Hidden Mobile Guards in Simple Polygons[*]

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Abstract

We consider guarding classes of simple polygons using mobile guards (polygon edges and diagonals) under the constraint that no two guards may see each other. In contrast to most other art gallery problems, existence is the primary question: does a specific type of polygon admit *some* guard set? Types include simple polygons and the subclasses of orthogonal, monotone, and starshaped polygons. Additionally, guards may either exclude or include the endpoints (so-called *open* and *closed* guards). We provide a nearly complete set of answers to existence questions of open and closed edge, diagonal, and mobile guards in simple, orthogonal, monotone, and starshaped polygons, with some surprising results. For instance, every monotone or starshaped polygon can be guarded using hidden open mobile (edge or diagonal) guards, but not necessarily with hidden open edge or hidden open diagonal guards.

1 Definitions

We define the *boundary* of a polygon P (denoted ∂P) as a simple polygonal chain consisting of a sequence of vertices specified in counterclockwise order, and the open set enclosed by ∂P to be the *interior* of P (denoted $\text{int}(P)$). An *edge* $e = \overline{pq}$ of the polygon is an interval of ∂P between consecutive vertices p, q, and a *diagonal* $d = \overline{rs}$ of P is a straight line segment between non-consecutive vertices r, s of ∂P such $d - \{r, s\} \in \text{int}(P)$, i.e. the portion of d excluding its endpoints lies in the interior of P.

We consider guarding $\text{int}(P)$ using a subset of the edges and diagonals of P. A guard g *sees* or *guards* a location l in the polygon if l is *weakly visible* [1] from the guard: there exists a point $p \in g$ such that the interior of the segment lp lies in the interior of the polygon. Edges and diagonals selected as guards are called *edge guards* and *diagonal guards*, respectively, and a *mobile guard* [9] is either an edge or a diagonal guard. If a set S of edges and diagonals of P is such that every location in the interior of P is seen by at least one guard in S, then S is a *guard set* of P and P is said to *admit* a guard set. A *closed guard set* includes the vertices at both ends of each edge or diagonal. If all endpoints are excluded, the guard set is called an *open guard set*.

In addition to *simple polygons* or simply *polygons*, we consider a number of special classes of polygons. An *orthogonal polygon* is a polygon that can be rotated such that all edges are parallel to the x- or y-axis. A *monotone polygon* is a polygon that can be rotated such that the portion of the polygon intersecting any vertical line consists of a connected interval. A *starshaped polygon* is a polygon that can be translated such that an interior point coincides with the origin and sees all locations in the interior of the polygon, and the *kernel* of the polygon is the set of all points in the polygon with this property. These three classes (along with convex and spiral polygons) are described by O'Rourke [10] in the context of guarding problems as being "usefully distinguished in the literature."

Finally, we add the constraint that a guard set is *hidden*: no pair of guards in the set see each other. Here a pair of guards g_1, g_2 in a polygon P can see each other if there exists a pair of points $p \in g_1, q \in g_2$ such that $pq - \{p, q\} \in \text{int}(P)$.

2 Introduction

Edge, diagonal, and mobile guards in polygons have been studied extensively in the past. Avis and Toussaint [1] considered the case where a single closed edge is sufficient to guard the entire polygon. Shortly after, Toussaint gave an example of a polygon whose smallest closed edge guard set is $\lfloor n/4 \rfloor$ [9] and conjectured that an edge guard set of this size is sufficient for any polygon. O'Rourke [9] showed that closed mobile guard sets of size $\lfloor n/4 \rfloor$ are sometimes necessary and always sufficient for polygons. For closed diagonal guards, Shermer [12] has shown that guard sets of size $\lfloor (2n+2)/7 \rfloor$ are necessary for some polygons, and no polygon requires a guard set of size greater than $\lfloor (n-1)/3 \rfloor$.

More recently, open edge guards were suggested by Viglietta [14] and studied by Benbernou et al. [2] and Tóth et al. [13], who showed that open edge guard sets

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of size $\lfloor n/3 \rfloor$ and $\lfloor n/2 \rfloor$ are sometimes necessary and always sufficient for simple polygons.

The study of hidden guards began with Shermer [11] who gave several results, including examples of polygons that are not guardable using hidden vertex guards. The study of hidden edges has only been initiated recently by Kranakis et al. [6] who showed that computing the largest hidden open edge set in a polygon (ignoring guarding) cannot be approximated within an arbitrarily small constant factor unless P = NP. In the same theme, Kosowoski et al. [7] have studied cooperative mobile guards, where each guard is *required* to be seen by another guard. Such a constraint is the opposite of hiddenness, which *forbids* any guard from seeing any other guard.

Here we evaluate the existence of hidden edge, diagonal, and mobile guard sets for simple polygon classes. A summary of results is seen in Table 1.

Guard class		Polygon class			
Inclusion	Type	Simple	Ortho	Mono	Star
Open	Edge	No	Yes	No	No
	Diagonal	No	No	No	No
	Mobile	No	Yes	Yes	Yes
Closed	Edge	No	No	No	No
	Diagonal	No	No	No	No
	Mobile	No	No	No	?

Table 1: New results in this paper. Entries indicate whether a hidden guard set exists for every polygon in the class.

3 Open edge guards

Recall open edge guards are edges of the polygon excluding the endpoints.

Lemma 1 *There exists a monotone polygon that does not admit a hidden open edge guard set.*

Proof. See Figure 1. We refer to the convex regions bounded by three edges in the upper left and right portions of the polygon as *ears*. Consider guarding the pair of ear regions without using any of the three edges that form each ear. The cases resulting from these attempts are seen in Figure 2. In each case, any maximal combination of non-ear edges fails to guard either ear completely. Moreover, a portion of the remaining unguarded region in each ear is not visible from any edge of the other ear. Thus any guard set contains one of the three edges in each ear. Also, every pair of ear edges in the same ear see each other, so any guard set contains exactly one edge in each ear.

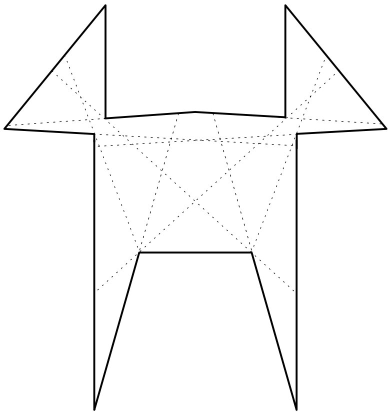

Figure 1: A monotone polygon that does not admit a hidden open edge guard set.

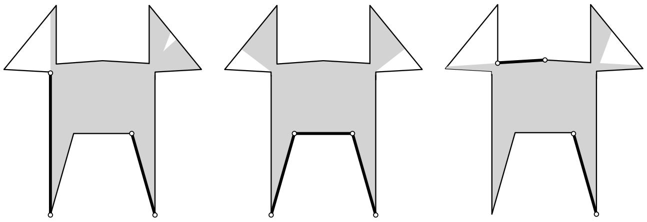

Figure 2: All maximal combinations of open edge guards that exclude the six ear edges.

Next, consider possible ear-edge pairs containing one edge from each ear. In Figure 3 it is shown that for each such ear-edge pair, the pair cannot be augmented to form a hidden open edge guard set for the polygon. Thus the polygon cannot be guarded with hidden open edge guards. ☐

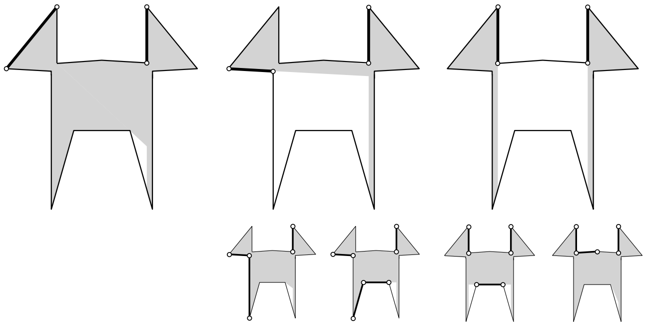

Figure 3: All combinations of ear edge pairs and the maximal hidden sets containing each ear edge pair.

Lemma 2 *There exists a starshaped polygon that does not admit a hidden open edge guard set.*

Proof. See Figure 4. The polygon consists of a central convex region with numerous spikes emanating from it. Figure 6 provides a labeled version of the polygon, with two sets of four large spikes each ($\{a_i\}$ and $\{b_i\}$) and

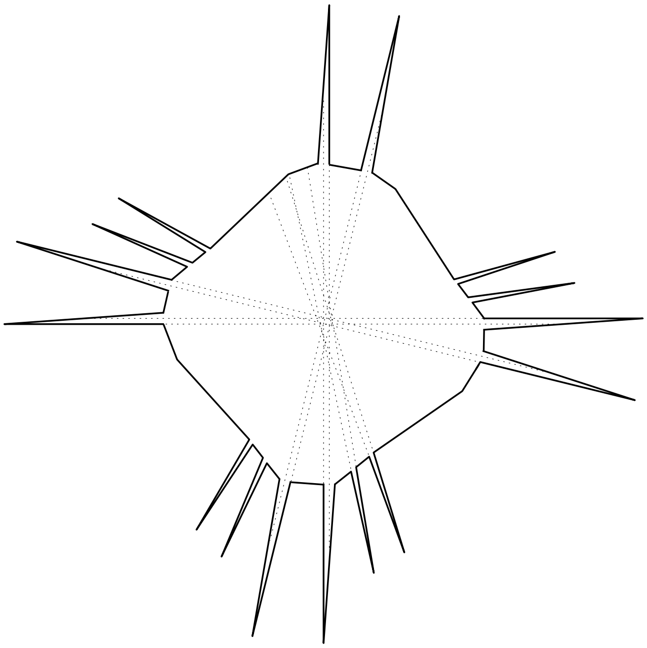

Figure 4: A starshaped polygon that does not admit a hidden open edge guard set.

four sets of two small spikes each ($\{c_1, c_2\}$ forms one such set). Call edges on the central convex region *central edges* and the spike pairs $\{a_1, a_3\}$, $\{a_2, a_4\}$, $\{b_1, b_3\}$, $\{b_2, b_4\}$ *opposing spike pairs.* Consider guarding the four spikes $\{a_i\}$ without using central edges (see Figure 5).

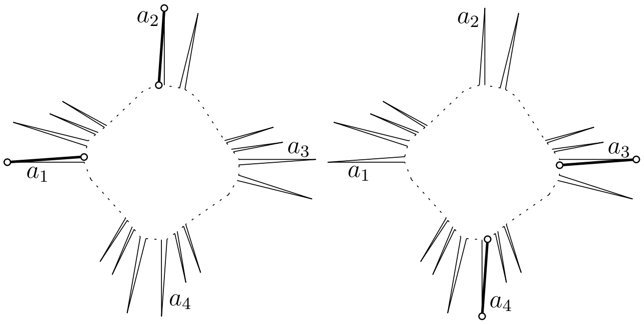

Figure 5: The two possible guardings of the four spike $\{a_i\}$ without using central edges (dotted).

Only one edge per opposing spike pair may be in any hidden edge guard set, as all four edges of an opposing spike pair see each other. Each spike has two asymmetric edges; one is able to guard the entire opposing spike pair, while the other is not. Each spike also contains a location not seem by any spike edge not in the spike's opposing spike pair. Finally, a pair of edges from a_1 and a_4 see each other, as do a pair in a_2 and a_3. So any hidden edge guard set for the opposing spike pairs $\{a_1, a_3\}$ and $\{a_2, a_4\}$ that does not include central edges consists of one of two pairs seen in Figure 5.

Now consider guarding the entire polygon. Any central edge guards the interior of at most one spike from $\{a_i\}$ or $\{b_i\}$. So one of the two spike sets $\{a_i\}$ and $\{b_i\}$ must be guarded without using central edges. Without loss of generality, assume the $\{a_i\}$ set is guarded in this way. Then one of the pairs of edges seen in Figure 5 must be in the guard set. Again without loss of generality, assume the edge pair of a_1 and a_2 are selected, as in Figure 6. Then there exist two spikes c_1 and c_2 whose edges are both seen by the guard edges in spikes a_1 and a_2, but portions of the interiors of c_1 and c_2 remain unguarded. The only edges sufficient to guard the interiors of c_1 and c_2 are the central edges e_1 and e_2. However, e_1 and e_2 each guard the interior of only one spike. Thus a portion of the interior of either c_1 or c_2 must remain unguarded, and the polygon cannot be guarded using hidden open edge guards. □

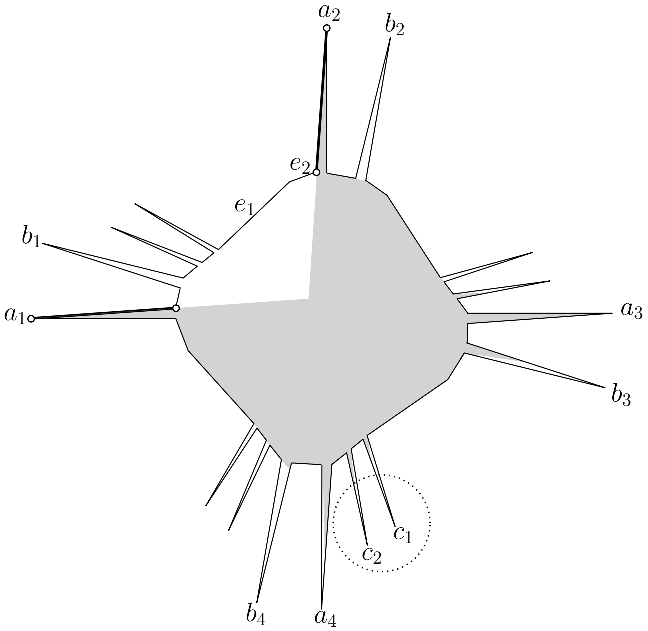

Figure 6: A incomplete but necessary set of guard edges and the region they guard. The interiors of c_1 and c_2 remain partially unguarded and cannot be guarded with a hidden open edge set.

Lemma 3 *Every orthogonal polygon admits a hidden open edge guard set.*

Omitted proofs can be found in the full version[1] of this paper.

4 Open diagonal guards

Lemma 4 *There exists a monotone and starshaped polygon that does not admit a hidden open diagonal guard set.*

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[1] `http://arxiv.org/pdf/1206.1803v1`

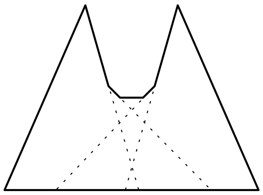

Figure 7: A monotone and starshaped polygon that does not admit a hidden open diagonal guard set.

Lemma 5 *There exists an orthogonal polygon that does not admit a hidden open diagonal guard set.*

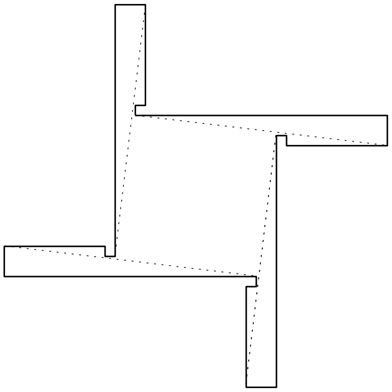

Figure 8: An orthogonal polygon that does not admit a hidden open diagonal guard set.

5 Open mobile guards

Lemma 6 *There exists a simple polygon that does not admit a hidden open mobile guard set.*

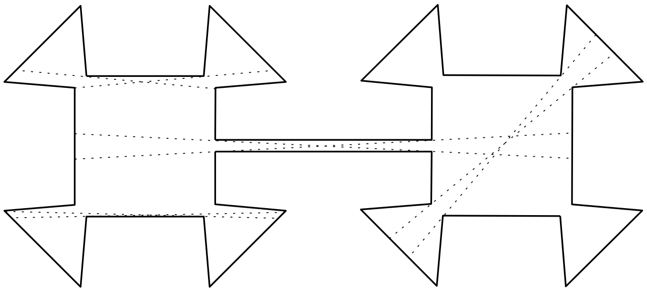

Figure 9: A simple polygon that does not admit a hidden open mobile guard set.

Observation 1 *Let g be a geodesic path between a pair of vertices p, q in a polygon P. Then the interiors of the edges g form a set of hidden open mobile guards in P.*

We refer to such a guard set for a path g as the *open mobile guard set induced by g.*

Lemma 7 *Every monotone polygon admits a hidden open mobile guard set.*

A natural approach to finding an open mobile guard set for a starshaped polygon is to look for a mobile guard that intersects the *kernel* of the polygon. Unfortunately such a guard may not exist as noted in [10] (see Figure 10).

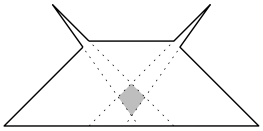

Figure 10: A starshaped polygon with no edge or diagonal intersecting its kernel (gray).

The following lemma is used in the proof of Lemma 9.

Lemma 8 *Let P be a starshaped polygon translated so that the origin lies in the kernel of P, and let v, v' be consecutive reflex vertices such that angle between the rays from v and v' through the origin (sweeping from v to v') is at most π. If a geodesic path $g \in P$ intersects both rays either before or after they intersect the origin, then g guards the subpolygon R bounded by the portions of the two rays before they intersect the origin, and the portion of ∂P from v to v'.*

Lemma 9 *Every starshaped polygon admits a hidden open mobile guard set.*

Proof. Let P be a given starshaped polygon translated so that the origin lies in the kernel of P. Consider shooting rays from each reflex vertex through the origin as seen in the left portion of Figure 11. Find a double wedge W formed by a consecutive pair of these rays such that each wedge is coincident to exactly one reflex vertex (which we call u and u') as seen in right portion of Figure 11 as a dark gray region. Such a double wedge is formed by every pair of consecutive intersections of rays along ∂P such that one intersection is the start of a ray (at a reflex vertex of P), and the other is the termination of a ray.

For every consecutive pair of reflex vertices v, v' on ∂P, the rays from v and v' through the origin lie entirely in $P - W$. Two pairs are an exception: the two pairs containing u and u' that form a pair of wedges, each containing half of the double wedge W (seen as the dark gray double wedge extended with two light gray wedges in the right portion of Figure 11). For all remaining pairs, the geodesic path from u to u' intersects both rays either before or after they have passed through the origin. Therefore, by Lemma 8, the hidden open mobile guard set induced by g sees the entire polygon except (possibly) the pair of wedges bounded by two pairs of consecutive reflex vertices adjacent to u and u'.

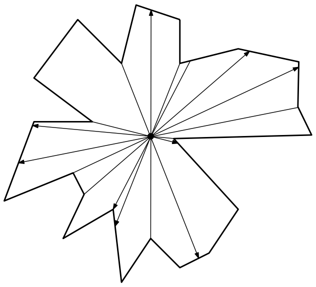

Figure 11: Left: a starshaped polygon with rays from each reflex vertex through the origin. Right: the polygon and a double wedge W (dark gray) with one reflex vertex (u or u') incident to each wedge. The light and dark gray regions together form the subpolygons possibly left unguarded by the hidden open mobile guard set induced by a geodesic path from u to u'.

It may be the case that the two remaining wedges are actually a single non-convex subpolygon with reflex vertex at the origin (see Figure 12).

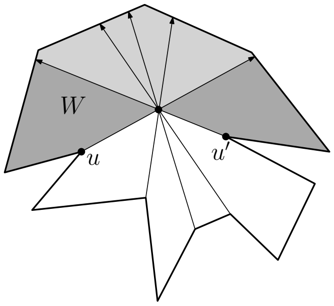

Figure 12: A polygon and double wedge W (dark gray region) where the region not necessarily guarded by the hidden open mobile guard set induced by the geodesic path from u to u' is actually a single non-convex polygon (light and dark gray regions combined) bounded by u and u'.

In this situation the subpolygon can be bisected into two convex subpolygons by a ray bisecting the reflex angle at the origin.

Recall that each convex subpolygon has a vertex u or u' in common with the geodesic's final edge (see Figure 13). If the interior angle formed by these two edges is at most π, then the subpolygon is seen by the interior of the final edge of the geodesic. If not, the geodesic can be extended to include an edge of ∂P in the subpolygon that guards the subpolygon completely.

Thus the hidden mobile guard set induced by the geodesic described guards P. □

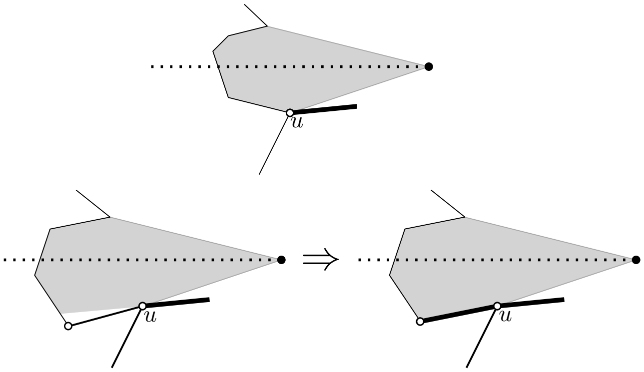

Figure 13: The two cases of guarding the remaining subpolygons. In the case shown in the upper part of the figure, the existing geodesic is sufficient to guard the wedge. In the second case, the geodesic leaves a portion of the wedge unguarded and must be extended.

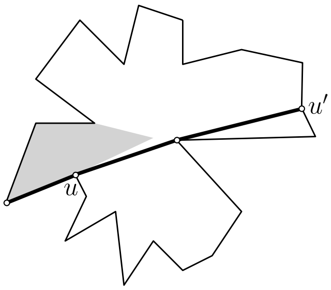

Figure 14: A polygon with a geodesic path inducing a hidden open mobile guard set for the polygon. The initial geodesic from u to u' leaves the gray region incident to u partially unguarded, so the geodesic is extended by one edge.

Computing such a guard set for a polygon with n edges can be done in $O(n)$ time, as each step takes at most $O(n)$ time: 1. compute a point in the kernel of the polygon ($O(n)$ time by Lee and Preparata [8]). 2. find a separating angle θ ($O(n)$ time). 3. triangulate the polygon and find a geodesic between the reflex vertices u and u' ($O(n)$ time by Fournier and Montuno [3] and Guibas et al. [5]). 4. check whether the two remaining subpolygons are already covered by the geodesic, and extend the geodesic by an additional edge if necessary ($O(1)$ time).

6 Closed edge and diagonal guards

In the next section we present orthogonal and monotone polygons that do not admit hidden closed mobile guard sets. Note that these polygons also serve as examples of polygons that do not admit hidden closed edge or hidden closed diagonal guards. For starshaped polygons no such example is known.

Lemma 10 *There exists a starshaped polygon that does not admit a hidden closed edge guard set.*

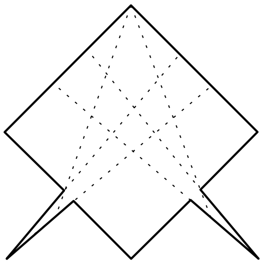

Figure 15: A starshaped polygon that does not admit a hidden closed edge guard set.

Lemma 11 *There exists a starshaped polygon polygon that does not admit a hidden closed diagonal guard set.*

7 Closed mobile guards

Lemma 12 *There exists an orthgonal polygon that does not admit a hidden closed mobile guard set.*

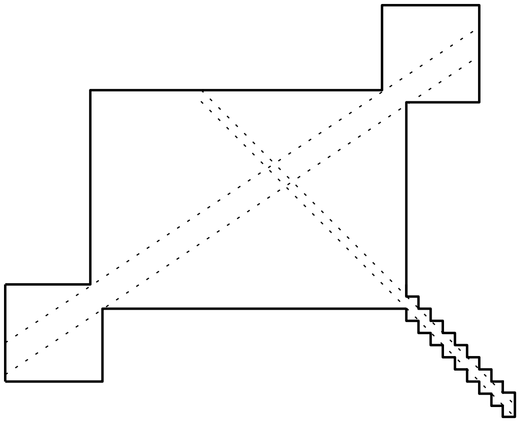

Figure 16: An orthogonal polygon that cannot be guarded using hidden closed mobile guards.

Lemma 13 *There exists a monotone polygon that does not admit a hidden closed mobile guard set.*

Conjecture 1 *Every starshaped polygon admits a hidden closed mobile guard set.*

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