# Finding Shadows among Disks

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## Abstract

Given a set of $n$ non-overlapping unit disks in the plane, a line $\ell$ is called blocked if it intersects at least one of the disks and a point $p$ is called a shadow point if all lines containing $p$ are blocked. In addition, a maximal closed set of shadow points is called a shadow region. We derive properties of shadow regions, and present an $\mathcal{O}(n^4)$ algorithm that outputs all shadow regions. We prove that the number of shadow regions is $\Omega(n^4)$ for some instances, which implies that the worst-case time complexity of the presented algorithm is optimal.


## 1 Introduction

Let $\mathcal{D}$ be a set of $n$ closed and non-overlapping unit disks, i.e., disks with radius 1, in the two-dimensional plane. A line $\ell$ is called *blocked* if it intersects at least one of the disks in $\mathcal{D}$. A point $p$ is called a *shadow point*, if all lines containing $p$ are blocked. A point that is not a shadow point, is called a *light point*.

For a light point $p$ it holds that there is at least one line in the plane that does not intersect any of the disks in $\mathcal{D}$. It follows that all the points outside the convex hull spanned by the disks are light points. In other words, all shadow points defined by the disks in $\mathcal{D}$ are inside the convex hull spanned by the disks, denoted as $H(\mathcal{D})$.

A closed shape $S$ in the plane is a *shadow region* if each point in $S$ is a shadow point and if $S$ is maximal in the sense that there is no shape $S'$ containing only shadow points for which $S \subset S'$. It follows that the collection of shadow regions partitions the set of shadow points. By definition, each disk $\delta \in \mathcal{D}$ is contained in a shadow region.

**Shadow Regions Problem**. *Given $\mathcal{D}$, determine the set $\mathcal{S}$ of shadow regions in the plane.*

In other words, we are interested in designing an efficient algorithm that outputs the set of all shadow regions, for a given set $\mathcal{D}$ of disks. Figure 1 illustrates a set of 17 shadow regions defined by 14 randomly positioned unit disks.

**Motivation**. Hollemans et al. [4] describe a method for detecting objects. It uses light emitters and sensors placed on the boundary of a rectangular detection area.

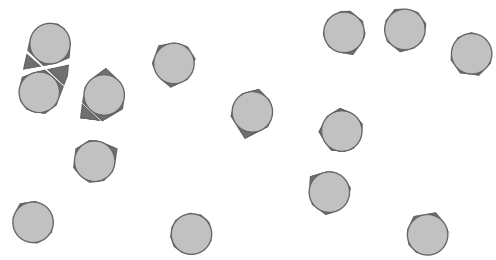

Figure 1: The shadow regions defined by 14 unit disks.

The shadow regions problem is related to the accuracy of the method in the following way. Each sensor continuously determines the set of emitters from which it receives light and the set of emitters from which it does not receive light because the line of sight is blocked by an object. Using this information, one can determine the set of shadow regions. As each object is located in a shadow region, this gives an approximation of the placement of the objects. Ideally, we have $n$ shadow regions, each with a size that is exactly equal to the object it contains. However, this ideal situation will not occur since, besides being part of an object, a point can also be a shadow point because: (1) the density of emitters and sensors is too low, and (2) all lines going through the point can be blocked by surrounding objects. The shadow points resulting from the latter cause are an intrinsic shortcoming of the method. By subtracting the objects from the solution of the shadow regions problem we get the shadow areas where detection fails due to this occlusion.

**Related work**. The problem considered by Dumitrescu and Jiang in [3] is to some extent related to the shadow regions problem. The authors show the existence of dark points [10] in maximal disk packings. A point is called dark within a set of disks if any ray with apex in that point intersects at least one of the disks. Note that any dark point is by definition a shadow point, but not vice versa. In addition, they present an algorithm for finding all of the dark points that are on the boundary of disks in a given set. While these authors' interest is in the dark points, we focus on the shadow points. Furthermore, we present an optimal algorithm to determine all shadow points in the plane defined by the disks, not only the ones on the disk boundaries. The problem of detecting circular objects in the plane is considered by Jovanović, Korst and Pronk in [5], where the

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authors present two algorithms to approximate the objects by convex polygons, using a finite set of line segments, defined by a given set of emitters and sensors. More remotely related problems are the problems on illumination of convex bodies [9, 11] and the visibility problems concerning hiding or blocking points and unit disks by a set of unit disks [6, 7, 8].

## 2 Introducing shadow regions

Let $\ell$ be a line in the plane such that it intersects the convex hull $H(\mathcal{D})$ of disks. The line $\ell$ is called a *defining line* for a shadow region $S$ if it contains an edge of $S$.

**Lemma 1** *Let $\ell$ be a defining line for a shadow region $S$. Then the following holds:*

- *$\ell$ does not intersect any disk in $\mathcal{D}$ in more than one point*
- *$\ell$ is tangent to at least two disks in $\mathcal{D}$*
- *$\ell$ is not tangent to any three disks $\delta_1$, $\delta_2$ and $\delta_3$, where $\delta_1$ and $\delta_2$ are on the same side of $\ell$ and $\delta_3$ is such that its point of tangency with $\ell$ is between the points of tangency of $\delta_1$ and $\delta_2$ with $\ell$.*

**Proof.** We prove the lemma by contradiction. Hence, suppose that $\ell$ is a defining line for a shadow region $S$ and suppose that (1) $\ell$ intersects a disk in $\mathcal{D}$ in more than one point, (2) $\ell$ does not intersect with any disk, (3) $\ell$ is tangent to exactly one disk or (4) $\ell$ is tangent to three disks $\delta_1$, $\delta_2$ and $\delta_3$, where $\delta_3$ is between $\delta_1$ and $\delta_2$ assuming that the disks are ordered by their points of tangency with $\ell$. We now show that all these cases yield a contradiction.

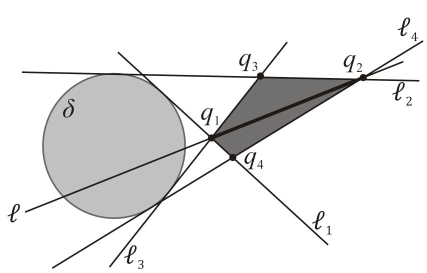


Figure 2: The line $\ell$ that intersects $\delta$ in more than one point cannot define a shadow region.

1. Let $\ell$ intersect a disk $\delta \in \mathcal{D}$ in more than one point, and let $\ell$ be a defining line for a shadow region $S$, i.e., $\ell$ contains an edge $(q_1, q_2)$ of $S$; see Figure 2. Let $\ell_1$ and $\ell_3$ be two lines that contain point $q_1$ and that are tangent to $\delta$, and let $\ell_2$ and $\ell_4$ be two lines that contain point $q_2$ and that are tangent to $\delta$. We denote $q_3 = \ell_2 \cap \ell_3$ and $q_4 = \ell_4 \cap \ell_1$. If we chose an arbitrary point $p$ inside the quadrilateral $q_1q_3q_2q_4$, then each line containing $p$ is either blocked by the disk $\delta$, or it intersects the edge $(q_1, q_2)$. By assumption, $(q_1, q_2)$ contains only shadow points, thus, each line intersecting $(q_1, q_2)$ is blocked. Hence, quadrilateral $q_1q_3q_2q_4$ is a shadow region, which is in contradiction with $(q_1, q_2)$ being an edge of a shadow region.

2. If $\ell$ does not intersect any disk in $\mathcal{D}$, then $\ell$ contains only light points. Hence, $\ell$ does not define any shadow region.

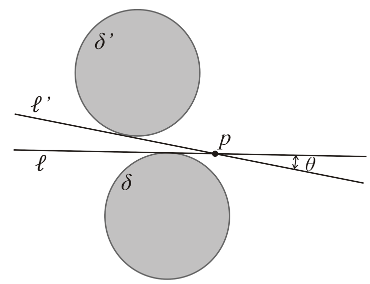


Figure 3: The line $\ell$ that is not tangent to more than one disk cannot contain the shadow point $p \notin \delta$.

3. Let $\ell$ be tangent to a disk $\delta \in \mathcal{D}$, such that it does not intersect any other disk in $\mathcal{D}$. In order for $\ell$ to be a defining line it should contain a shadow point $p$ outside $\delta$. Let $p \in \ell$ with $p \notin \delta$ be a shadow point; see Figure 3. Since $\ell$ is not tangent to any other disk in $\mathcal{D}$, $\ell$ can be rotated around $p$ over some angle $\theta$, in the direction away from the disk $\delta$, until it becomes tangent to some disk $\delta' \in \mathcal{D}$. We denote the rotated line as $\ell'$. Then, any line containing $p$ that is inside the angle $\theta$ between $\ell$ and $\ell'$ is not blocked, which is in contradiction with $p$ being a shadow point.

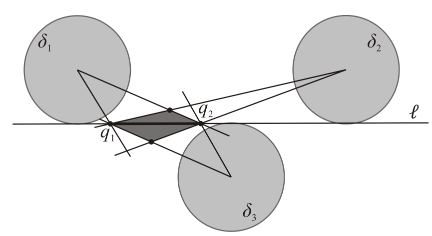


Figure 4: The line $\ell$ that is tangent to three disks ordered as illustrated cannot define a shadow region.

4. Let $\ell$ be a line tangent to three disks $\delta_1$, $\delta_2$ and $\delta_3$, where $\delta_3$ is between $\delta_1$ and $\delta_2$ when the disks are ordered by their points of tangency with $\ell$; see Figure 4. Furthermore, let $\ell$ contain a shadow edge $(q_1, q_2)$. There are several cases of different positioning of the shadow edge $(q_1, q_2)$ that need to be considered. Here, we prove only the case when

$(q_1, q_2)$ is between the points of tangency of $\delta_1$ and $\delta_3$. The other cases can be proved in a similar fashion. Now, let a shadow edge $(q_1, q_2)$ be between the points of tangency of $\delta_1$ and $\delta_3$. We connect the centers of the disks with the points $q_1$ and $q_2$, defining in that way six lines. Then, the smallest quadrilateral defined by these lines that has $(q_1, q_2)$ as its diagonal is a shadow region, which is in contradiction with $(q_1, q_2)$ being an edge of a shadow region.

□

Now, let us take a look at a small example of $\mathcal{D}$ consisting of only three disks, so that we can get a notion on the size, shape and the number of shadow regions defined by the disks. Each two non-tangent disks define four common tangent lines: a pair of parallel tangent lines and a pair of crossing (intersecting) tangent lines. The four tangent lines define four shadow areas that are attached to the disks; see Figure 5. By definition, a disk and all its attached shadow areas represent one shadow region. Note that the size of these shadow regions depends on the distance between the disks: the closer the disks, the larger the shadow regions. Depending on the

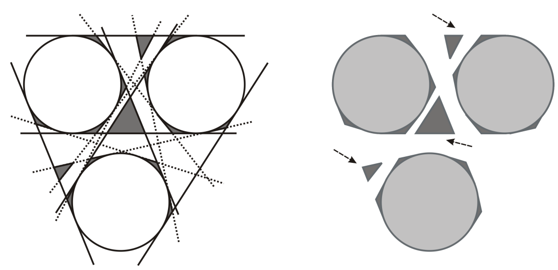

Figure 5: The shadow regions defined by 3 unit disks; the arrows point at the free shadow areas.

mutual distance, the three disks may define one or more *free* shadow regions, i.e., shadow regions that are not attached to any of the disks; see Figure 5. A free shadow region is bounded by line segments only, thus, it has the shape of a polygon. It can be shown that three disks can define at most 4 free shadow regions, which implies that they can define 1 to 7 shadow regions in total.

A shadow region can be formally represented by a cyclic sequence of points $p_0, p_1, \ldots, p_k$, where each two neighboring points are connected by either a line segment or a circular arc of radius 1.

Generally, $n$ disks define at most $2n(n-1)$ common tangent lines, which can partition the plane into $\mathcal{O}(n^4)$ non-overlapping convex polygons that contain either shadow points only or light points only. In Section 5, we will prove that there are instances for which the number of shadow regions defined by $n$ disks is $\Omega(n^4)$.

**Lemma 2** *A shadow region is convex.*

**Proof.** We prove the lemma by contradiction. Hence, assume that a shadow region $S$ is not convex. Let $p$ be a light point inside the convex hull $H(S)$ of $S$ and outside $S$. Each line containing $p$ intersects the shadow region $S$, which implies that it is blocked. This implies that $p$ is a shadow point, which is in contradiction with the assumption of $p$ being a light point. □

As a consequence of Lemma 1, in the process of determining the shadow regions, we consider only the set $\mathcal{T}$ of defining lines.

Let $t \in \mathcal{T}$ be a line tangent to two disks $\delta_1$ and $\delta_2$ in $\mathcal{D}$. The points of tangency between the line $t$ and the disks $\delta_1$ and $\delta_2$ divide $t$ into three parts: one line segment denoted by $s$, and two rays denoted by $r$ and $r'$.

**Lemma 3** *If disks $\delta_1$ and $\delta_2$ are not on the same side of $t$, line segment $s$ does not define a shadow region. If disks $\delta_1$ and $\delta_2$ are on the same side of $t$, the rays $r$ and $r'$ do not define a shadow region.*

**Proof.** We prove only the case of disks $\delta_1$ and $\delta_2$ being on different sides of $t$. The other case can be proved in the similar way.

Let $t$ be a crossing tangent line for disks $\delta_1$ and $\delta_2$ and let the line segment $s$ contain a shadow point $p$, such that $p \notin \delta_1$ and $p \notin \delta_2$. If $t$ is not a tangent line for any other disk except $\delta_1$ and $\delta_2$, then $t$ can be rotated around point $p$ over some angle $\theta$, in the direction that it does not intersect the disks $\delta_1$ and $\delta_2$, until it becomes tangent to some disk $\delta_3 \in \mathcal{D}$ or eventually, to both $\delta_1$ and $\delta_2$. Then, any line containing $p$ "inside" $\theta$ is not blocked, which is in contradiction with $p$ being a shadow point. If $t$ is also a tangent line to a disk $\delta_4$, so that it is not possible to rotate $t$ around $p$ over any angle $\theta$ without intersecting at least one of the disks $\delta_1$, $\delta_2$ and $\delta_4$, then from the third condition of Lemma 1 $t$ is not a defining line for any shadow region. □

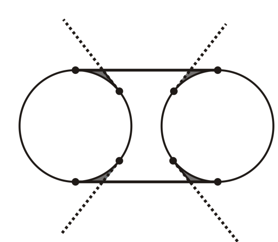

Figure 6: Parts of the tangent lines that define the shadow regions.

From Lemma 3, each crossing tangent line may be involved in the definition of shadow regions through the pair of rays with apices in the points of tangency. The parallel tangent lines are involved in the definition of shadow regions through the line segments connecting the points of tangency; see Figure 6.

## 3 Modelling light corridors

Let $\mathcal{L}$ be the set of all lines in the plane that do not intersect any disk, hence, $\mathcal{L}$ is the set of lines that only contain light points. Set $\mathcal{L}$ can be partitioned into two subsets, dividing lines and non-dividing lines. For a non-dividing line all disks are on the same side of that line. Each dividing line specifies a bipartition of the set of disks into non-empty sets. All dividing lines specifying the same bipartition of disks in $\mathcal{D}$ form a light corridor; see Figure 7. Note that each light point is contained in one or more light corridors. This means that the collection of shadow regions is given by the difference between $H(\mathcal{D})$ and the union of all light corridors.

Let $\mathcal{T}$ be the set of all defining lines. A light corridor can be characterized by its two crossing tangent lines $t$ and $t'$ in $\mathcal{T}$ that are clockwise fixed and counterclockwise fixed, respectively; see Figure 7. For $t$ this means that it cannot be rotated in clockwise direction around any point on the line over any angle $\theta$ such that it does not intersect at least one of the disks in more than one point. An analogous interpretation holds for $t'$. These lines define the "in" and "out" of the corridor through $H(\mathcal{D})$. Inside the convex hull $H(\mathcal{D})$, each light corridor is an open non-convex area, bounded by a set of line segments and a set of circular arcs of radius 1.

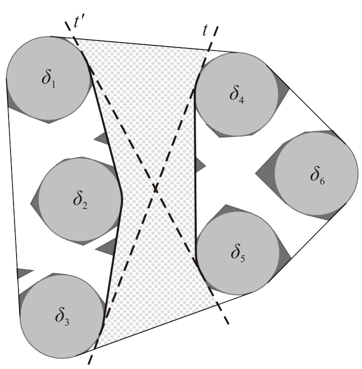


Figure 7: An example of a light corridor.

**Lemma 4** *The number of light corridors defined by $n$ non-overlapping unit disks is at most $\frac{n(n-1)}{2}$.*

**Proof.** The $n$ disks define at most $n(n-1)$ crossing tangent lines in $\mathcal{T}$. Each crossing tangent line in $\mathcal{T}$ defines one bipartition of disks, which corresponds to exactly one light corridor. Hence, the number of light corridors is not larger than the number of crossing tangent lines in $\mathcal{T}$. Moreover, each light corridor is characterized by a pair of crossing tangent lines, which implies that the number of light corridors is at most $n(n-1)/2$. $\square$

## 4 Algorithm

In this section, we present an algorithm for determining the set of all shadow regions defined by $n$ non-overlapping unit disks. We give the algorithm in a step-by-step manner and discuss its overall time complexity.

The algorithm for determining all shadow regions defined by $n$ unit disks consists of the following four main steps:

1. Determine the convex hull $H(\mathcal{D})$;
2. Determine the set $\mathcal{T}$ of all defining tangent lines;
3. Determine all light corridors inside $H(\mathcal{D})$;
4. Determine the union $U$ of all light corridors and next, the set of all shadow regions, by finding the set difference between $H(\mathcal{D})$ and $U$.

Let us now take a closer look at each step of the algorithm and its worst-case time complexity. In the first step, it is needed to compute first the convex hull of the disk centers, and then to compute an offset polygon, which can be done in $\mathcal{O}(n \log n)$ time [2].

As defined in Section 2, the set $\mathcal{T}$ of defining lines are the lines tangent to at least two disks in $\mathcal{D}$ that do not intersect any of the disks in $\mathcal{D}$ in more than one point. Now, we can determine the set $\mathcal{T}$ of all defining lines in $\mathcal{O}(n^2 \log n)$ time, as follows. For each disk in $\mathcal{D}$, we sort radially the other $n-1$ disks, which takes $\mathcal{O}(n \log n)$ time. This structure allows to find all defining lines of one disk in linear time. In addition to each defining line determined, we keep the information on tangent disks and the tangency points, the type of the tangent line, i.e., whether it is a crossing line or not, and the part(s) of the line which are involved in the definition of the shadow regions, i.e., the rays or the line segment, as explained in Lemma 3. Hence, it takes $\mathcal{O}(n \log n)$ time to determine all defining lines of one disk and all the additional properties. Therefore, finding the set $\mathcal{T}$ of defining tangent lines for all $n$ disks takes $\mathcal{O}(n^2 \log n)$ time.

In order to determine the set of all light corridors, for each of the disks we need the sorted list of all its points of tangency, in a cyclic order. Such a list can be determined in $\mathcal{O}(n^2 \log n)$ time since all the defining lines are determined, hence, all the points of tangency for each of the disks.

As mentioned in the proof of Lemma 4, a crossing tangent line in $\mathcal{T}$ characterizes one light corridor. Starting with a crossing line from $\mathcal{T}$, we determine the corresponding light corridor as follows. We start by including one ray of the chosen crossing line. Then, we simply look up the corresponding point of tangency on the tangent disk and take the successor point of tangency from the sorted list of points for that disk. That point is a starting point for either a line segment, or a ray of some other tangent line. In the case of a starting point of a line segment, we look up the ending point on the next disk, etc. The computation of one side of the corridor is finished when a ray occurs in the sequence. The other side is determined in the same way, starting with the other ray of the originally chosen crossing tangent line.

From Lemma 4, the number of light corridors is $\mathcal{O}(n^2)$. In addition, the number of all defining tangent lines is also quadratic in the number of disks, which implies that the total number of all rays (2 rays per crossing tangent) and line segments (1 line segment per parallel tangent) together is also $\mathcal{O}(n^2)$. In this way, amortized over all iterations, the light corridors can be determined in $\mathcal{O}(n^2)$ time, which implies that the total time complexity of the third step of the algorithm is $\mathcal{O}(n^2 \log n)$.

The problem of determining the union $U$ of all light corridors comes down to the problem of finding the intersections of a set of line segments and circular arcs. This is a well-known and extensively studied problem [2]. Using the deterministic algorithm by Balaban [1], the intersections of $N$ line or curve segments can be determined in $\mathcal{O}(N \log N + K)$ time, where $K$ is the number of intersecting pairs. Given that we have $\mathcal{O}(n^2)$ line segments and circular arcs, the number $K$ of intersecting pairs is $\mathcal{O}(n^4)$. Therefore, using this algorithm, the union $U$ of all light corridors can be determined in $\mathcal{O}(n^4)$ time. The set of all shadow regions is then simply determined as a complement set of $U$ within the convex hull $H(\mathcal{D})$.

With the discussion above, we get to the following result.

**Theorem 5** *The set of all shadow regions defined by $n$ non-overlapping unit disks can be determined in $\mathcal{O}(n^4)$ time.*

## 5 Determining the number of shadow regions

In the previous section we presented an $\mathcal{O}(n^4)$ algorithm for deriving all shadow regions created by a set of $n$ disks. From this it follows that a set of $n$ disks defines $\mathcal{O}(n^4)$ shadow regions. In this section we show that this bound is tight, i.e., that problem instances exist with $\Omega(n^4)$ shadow regions. This implies the interesting result that the $\mathcal{O}(n^4)$ worst-case time complexity of the presented algorithm is optimal.

To construct a problem instance with $\Omega(n^4)$ shadow regions, we place the disks in two "columns", where each column contains $n$ equidistant disks, such that each disk of one column is directly opposite to a disk of the other column. The idea behind the construction is to obtain a quadratic number of thin light corridors that pass between the disks of the two columns, i.e., in the left to right direction. If these corridors do not intersect within some finite area of width $w$, then adding another $2n$ disks that, in the same way, create quadratic number of light corridors in the top-bottom direction, results in $\Theta(n^4)$ shadow regions; see an illustration in Figure 8. If we need to add only linear number of mutually tangent disks to block the light corridors that come from other (e.g., diagonal) directions, we then have a linear number of disks creating $\Theta(n^4)$ shadow regions.

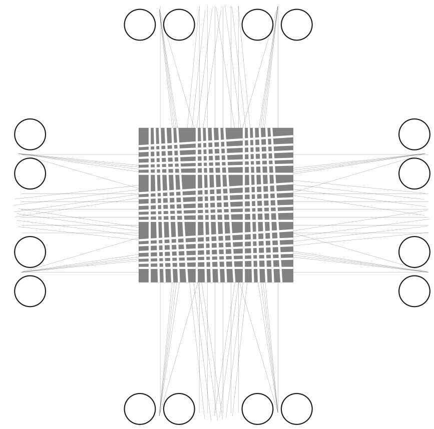

Figure 8: Constructing $\Omega(n^4)$ shadow regions with a linear number of disks - an illustration.

Let $L$ be the line connecting the centers $O_1, O_2, \ldots, O_n$ of the disks $\delta_1, \delta_2, \ldots, \delta_n$ in the left column and, in the same fashion, let $R$ be the line connecting the centers $O'_1, O'_2, \ldots, O'_n$ of the disks $\delta'_1, \delta'_2, \ldots, \delta'_n$ in the right column; see Figure 9. Furthermore, let $h$ denote the distance between the columns, i.e., the distance between $L$ and $R$, and let $d$ denote the distance between two neighboring disks in one column, measured from center to center. Given $h$, the distance $d$ is chosen so that the top two disks of one column and the bottom two disks of the other column are all tangent to the same line. In this way, there is no light corridor defined by these top-bottom pairs of disks, however, there is exactly one light corridor between any other two pairs of neighboring disks in different columns. From the congruence of the two gray triangles in Figure 9, the relation between the distances $d$ and $h$ is given by

$$d = \frac{2h}{\sqrt{h^2 - 4(n-2)^2}} \tag{1}$$

For the time being, we only consider the light corridors between pairs $(\delta_i, \delta_{i+1})$ of neighboring disks from the left column and pairs $(\delta'_j, \delta'_{j+1})$ of neighboring disks from the right column, where $i, j \in \{1, \ldots, n-1\}$.

The distance $d$ between the neighboring disks determines the width of the corridors. From Equation (1), we get that if $h \to \infty$, then $d \to 2$. Furthermore, using elementary calculus, it can be shown that increasing

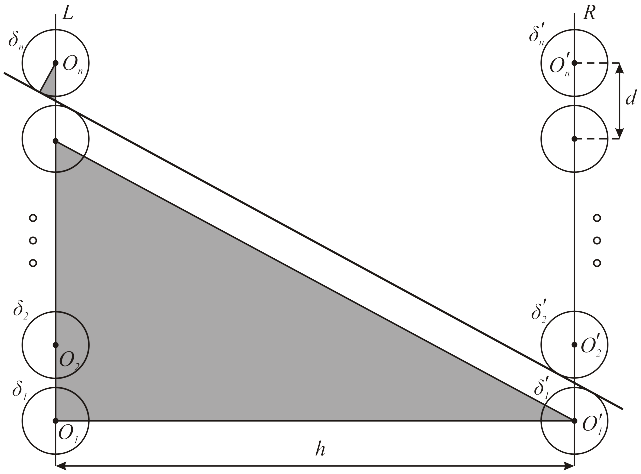


Figure 9: The columns of disks, each column containing $n$ disks.

the distance $h$ between the columns results in decreasing the width of the corridors. Note that the corridors are not all of the same width, i.e., the longer corridors are thinner than the shorter corridors.

It remains to be shown that there is an area between the columns where no two corridors intersect. Furthermore, we want to show that for some $h$, the width $w$ of that area can be at least $nd$. In this way, overlapping (or intersecting) this area containing the left to right corridors with the area containing the top to bottom non-intersecting corridors, results in creating $\Theta(n^4)$ shadow regions.

In Section 3 we showed that each light corridor is characterized by a pair of two crossing tangent lines. In this special case of disks being placed in two columns, one can easily show that, between the columns, each corridor is bounded by a pair of parallel line segments. Considering the left column as the beginning and the right column as the end of the corridors, among the intersection points of the corridors' bounding line segments, we can distinguish two subsets of points: the splitting points and the meeting points; see Figure 10. The splitting point of two light corridors that begin between the same pair of disks is the common (intersection) point of these corridors furthest from $L$. In a similar way, the meeting point of two corridors that do not begin between the same pair of disks is the intersection point of these two corridors closest to $L$. Let $P_s$ denote the vertical line containing the splitting point(s) furthest from $L$ and let $P_m$ denote the vertical line containing the meeting point(s) closest to $L$. Clearly, if the distance $\bar{h}_s$ between $P_s$ and $L$ is smaller than the distance $\bar{h}_m$ between $P_m$ and $L$, the area between the two vertical lines $P_s$ and $P_m$, gives an area inside which no two corridors intersect. In addition, the width $w$ of the

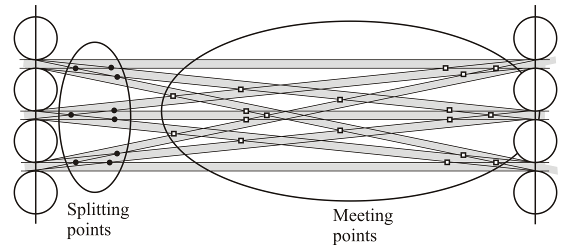


Figure 10: The splitting points and the meeting points of nine light corridors passing between eight disks in the columns.

area is given by

$$w = \bar{h}_m - \bar{h}_s \tag{2}$$

Next, we express the distance $\bar{h}_s$ as a function of the distance $h$ between the columns of disks. Let us consider only the $n-1$ light corridors that all begin between one pair of neighboring disks in the left column. One can show that among the splitting points of these corridors, the splitting point furthest from $L$, is the splitting point of two neighboring light corridors, i.e., the corridors that end between the neighboring pairs of disks in the right column. Let $h_s$ be the distance from the splitting point $P$ of an arbitrary pair of neighboring corridors to the line $L$. By definition, $\bar{h}_s$ is the maximum of all distances $h_s$ of the splitting points of all neighboring corridors.

**Lemma 6** *For an arbitrary pair of neighboring light corridors $C_j$ and $C_{j+1}$, it holds that*

$$\lim_{h\to\infty} h_s = 0.$$

**Proof.** Let $C_j$ and $C_{j+1}$ be two neighboring corridors that begin between the disks $\delta_j$ and $\delta_{j+1}$ with centers $O_j$ and $O_{j+1}$ on $L$, respectively. Furthermore, let $C_j$ end between the disks $\delta'_{j+k-1}$ and $\delta'_{j+k}$ and let $C_{j+1}$ end between the disks $\delta'_{j+k}$ and $\delta'_{j+k+1}$.

The splitting point $P$ of the corridors $C_j$ and $C_{j+1}$ is the intersection point of the tangent lines $t_j$ and $t_{j+1}$.The point of tangency between the disk $\delta_j$ and the line $t_j$ is denoted as $T_j$, while the point of tangency between the disk $\delta_{j+1}$ and the line $t_{j+1}$ is denoted as $T_{j+1}$. In addition, we denote $L \cap t_j = P_j$ and $L \cap t_{j+1} = P_{j+1}$.

Since $\angle O_{j+1}O'_{j+k}O_{j+k} = \angle P_{j+1}O_{j+1}T_{j+1}$ and $\angle O_jO'_{j+k}O_{j+k} = \angle P_jO_jT_j$, and by denoting

$$\alpha = \angle P_jO_jT_j, \quad \beta = \angle P_{j+1}O_{j+1}T_{j+1},$$
$$z = P_jP_{j+1}, \quad z_j = O_jP_j, \quad z_{j+1} = O_{j+1}P_{j+1},$$

we can express the relations

$$\begin{aligned} z_j &= \frac{1}{\cos\alpha}, \quad z_{j+1} = \frac{1}{\cos\beta}, \\ z &= d - (z_j + z_{j+1}) = d - (\frac{1}{\cos\alpha} + \frac{1}{\cos\beta}) \end{aligned} \tag{3}$$

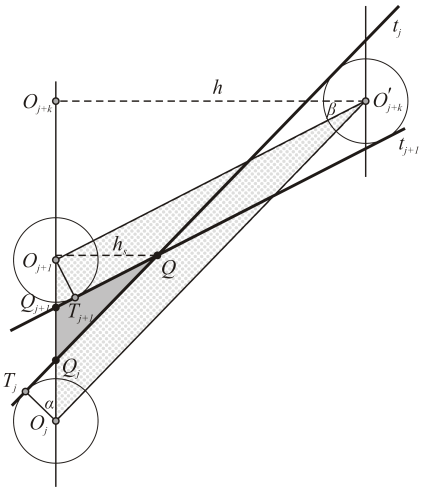


Figure 11: Deriving the distance $h_s$ from the splitting point $P$ to the line $L$ - an illustration.

From (3) and $\triangle PP_jP_{j+1} \sim \triangle O_jO'_{j+k}O_{j+1}$ we derive the distance $h_s$ as follows.

$$h_s = \frac{z \cdot h}{d}. \tag{4}$$

Since $O_jO_{j+k} = k \cdot d$ and

$$\cos\alpha = \frac{h}{\sqrt{h^2 + k^2d^2}}, \qquad \cos\beta = \frac{h}{\sqrt{h^2 + (k-1)^2d^2}},$$

we have

$$h_s = h - \frac{\sqrt{h^2+k^2d^2} + \sqrt{h^2+(k-1)^2d^2}}{d} \tag{5}$$

From Equation (1) in Section 5 and (5) we have

$$h_s = h - \frac{\sqrt{h^2+4(k^2-(n-2)^2)} + \sqrt{h^2+4((k-1)^2-(n-2)^2)}}{2} \tag{6}$$

Since $k \leq n-1$, expressing the limit for $h_s$ in (6) when $h \to \infty$ implies the final result, i.e.,

$$\lim_{h\to\infty} h_s = 0.$$

□

In other words, for $h$ large enough, all light corridors split on distance $\epsilon$ from the line $L$ and "enter" the area in which they do not intersect.

From Equation (2), to determine the width $w$ of the area where the light corridors do not intersect, besides the distance $\bar{h}_s$, we also need to determine the distance $\bar{h}_m$, i.e., the distance from the closest meeting point(s) to the line $L$. We first determine the light corridors that define the closest meeting point(s).

One can show that the light corridors that define the closest meeting point(s) begin between two neighboring pairs of disks; see Figure 10. More precisely, the bottom-most corridor $C_b$ of all corridors beginning between the pair of disks $(\delta_{j+1}, \delta_j)$ and the top-most corridor $C_t$ of all corridors beginning between the pair of disks $(\delta_j, \delta_{j-1})$ define (one of) the closest meeting point(s) to the line $L$. Let $h_m$ be the distance from the splitting point $P'$ of the corridors $C_b$ and $C_t$ to the line $L$. Note that $C_b$ ends between the bottom pair of disks $(\delta'_1, \delta'_2)$ and $C_t$ ends between the top pair of disks $(\delta'_n, \delta'_{n-1})$ in the right column. In a similar way as Lemma 6, using elementary calculus, one can prove the following lemma.

**Lemma 7** *For the distance $h_m$, it holds that $h_m \to \infty$, when $h \to \infty$.*

From Lemma 6 and Lemma 7 and Equation (2), we can conclude that for $h$ large enough, the width $w$ of the area where corridors do not intersect can be of size $nd$. Note that the area is not in the middle between the columns. Instead, we have two such areas of non-intersecting corridors adjacent to the left and to the right column, respectively. In the next step of the construction, we add

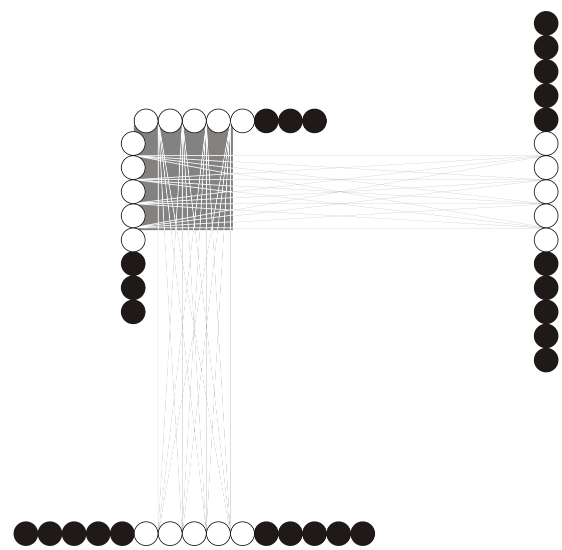

Figure 12: A linear number of unit disks defining $\Theta(n^4)$ shadow regions - the thin light corridors pass between the white disks; the black disks are mutually tangent, hence, representing the blocking disks.

$2n$ disks organized in two rows that are on the top and the bottom side, as we mentioned earlier in this section, and such that the areas of non-intersecting corridors completely overlap. Each of the $\mathcal{O}(n^2)$ light corridors in the left to right direction intersects each of the $\mathcal{O}(n^2)$ light corridors in the top to bottom direction. Hence, they partition the square area of size $(nd)^2$ into $\Theta(n^4)$ regions. In order for these regions to be the *shadow* regions, the light coming from directions different than left, right, top or bottom must be blocked. Therefore, in addition to the $4n$ disks used in this construction, we "close the gaps" by extending, for example, the top row and the left column by $\lceil \frac{n}{2} \rceil$ tangent disks each and the right column and the bottom row with $2n$ tangent disks each; see Figure 12. These blocking disks ensure that there are no additional corridors intersecting the area partitioned into shadow regions by the constructed light corridors.

## 6 Concluding remarks

We considered the problem of determining all shadow regions defined by a set of $n$ non-overlapping unit disks in the plane. We discussed the basic properties of the shadow regions and we presented an $\mathcal{O}(n^4)$ algorithm for determining them. We showed that the number of shadow regions can be $\Omega(n^4)$. Hence, the presented algorithm determines all the shadow regions in worst-case optimal time.

## References

[1] I.J. Balaban. An optimal algorithm for finding segment intersections. In *Proc. of the 11th ACM Symposium on Computational Geometry*, 1995, Vancouver, Canada, 211–219.

[2] M. de Berg, M. van Kreveld, M. Overmars, and O. Schwarzkopf. *Computational Geometry: Algorithms and Applications*, 2nd Edition, 2000, Springer.

[3] A. Dumitrescu, and M. Jiang. The forest hiding problem. In *Proc. of the 21st ACM-SIAM Symposium on Discrete Algorithms*, 2010, Austin, Texas, USA.

[4] G. Hollemans, T. Bergman, V. Buil, K. van Gelder, M. Groten, J. Hoonhout, T. Lashina, E. van Loenen, and S. van de Wijdeven. Entertaible: multi-user multi-object concurrent input. In *Proc. of the 19th Annual ACM Symposium on User Interface Software and Technology*, 2006, Montreux, Switzerland, 55–56.

[5] N. Jovanović, J. Korst, and V. Pronk. Object detection in flatland. In *Proc. of the 3rd International Conference on Advanced Engineering Computing and Applications in Sciences*, 2009, Sliema, Malta.

[6] N. Jovanović, J. Korst, and A.J.E.M. Janssen. Minimum blocking sets of circles for a set of lines in the plane. In *Proc. of the 20th Canadian Conference on Computational Geometry*, 2008, Montréal, Canada, 91–94.

[7] N. Jovanović, J. Korst, R. Clout, V. Pronk, and L. Tolhuizen. Candle in the woods: asymptotic bounds on minimum blocking sets. In *Proc. of the 25th ACM Symposium on Computational Geometry*, 2009, Aarhus, Denmark, 148–152.

[8] N. Jovanović, J. Korst, Z. Aleksovski, and R. Jovanović. Hiding in the crowd: asymptotic bounds on minimum blocking sets. In *Proc. of the 26th European Workshop on Computational Geometry*, 2010, Dortmund, Germany, 197–200.

[9] H. Martini, and V. Soltan. Combinatorial problems on the illumination of convex bodies. *Aequationes Mathematicae 57*, 1999, 121–152.

[10] J. Mitchell. Dark points among disks. In *Open Problems from the 2007 Fall Workshop in Computational Geometry*, Hawthorne, New York, USA.

[11] L. Szabo, and Z. Ujvary-Menyhart. Clouds of planar convex bodies. In *Aequationes Mathematicae 63*, 2002, 292–302.