Point-Set Embedding in Three Dimensions *

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Abstract

Given a graph G with n vertices and m edges, and a set P of n points on a three-dimensional integer grid, the 3D Point-Set Embeddability problem is to determine a (three-dimensional) crossing-free drawing of G with vertices located at P and with edges drawn as polylines with bend-points at integer grid points. We solve a variant of the problem in which the points of P lie on a plane. The resulting drawing lies in a bounding box of reasonable volume and uses at most $O(\log m)$ bends per edge.

If a particular point-set P is not specified, we show that the graph G can be drawn crossing-free with at most $O(\log m)$ bends per edge in a volume bounded by $O((n + m) \log m)$. Our construction is asymptotically similar to previously known drawings, however avoids a possibly non-polynomial preprocessing step.

1 Introduction

The two-dimensional (2D) graph drawing literature is extensive. Drawing graphs in three dimensions (3D) has been considered by several authors under a variety of models. One natural model is to draw vertices as points at integer-valued grid points in a 3D Cartesian coordinate system and represent edges as straight line segments between adjacent vertices, with no pair of edges intersecting.

Cohen, Eades, Lin and Ruskey [4] showed that it is possible to draw *any* graph in this model, and indeed the complete graph K_n is drawable within a bounding box of volume $\Theta(n^3)$. Restricted classes of graphs may however be drawn in smaller asymptotic volume. For example, Calamonieri and Sterbini [3] showed that all 2-, 3-, and 4-colourable graphs can be drawn in $O(n^2)$ volume. Pach, Thiele and Tóth [17] showed a volume bound of $\Theta(n^2)$ for r-colourable graphs, where r is a constant. Dujmović, Morin and Wood [9] investigated the connection of bounded tree-width to 3D layouts. For outerplanar graphs, Felsner, Liotta and Wismath [13] described a 3D drawing in $\Theta(n)$ volume. Establishing tight volume bounds for the class of planar graphs remains an open problem. An upper bound of $O(n^{1.5})$ was established by Dujmović and Wood [10]. Recently, di Battista, Frati and Pach [8] improved the volume bound for planar graphs to $O(n \log^{16} n)$.

In two-dimensional graph drawing, the effect of allowing bends in edges has been well studied. For example, Kaufmann and Wiese [15] showed that all planar graphs can be drawn with only two bends per edge and all vertices located on a straight line.

The consequences of allowing bends in 3D has received less attention. The model considered here draws both vertices and bend points of edges at integer grid points. A simple one-bend construction achieving a volume of $O(n{\cdot}m)$ uses two skew lines – one for the vertices and one for a single bend-point on each edge. Bose, Czyzowicz, Morin, and Wood [1], showed that the number of edges in a graph provides an asymptotic lower bound on the volume regardless of the number of bends permitted, thus establishing $\Omega(n^2)$ as the lower bound on the volume for K_n. This lower bound was explicitly achieved by Dyck, Joevenazzo, Nickle, Wilsdon and Wismath [12] who presented a construction with at most two bends per edge. The upper bound is also a consequence of a more general result of Dujmović and Wood [11]. Morin and Wood [16], presented a one-bend drawing of K_n that achieves $O(n^3/\log^2 n)$ volume and in [5] the gap between this result and the $\Omega(n^2)$ lower bound was narrowed to achieve a one-bend drawing with volume $O(n^{2.5})$.

It is also interesting to consider the volume of classes of graphs when bends are allowed. Dujmović and Wood [11] showed that in general, a volume of $O(n{+}m \log q)$ is achievable with $O(\log q)$ bends per edge, where q is the queue number of the graph and thus $q \leq n$. A recent result of di Battista et al. [8] on the queue number for planar graphs thus implies a volume of $O(n \log \log n)$ with $O(\log \log n)$ bends per edge for planar graphs. Both of these results implicitly rely on bounding the queue number of the given graph and obtaining an initial ordering of the vertices that achieves the queue layout. It is known that determining the queue number of a graph is in general NP-Complete [14].

In this paper, we describe a three-dimensional drawing technique that is asymptotically competitive with previous volume bounds for some classes of graphs, including planar graphs, at the expense of a relatively large number of bends per edge. For planar graphs, the

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volume of our drawing is bounded by $O(n \log n)$ with at most $O(\log n)$ bends per edge. Previous results were primarily existential, whereas our technique is constructive and does not rely on computing a queue layout of the graph. Our construction achieves the stated bounds independent of the vertex ordering.

Furthermore, our construction can be extended to resolve an interesting three-dimensional point-set embeddability problem. For example, we show that if P is a set of n points given on a plane and in a bounding box of size W by H, then any graph with n vertices and m edges can be drawn crossing-free on P within a bounding box of dimensions $\max(W, m) \times (H+3) \times (2+\log m)$ and with at most $O(\log m)$ bends per edge.

1.1 Point-Set Embedding

Embedding a graph onto a specified point-set in 2D has been considered in various models. We formulate a variant which includes consideration of the resulting area and number of bends per edge.

2DPSE: Given a planar graph G with n vertices, $V = \{v_1, v_2, \ldots v_n\}$, and given a set of n distinct points $P = \{p_1, p_2, \ldots p_n\}$ each with integer coordinates in the plane, can G be drawn crossing-free on P with v_i at p_i and with a number of bends polynomial in n and in an area polynomial in n and the dimension of P?

If the bijection is relaxed so that each vertex v_i is mapped to any point p_j, the embedding is said to be *without mapping*, and if a specific bijection is provided, the embedding is said to be *with mapping*.

We now formulate a version of the 2DPSE problem for three dimensions which also constrains the bends and volume of the resulting drawing.

3DPSE: Given a graph G with n vertices, $V = \{v_1, v_2, \ldots v_n\}$, and given a set of n distinct points $P = \{p_1, p_2, \ldots p_n\}$ each with integer coordinates in three dimensions, can G be drawn crossing-free on P with v_i at p_i and with a number of bends polynomial in n and in an volume polynomial in n and the dimension of P?

This general problem remains open. The existence version of the problem, ignoring bend and volume constraints is resolved in section 4 where we also present a version of the 3DPSE problem that is solvable via a modification of the construction we present in section 2. But here we first review the relevant results from 2D.

In two dimensions, Cabello [2] showed that determining whether there exists a straight-line drawing of planar graph G on P *without mapping* is NP-hard. Pach and Wenger [18] proved that for the *with mapping* version, $O(n^2)$ bends may be required. The Kaufmann and Wiese [15] result establishes that two bends are always sufficient for the *without mapping* version of the problem, however the bend-points that are computed are not required to have integer coordinates and the resulting area appears to be inherently exponential. Di Giacomo et al. [6] investigated a version of the point-set embeddability problem in which some of the edges of the graph are specified to be straight-line segments. They showed that some edges may then require $O(2^n)$ bends. These two results thus motivate the 3DPSE problem in which both the bends and volume are constrained.

2 Definitions and Preliminaries

Given an undirected graph G of n vertices and m edges, a 3D grid drawing of G maps each vertex to a distinct point of $\mathbb{Z}^3$, and each edge of G to a polyline between its associated endpoints. The *bend points* of each edge are also located at distinct integer grid points. No pair of polylines representing edges is permitted to cross except at common endpoints.

The *volume* of such a drawing is typically defined in terms of a smallest bounding box containing the drawing and with sides orthogonal to one of the coordinate axes. If such a box B has width w, length l and height h, then we refer to the *dimensions* of B as $(w+1) \times (l+1) \times (h+1)$ and define the volume of B as $(w+1) \cdot (l+1) \cdot (h+1)$.

The concept of *track drawing* has been used by several authors with slightly different definitions. Here we follow the notation of [7]. Let $G = (V, E)$ be an undirected graph. A *t-track assignment* of G consists of a partition of V into t sets $V_0, \ldots V_{t-1}$, called *tracks* and a total order $\leq_i$ for each set V_i. An *overlap* on track t_i occurs if there is an edge (u, w) and a vertex v with $u <_i v <_i w$. An *X-crossing* occurs if there are two edges (u, v) and (w, z) with $u, w \in V_i$, $v, z \in V_j$, $u <_i w$ and $z <_j v$. A t-track assignment with no overlaps and no X-crossing is called a *t-track layout*.

In a *subdivision* of a graph G, at least one edge (u, v) of G is replaced by a path $u, d_1, d_2, \ldots d_k, v$, with $k \geq 1$. The internal vertices on the path are called *division vertices*.

For a specific ordering of the vertices of a graph, a subset E' forms a *queue* if for each edge $(v_i, v_l) \in E'$ there is no edge $(v_j, v_k) \in E'$ with $i < j < k < l$. I.e. a FIFO invariant holds and no pair of edges *nest*. If, for a specific vertex ordering, the edges of G can be partitioned into q queues, then the partition is called a *queue layout* of G. The *queue number* of a graph is the minimum over all vertex orderings of the minimum cardinality queue layout. Determining the queue number of a graph is in general NP-Hard, however many properties of queue layouts are known – see [10] for an overview of relevant results.

2.1 Perfect Matching Layouts

The technique developed in section 3 places all vertices collinearly and with edges arranged to avoid intersec-

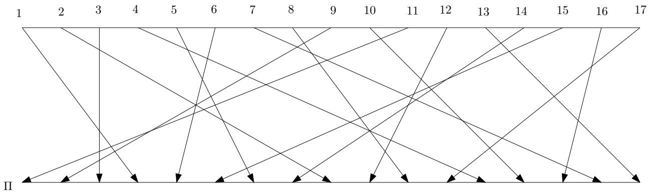

Figure 1: A perfect matching

tions. A critical device used in our construction involves a track layout of a (subdivision of a) perfect matching. See Fig. 1. We redraw the matching on a sequence of tracks to eliminate any X-crossings. Such a track layout can then be drawn in 3D without edge crossings. Our technique is similar in spirit to that used in circuit design, for example the radix-k butterfly layout for FFTs.

Lemma 1 *Let $m = 2^k$, where k is an integer. A perfect matching between two sets of m points can be drawn on $3k+1$ tracks and with at most $2k$ bends per edge and no X-crossings.*

Proof. Assume the perfect matching is defined by a permutation $\pi(i)$ for $0 \le i < m$. On each track there are potentially points numbered $0, 1, 2, ..., m-1$ There are $3k$ tracks.

Construction: Every point i on track 0 must be connected (via a polyline) to point $\pi(i)$ on track $3k$. Divide the points on track $3k$ in two equal sized intervals of size $m/2$, and into 4 equal sized intervals of size $m/4$, etc. If $\pi(i)$ is in the first/second interval of size $m/2$ of track $3k$, we place i on the first/second interval of size $m/2$ of track 3. If $\pi(i)$ is in the first/second/third/fourth interval of size $m/4$ of track $3k$, we place i on the first/second/third/fourth interval of size $m/4$ of track 6. In general if $\pi(i)$ is in the h-th interval of size $m/(2^j)$ of track $3k$, we place i on the h-th interval of size $m/(2^j)$ track $3j$.

The above construction yields a track layout of a subdivision of the given matching. Furthermore, the resulting track layout contains no X-crossings. The proof is by induction on k.

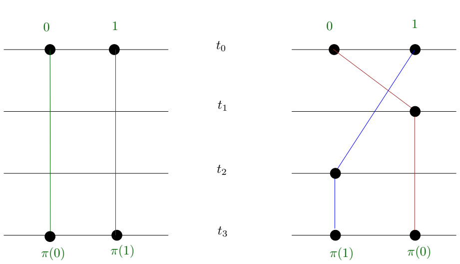

Figure 2: Base case – the two possible matches.

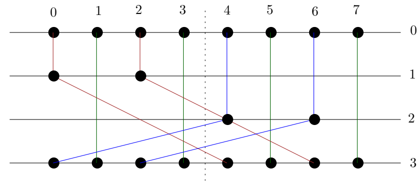

Figure 3: Inductive case. Bend-points shown in black. A simple 3D layout places tracks 0 and 3 in the plane $Z = 0$, track 1 in $Z = -1$, and track 2 in $Z = +1$ which thus eliminates crossings.

Base case, $k = 1$: See Fig. 2. If $\pi(0) = 0$, we connect points 0 and 1 on track 0 to points 0 and 1 on track 3. If $\pi(0) = 1$, we connect point 0 on track 0 via a point on track 1 to point $\pi(0)$ on track 3. We connect point 1 on track 0 via a point on track 2 to point $\pi(1)$ on track 3. Thus the potential X-crossing formed by 0, $\pi(0)$ and 1, $\pi(1)$ is removed.

Inductive case, $k > 1$: See Fig. 3. If ($i < m/2$ and $\pi(i) < m/2$) or ($i \ge m/2$ and $\pi(i) \ge m/2$), we connect point i on track 0 to point i on track 3. If ($i < m/2$ and $\pi(i) \ge m/2$) we connect point i on track 0 to point i on track 1. We connect the points in track 1 to the remaining points in the second half on track 3 so that the points have the same order on both tracks. If ($i \ge m/2$ and $\pi(i) < m/2$) we connect point i on track 0 to point i on track 2. We connect the points in track 2 to the remaining points in the first half on track 3 so that the points have the same order on both tracks. There are no crossings involved within each of these three cases, since the same order is maintained within each case. A crossing occurring between two different cases is not an X-crossing since each case involves different tracks – an intersection can only occur between a pair of edges involving two of tracks 0,3, tracks 1,3, and tracks 2,3. Tracks 1, 2 and 3 contain at most 2 subdivision vertices per edge which correspond to bend-points.

We can now recursively connect the 2^{k-1} points in the first half of track 3 to the first half of track $3k$, using the first halves of tracks $4, 5, 6, ..., 3k$ and at most $2(k-1)$ bends per edge. And we can recursively connect 2^{k-1} points in the second half of track 3 to the second half of track $3k$, using the second halves of tracks $4, 5, 6, ..., 3k$ and with at most $2(k-1)$ bends per edge. □

This matching construction will be used in section 3 but it is instructive to note a simple three-dimensional drawing with all points on three parallel planes. Since there are no X-crossings in the resulting layout, a 3D drawing is easily constructed by placing each track t_i as follows:

- if $i = 0 (\text{mod } 3)$ then track t_i is in the plane $Z = 0$
- if $i = 1 (\text{mod } 3)$ then track t_i is in the plane $Z = -1$
- if $i = 2 (\text{mod } 3)$ then track t_i is in the plane $Z = +1$

3 Three-Dimensional Drawing of a Given Graph

Given an arbitrary graph with n vertices and m edges, we outline a technique to draw the graph in three dimensions. As a first step, a track layout of (a subdivision of) the graph is produced, and subsequently a three dimensional drawing is produced.

Each vertex of the graph is placed on track t_0 with coordinates for v_i at $(i, 0)$. A matching is now created based on the edges. For each edge (i, j), $i < j$, a point is placed on track t_1 at consecutive integer x values. The order of the edges is lexicographic – all edges for vertex v_i come before those of v_{i+1}, and the point for edge (i, j) comes before $(i, j + 1)$. Thus there are m points on track t_1; each point representing edge (i, j) is joined by a line segment to v_i on track t_0. Label the points $1,...m$. Fig. 4 shows this preliminary step for K_6 but with a modified labeling.

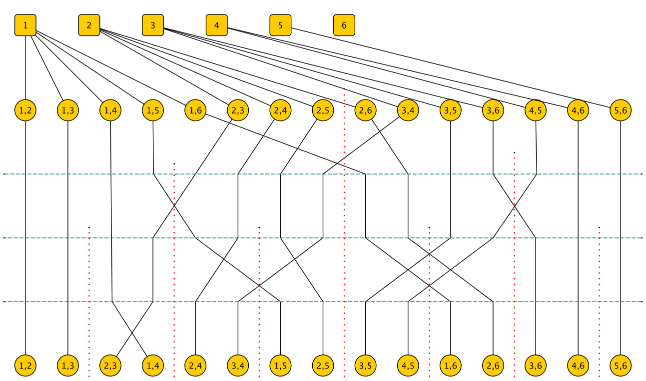

Figure 4: The matching for the 15 edges of K_6. The intervals are 8, 4, and 2. Vertices are labeled 1,2,...6 and the label for edge (i, j) is denoted as i, j. X-crossings in the matching are removed as in Lemma 2.1. Edges from the lowest track to the vertices are not displayed.

Similarly, on track t_*, m points are located as follows. For each edge (i, j) with $i < j$, points for v_j come before those of v_{j+1} and those from v_i are before those of v_{i+1}. The labels for the points on track t_* are determined from the associated edge point on track t_1; if edge (i, j) has label α on track t_1, then it maintains that label on track t_*. Each edge point on track t_* associated with an edge (i, j) is joined by a line segment to v_j on track t_0. The resulting perfect matching between the edge points on the two tracks t_1 and t_* can be processed as in the previous section. The track drawing thus constructed has $3 \log m + 2$ tracks and no X-crossings and no overlaps. The width of the drawing is $\max(n, m)$.

One method to convert the final track drawing into a three-dimensional drawing is to use the technique described in [7] which would automatically yield a drawing of volume $O((n + m) \log^3 m)$, however a more compact drawing can be achieved as follows.

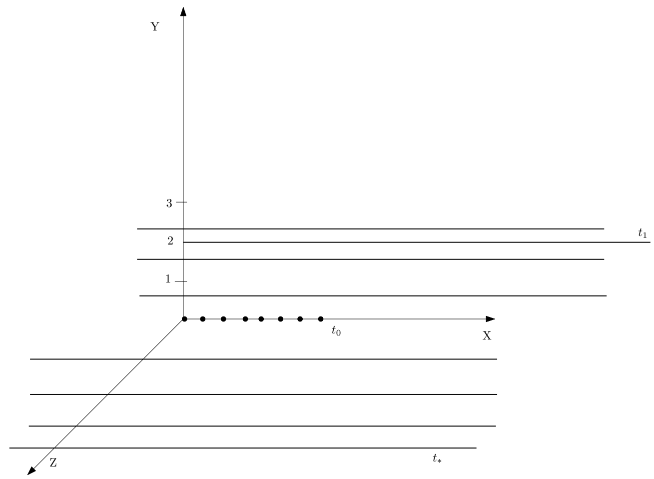

Figure 5: Sketch of track construction – only the first and last sets of tracks for the j groups are shown.

Place track t_0 along the x-axis $(y = 0, z = 0)$. Place track t_1 parallel to t_0 but at $y = 2, z = 0$ and then each group j of 3 tracks for $0 \leq j < \log m$ at: $y = 1, z = j+1$; $y = 3, z = j + 1$; $y = 2, z = j + 1$.

The track t_* is placed at $y = 1, z = \log m + 1$ and connected to each point on the final track which is at $y = 2, z = \log m$. Finally, each point on track t_* is joined to the associated vertex on track t_0. Fig. 5 sketches the layout of the tracks. Two views of K_6 drawn with this technique are provided in Figs. 6 and 7.

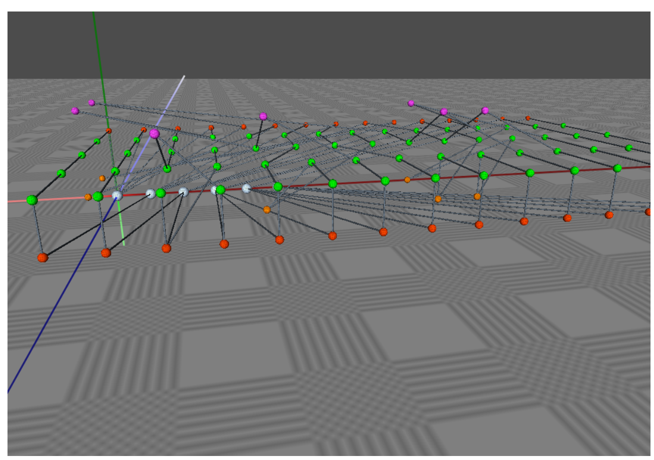

Figure 6: K_6 drawn in 3D.

The volume is $O((n+m) \log m)$. Each point produced in the perfect matching construction may contribute a bend point and there are at most $2 \log m$ such points. The construction outlined above can be summarized in the following theorem.

Theorem 2 *An arbitrary graph with n vertices and m*

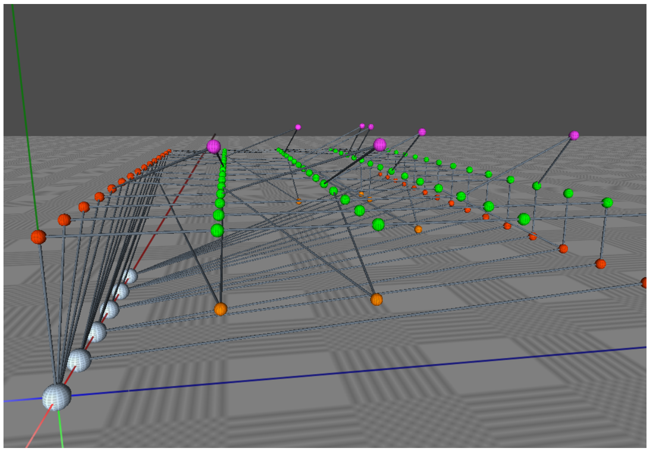

Figure 7: K_6 drawn in 3D. Vertices are white, bend-vertices on tracks t_0 and t_* are red.

edges can be drawn crossing-free in three dimensions in volume $O((m+n)\log m)$ with at most $O(\log m)$ bends per edge.

For planar graphs $m \leq 3n-6$. Hence the following corollary is immediate.

Corollary 3 *A planar graph with n vertices can be drawn crossing-free in three dimensions with a volume of $O(n \log n)$, and with at most $O(\log n)$ bends per edge.*

3.1 Modified Construction

We now modify the previous construction to obtain a drawing with asymptotically similar volume but slightly improved bend complexity. The modified drawing will be used in section 5.

The previous construction has dimensions $m \times 4 \times (1+\log m)$ and indeed is contained in an infinite wedge-shaped region defined by t_0, and the two half-planes through t_0,t_1 and t_0,t_*. We now partition the edges of G into groups of cardinality n, and draw each group in a separate wedge. Let $z = \lceil m/n \rceil$. Arbitrarily partition the edges of E into $E_1, \cdots, E_z$. Each set of edges E_i is drawn in a separate (infinite) wedge bounded by t_0, and the two half-planes containing t_0 and the tracks t_1^i at $y=1$, $z=i(1+\log n)$ and t_*^i at $y=1$, $z=i(1+\log n)+\log n$. The intersection of these wedges is exactly the track at t_0 containing the vertices of G. The volume of the resulting drawing is $O(n \cdot z \log n)$ which is $O(m \log n)$, and there are $O(\log n)$ bends. Note that these results match asymptotically those of Dujmović, and Wood [10], however our construction is independent of the vertex ordering, whereas their construction requires knowledge of a queue layout. In Section 5 we discuss this issue in more detail.

4 Three Dimensional Point-Set Embedding

Our first result is that the existence version of 3DPSE is always solvable if the volume of the drawing is unconstrained.

Theorem 4 *The complete graph K_n can be drawn crossing-free on any set of n distinct grid points in 3D, with at most 3 bends per edge.*

The proof that K_n can be drawn on any point set of cardinality n relies on the following lemma.

Lemma 5 *In any 3D grid drawing of a finite graph G, each point corresponding to a vertex is visible to a countably infinite number of integer grid points.*

Proof. Without loss of generality, consider a vertex point p at coordinates $(0,0,0)$. Consider the lines defined at $(1, y', *)$. For a particular value y', all the grid points on the line can only be blocked by a line segment(s) that lies in the plane formed by the line and p. To block all lines formed by integer values of y' would require an infinite number of line segment edges. □

A simple construction of K_n for any given point set P is now possible. Choose an arbitrary pair of vertex points that have not yet been joined in the drawing. For each point, determine a visible grid point that lies outside the bounding box of the current drawing and join each point to its visible neighbour. These 2 grid points can be joined by determining a mutually visible grid point and thus creating an edge consisting of 3 bends.

Clearly, if no bends are allowed, K_n may be undrawable on the specified point-set.

Since the 3DPSE problem remains open when the bends and volume are both constrained, it is natural to consider constraints on the point-set in this context. We now consider a three dimensional version of the point-set embeddability problem *with mapping.*

3DPSE$_p$: Given a graph G with n vertices, $V = \{v_1, v_2, \ldots v_n\}$, and given a set of n distinct points $P = \{p_1, p_2, \ldots p_n\}$ each with integer coordinates in the XY plane, can G be drawn crossing-free on P with v_i at p_i and with a polynomial number of bends (with integer coordinates) and in a polynomial volume?

Remark: Since our construction does not rely on properties of the graph, we assume the *with mapping* version of the problem. Our solution trivially solves the *without mapping* version of the problem by creating an arbitrary mapping.

Theorem 6 *Let G be an arbitrary graph with n vertices and m edges and let P be a set of n points each with integer coordinates in the XY plane in a bounding box of size $W \times H$, with $W \geq H$. Then G can be drawn crossing-free on P within a bounding box of dimensions $max(W,m) \times (H+3) \times (2+\log m)$ and with at most $O(\log m)$ bends per edge.*

Proof. Without loss of generality, we assume the points are labelled in order by X-coordinate, and then by Y-coordinate in the case of points with equal X-coordinate. We assume $X(p_0) = 0$ and $\min_i Y(p_i) = 0$. Then, $W = X(p_n)$ and $H = \max_i Y(p_i)$. See Fig. 8.

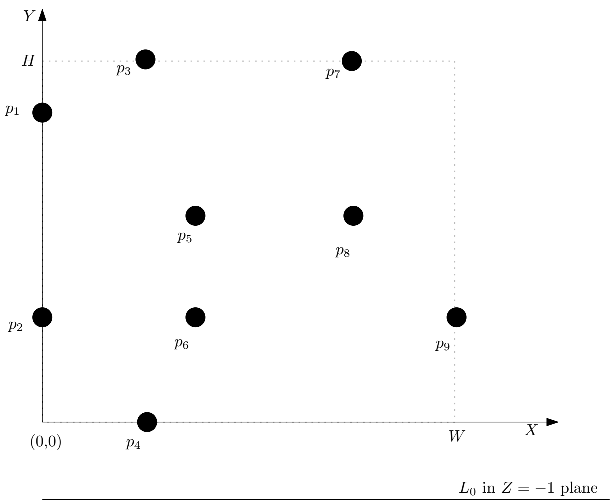

Figure 8: Points specified in the $Z = 0$ plane and projection of the line L_0

The drawing technique described in section 3 can now be almost directly applied, using two lines that host the required matching of the edge points. We construct a line L_0 from (0,-2,-1) to ($m-1$,-2,-1). The m edges of G are ordered lexicographically. That is, edge (v_i, v_j) precedes edge (v_i, v_{j+1}), and if $i < j$ then all edges from v_i precede those of v_j. More precisely, the following pseudocode specifies the ordering as drawn.

```
e:=0;
for i:=1 to n-1
  for j:= i+1 to n
    if (v_i, v_j) ∈ E(G) then
    { join p_i to (e,-2,-1);
    e++; }
```

The line L' from (0,-1, $\log m$) to $(m-1, -1, \log m)$ provides the matching, but for convenience a parallel line L'' is used from (0,0,1+$\log m$) to $(m-1, 0, 1+\log m)$ and joined to the vertex points as follows.

```
e:=0;
for j:=2 to n
  for i:= 1 to j-1
    if (v_i, v_j) ∈ E(G) then
    { join p_j to (e,0,1+log m);
    e++; }
```

Each point (i,0,1 + $\log m$) is joined to the corresponding point $(i, -1, \log m)$ on L'. The matching construction from L_0 to L' then lies between the planes $Y = -1$ and $Y = -3$.

We now prove there are no crossings in the graph as drawn. Consider the line segments between L_0 and the points P. A pair of segments that crossed would necessarily have two endpoints on L_0, one from each segment. Consider the set of all planes containing L_0 and intersecting a pair of points p_i and p_j. These two points must have the same Y-coordinate, in which case their edges cannot cross since all edges from p_i come strictly before any edges from p_j.

A similar argument holds for the segments between L'' and P. Finally, no segments in the matching intersect the segments in the previous two cases since they are separated in space by a plane. □

A simple consequence of our construction is that K_n can be drawn with the vertices located in an $\sqrt{n} \times \sqrt{n}$ 2D grid and with a volume of $m \cdot \sqrt{n} \cdot \log m$. Since $m = n(n-2)/2$, this volume is $O(n^{2.5} \log n)$.

Similarly, any planar graph can be drawn with vertices in an $\sqrt{n} \times \sqrt{n}$ 2D grid and with a volume of $O(n \log n)$. The aspect ratio of such a drawing is superior to previously published drawings, thus partially addressing the open problem presented in [8].

5 Appendix: Queues, 2-Bend Drawings, and 3DPSE$_l$

The modified approach to 3D drawing developed in section 3.1 can be improved in some situations. Consider a particular matching of the edge points from track t_1 to t_* as described in section 3. If these edges were partitioned into groups $E_1, \ldots, E_p$ such that within each group, no pair of edges crossed, then the matching step would be trivial and the wedge technique developed in section 3.1 could be adapted to create a 2-bend drawing in a bounding box of dimensions at most $m' \times 2 \times 2p$, where m' is the cardinality of the largest edge group.

The connection to queue layouts is explicitly stated in the following lemma.

Lemma 7 *For a particular vertex ordering, determining a partitioning into edge groups containing no crossings, is equivalent to determining a queue layout.*

Proof. The constraint for a valid queue in a set of edges is that no pair of edges nest: i.e. there is no pair of edges (v_i, v_l) and (v_j, v_k) with $i < j < k < l$. These are exactly the conditions for a crossing to appear in a matching of a subset of edges as created by our construction. □

Thus if a particular vertex ordering and optimum queue layout is known or computed, a 2-bend drawing can be produced that is of dimensions $m' \times 2 \times 2q$, where m' is the cardinality of the largest queue and q is the number of queues (i.e. the queue number).

Finally, we introduce a more constrained version of the 3DPSE problem.

3DPSE$_l$: Given a graph G with n vertices, $V = \{v_1, v_2, \ldots v_n\}$, and given a set of n distinct points $P = \{p_1, p_2, \ldots p_n\}$ each with integer coordinates, on a line, can G be drawn crossing-free on P with v_i at p_i and with a polynomial number of bends (with integer coordinates) and in a polynomial volume?

Note that for the *with mapping* variant of this problem, the construction of section 3 applies, whereas previous constructions, and the 2-bend construction outlined in this section, require a particular vertex ordering and thus address the *without mapping* variant since the ordering must be imposed on the point-set.

6 Conclusions and Open Problems

This paper presented a constructive technique to draw arbitrary graphs in three dimensions with low volume but with a non-constant number of bends. In particular for planar graphs the construction results in a volume of $O(n \log n)$ with $O(\log n)$ bends per edge. It remains an open problem to determine if planar graphs can be drawn in $O(n)$ volume with any number of bends.

Our solution to the 3DPSE$_p$ problem requires $O(\log m)$ bends per edge. Can the number of bends per edge be reduced to a constant while preserving a reasonable volume bound? The general 3D point-set embeddability problem in which the specified point-set is not constrained to a plane remains as an interesting open problem if the volume must be constrained.

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