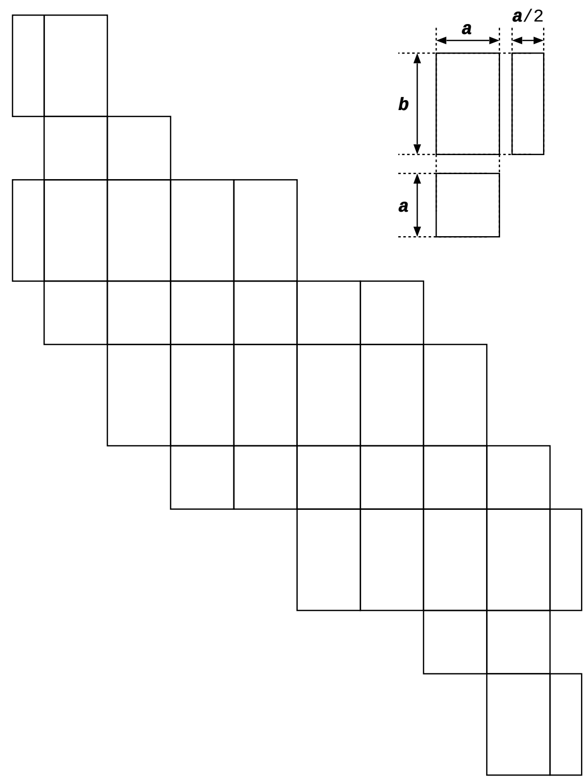

Figure 5: The base common development of two boxes of size $a \times b \times 8a$ and $a \times 2a \times (2a+3b)$.

To apply this idea, we choose a common development of two boxes of size $a \times b \times 8a$ and $a \times 2a \times (2a + 3b)$ in Figure 5. This is a modification of the common development of two boxes of size $1 \times 1 \times 8$ and $1 \times 2 \times 5$ in [6, Figure 5]. To apply the idea, we cut each of the top and bottom rectangles of size $a \times b$ into two congruent rectangles of size $a/2 \times b$. For any integers a and b, the orthogonal polygon in Figure 5 is a common development of two boxes of size $a \times b \times 8a$ and $a \times 2a \times (2a + 3b)$ (the two folding ways are drawn in bold lines in Figure 6).

The development in Figure 5 has useful properties for applying the idea in Figure 3: (1) we can adjust the size of the top and bottom rectangles to an arbitrary size, and (2) two folding ways share several folding lines. Especially, in Figure 6, each of the two connected gray areas is folded in the same way in both folding ways. Thus we attach the gadget from Figure 3 at this area letting $a = 7$ and $b = 8$. That is, we replace the rectangles of size $a/2 \times b$ by the rectangles A and B surrounded by the zig-zag lines in Figure 4.

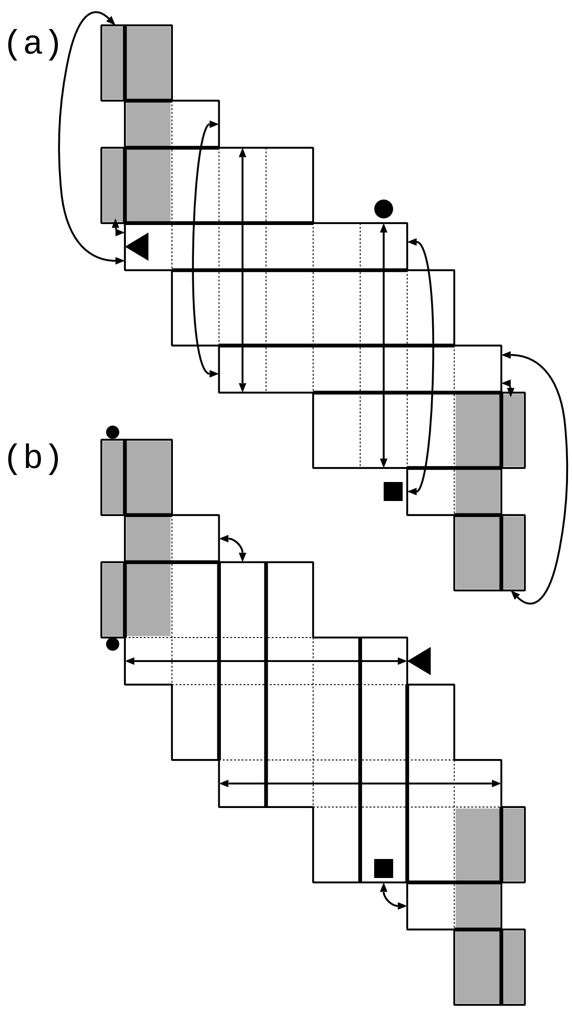

Figure 6: Some properties of the common development of two boxes of size $a \times b \times 8a$ and $a \times 2a \times (2a + 3b)$.

The only problem when applying the gadget is that the zig-zag lines propagate themselves according to the folding ways. That is, the zig-zag lines are glued to the different edges in some folding. For example, a zig-zag line at black triangle in Figure 6(a) is attached to the edge at black triangle in the folding way in Figure 6(b). Thus, these edges must consist of the same zig-zag pattern. On the other hand, this edge is attached to the edge at the black square in the folding way in Figure 6(a), which is attached to the black square in Figure 6(b). Thus, they also must have the same zig-zag pattern. Then the last edge is again attached to the edge with the black circle in Figure 6(a), and this is attached to the two edges with the smaller black circles in Figure 6(b). Then the loop of the propagation is closed, and we obtain the set of the edges that have to be represented by the zig-zag pattern.

Checking all the propagations, we finally obtain a common development of three different boxes of size $7 \times 8 \times 56$, $7 \times 14 \times 38$, and $2 \times 13 \times 58$ in Figure 2.

3 Generalization

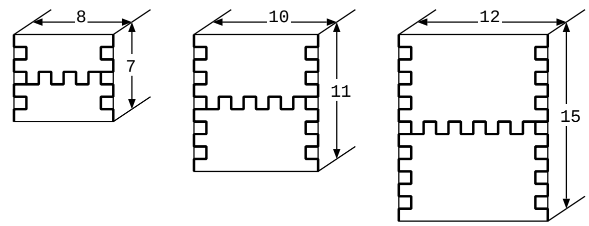

Figure 7: Generalization of the zig-zag cut.

In Section 2, we set $a = 7$ and $b = 8$, and change the rectangle of size 7×8 into 2×13. It is straightforward to generalize this method. For example, setting $a = 11$ and $b = 10$, we can change the rectangle of size 11×10 into 4×17 (see Figure 7). In general, for each integer $k = 0, 1, 2, \ldots$, setting $a = 4k+7$ and $b = 2(k+4)$, we can change the rectangle of size $a \times b$ to $2(k+1) \times (4k+13)$ in the same way as in Figure 4. The difference here from Figure 2 is in the number of turns of the zig-zags. Therefore, we have the following theorem immediately:

Theorem 1 *For each integer $k = 0, 1, 2, \ldots$, there is a common development that can fold into three different boxes of size $(4k+7) \times 2(k+4) \times 8(4k+7)$, $(4k+7) \times 2(4k+7) \times 2(7k+19)$, and $2(k+1) \times (4k+13) \times 2(16k+29)$.*

That is, there exists an infinite number of orthogonal polygons that can fold into three incongruent orthogonal boxes.

4 Concluding remarks

It is an open question if a polygon exists that can fold into four or more orthogonal boxes such that all of them have positive volume.

When two boxes of size $a \times b \times c$ and $a' \times b' \times c'$ share a common development, they satisfy a simple necessary condition $ab + bc + ca = a'b' + b'c' + c'a'$ since they have the same surface area. According to the experiments in [6], this necessary condition seems also sufficient for two boxes: for each pair of 3-tuples of integers satisfying the condition, there exist many common developments of two boxes of these size [6]. In this sense, the smallest possible surface area that can fold into three different boxes is 46; the area can produce three boxes of size $(1, 1, 11)$, $(1, 2, 7)$, and $(1, 3, 5)$. On the other hand, our construction produces a polygon of large surface area. The polygon in Figure 2 has area 1792. Applying the same idea to the different common development in [6], we also construct another smaller development of area

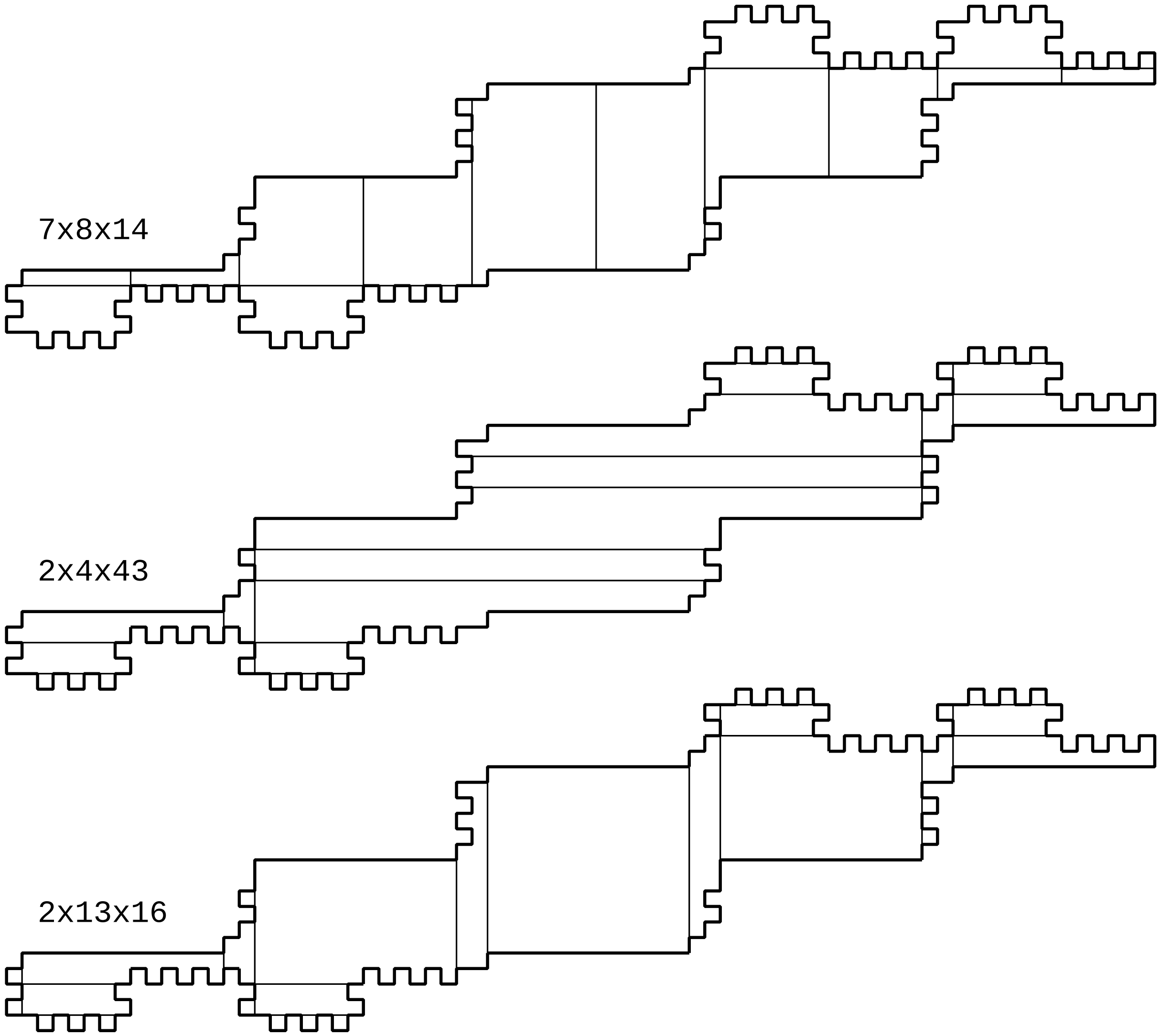

Figure 8: Another polygon that can fold into three boxes of size $7 \times 8 \times 14$, $2 \times 4 \times 43$, and $2 \times 13 \times 16$.

532 (Figure 8). Comparing to the results for two boxes, finding much smaller polygons would be a future work. Especially, is there a common development of area 46 that can fold into three boxes of size $(1, 1, 11)$, $(1, 2, 7)$, and $(1, 3, 5)$?

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