ON CERTAIN ARITHMETICAL FUNCTIONS INVOLVING EXPONENTIAL DIVISORS II.

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Dedicated to the memory of Professor M.V. Subbarao

1. Introduction

Let $n > 1$ be an integer of canonical form $n = p_1^{a_1} \cdots p_r^{a_r}$. The integer d is called an exponential divisor (e-divisor) of n if $d = p_1^{b_1} \cdots p_r^{b_r}$, where $b_1 \mid a_1$, ..., $b_r \mid a_r$, notation: $d \mid_e n$. By convention $1 \mid_e 1$. The integer $n > 1$ is called exponentially squarefree (e-squarefree) if all the exponents $a_1, ..., a_r$ are squarefree. The integer 1 is also considered to be e-squarefree.

The exponential convolution (e-convolution) of arithmetic functions is defined by

$$(f \odot g)(n) = \sum_{b_1 c_1 = a_1} \cdots \sum_{b_r c_r = a_r} f(p_1^{b_1} \cdots p_r^{b_r}) g(p_1^{c_1} \cdots p_r^{c_r}),$$

where $n = p_1^{a_1} \cdots p_r^{a_r}$.

These notions were introduced by M.V. Subbarao [8]. The e-convolution $\odot$ is commutative, associative and has the identity element μ^2, where μ is the Möbius function. Furthermore, a function f has an inverse with respect to $\odot$ iff $f(1) \neq 0$ and $f(p_1 \cdots p_s) \neq 0$ for any distinct primes $p_1, ..., p_s$.

The inverse with respect to $\odot$ of the constant 1 function is called the exponential analogue of the Möbius function and it is denoted by $\mu^{(e)}$. Hence for every $n \geq 1$,

$$\sum_{d \mid_e n} \mu^{(e)}(d) = \mu^2(n).$$

Here $\mu^{(e)}(1) = 1$ and for $n = p_1^{a_1} \cdots p_r^{a_r} > 1$,

$$\mu^{(e)}(n) = \mu(a_1) \cdots \mu(a_r).$$

Observe that $|\mu^{(e)}(n)| = 1$ or 0, according as n is e-squarefree or not. For properties and generalizations of the e-convolution see [8], [3].

Other arithmetic functions regarding e-divisors, for example the number and the sum of e-divisors of n, were investigated by several authors, see the references given in the first part [11] of the present paper, devoted to the study of functions involving the greatest common exponential divisor of integers.

An asymptotic formula for $\sum_{n \leq x} |\mu^{(e)}(n)|$ was established by M.V. Subbarao [8], improved by J. Wu [14], see also Part I of the present paper. We show that the corresponding error term can further be improved on the assumption of the Riemann hypothesis (RH), see Theorem 3.

In Theorem 2 we give a formula for $\sum_{n \leq x} \mu^{(e)}(n)$ without and with assuming RH. As far as we know there is no such result in the literature. We show that the error terms depend on estimates for the number of squarefree integers $\leq x$.

Consider now the exponential squarefree exponential divisors (e-squarefree e-divisors) of n. Here $d = p_1^{b_1} \cdots p_r^{a_r}$ is an e-squarefree e-divisor of $n = = p_1^{a_1} \cdots p_r^{a_r} > 1$, if $b_1 \mid a_1, ..., b_r \mid a_r$ and $b_1, ..., b_r$ are squarefree. Note that the integer 1 is e-squarefree and it is not an e-divisor of $n > 1$.

We introduce the functions $t^{(e)}$ and $\kappa^{(e)}$, where $t^{(e)}(n)$ and $\kappa^{(e)}(n)$ denote the number of e-squarefree e-divisors of n and the maximal e-squarefree e-divisor of n, respectively. These are the exponential analogues of the functions representing the number of squarefree divisors of n (i.e. $\theta(n) = 2^{\omega(n)}$, where $\omega(n) = r$) and the maximal squarefree divisor of n (the squarefree kernel $\kappa(n) = = \prod_{p|n} p$), respectively.

The functions $t^{(e)}$ and $\kappa^{(e)}$ are multiplicative and for $n = p_1^{a_1} \cdots p_r^{a_r} > 1$,

$$t^{(e)}(n) = 2^{\omega(a_1)} \cdots 2^{\omega(a_r)},$$

$$\kappa^{(e)}(n) = p_1^{\kappa(a_1)} \cdots p_r^{\kappa(a_r)}.$$

Asymptotic properties of the functions $t^{(e)}(n)$ and $\kappa^{(e)}(n)$ are given in Theorems 4, 5 and 7.

2. Results

The function $\mu^{(e)}$ is multiplicative and $\mu^{(e)}(p^a) = \mu(a)$ for every prime power p^a. Hence $\mu^{(e)}(n) \in \{-1, 0, 1\}$ for every $n \geq 1$ and for every prime p,

$\mu^{(e)}(p) = 1$, $\mu^{(e)}(p^2) = -1$, $\mu^{(e)}(p^3) = -1$, $\mu^{(e)}(p^4) = 0$,... .

According to a well-known result of H. Delange, cf. [1] Ch. 6, the function $\mu^{(e)}$ has a non-zero mean value given by

$$m(\mu^{(e)}) = \prod_p \left(1 + \sum_{a=2}^{\infty} \frac{\mu(a) - \mu(a-1)}{p^a} \right).$$

An asymptotic formula for $\mu^{(e)}$ can be obtained from the following general result, which may be known.

Theorem 1. *Let f be a complex valued multiplicative function such that $|f(n)| \leq 1$ for every $n \geq 1$ and $f(p) = 1$ for every prime p. Then*

$$\sum_{n \leq x} f(n) = m(f)x + O(x^{1/2} \log x),$$

where

$$m(f) = \prod_p \left(1 + \sum_{a=2}^{\infty} \frac{f(p^a) - f(p^{a-1})}{p^a} \right)$$

is the mean value of f.

Theorem 1 applies also for the multiplicative functions $f = \mu^{*(e)}$ and $f = F$, where $\mu^{*(e)}(p^a) = \mu^*(a) = (-1)^{\omega(a)}$ representing the unitary exponential Möbius function, cf. [3], and $F(p^a) = \lambda(a) = (-1)^{\Omega(a)}$ the Liouville function, with $\Omega(a)$ denoting the number of prime power divisors of a.

We prove for $\mu^{(e)}$ the following more precise result.

Theorem 2. *(i) The Dirichlet series of $\mu^{(e)}$ is of form*

$$\sum_{n=1}^{\infty} \frac{\mu^{(e)}(n)}{n^s} = \frac{\zeta(s)}{\zeta^2(2s)} U(s), \quad \text{Re } s > 1,$$

where $U(s) := \sum_{n=1}^{\infty} \frac{u(n)}{n^s}$ is absolutely convergent for Re $s > 1/5$.

(ii)

$$\sum_{n \leq x} \mu^{(e)}(n) = m(\mu^{(e)})x + O(x^{1/2} \exp(-c(\log x)^{\Delta})),$$

where $\Delta < 9/25 = 0,36$ and $c > 0$ are constants.

(iii) Assume RH. Let $1/4 < r < 1/3$ be an exponent such that $D(x) :=$
$:= \sum_{n \leq x} \mu^2(n) - x/\zeta(2) = O(x^{r+\varepsilon})$ *for every $\varepsilon > 0$. Then the error term in (ii)*
is $O(x^{(2-r)/(5-4r)+\varepsilon})$ for every $\varepsilon > 0$.

The best known value – to our knowledge – of r is $r = 17/54 \approx 0,314814$, obtained in [2], therefore the error term in (ii), assuming RH, is $O(x^{91/202+\varepsilon})$ for every $\varepsilon > 0$, where $91/202 \approx 0,450495$.

Theorem 3. *If RH is true, then*

$$\sum_{n \leq x} |\mu^{(e)}(n)| = \prod_p \left(1 + \sum_{a=4}^{\infty} \frac{\mu^2(a) - \mu^2(a-1)}{p^a} \right) x + O(x^{1/5+\varepsilon})$$

for every $\varepsilon > 0$.

The function $t^{(e)}$ is multiplicative and $t^{(e)}(p^a) = 2^{\omega(a)}$ for every prime power p^a. Here for every prime p, $t^{(e)}(p) = 1$, $t^{(e)}(p^2) = t^{(e)}(p^3) = t^{(e)}(p^4) = $ $= t^{(e)}(p^5) = 2, t^{(e)}(p^6) = 4, \dots$.

Theorem 4. *(i) The Dirichlet series of $t^{(e)}$ is of form*

$$\sum_{n=1}^{\infty} \frac{t^{(e)}(n)}{n^s} = \zeta(s)\zeta(2s)V(s), \quad \text{Re } s > 1,$$

where $V(s) = \sum_{n=1}^{\infty} \frac{v(n)}{n^s}$ is absolutely convergent for $\text{Re } s > 1/4$.

(ii)
$$\sum_{n \leq x} t^{(e)}(n) = C_1 x + C_2 x^{1/2} + O(x^{1/4+\varepsilon})$$

for every $\varepsilon > 0$, where C_1, C_2 are constants given by

$$C_1 := \prod_p \left(1 + \sum_{a=6}^{\infty} \frac{2^{\omega(a)} - 2^{\omega(a-1)}}{p^a} \right),$$

$$C_2 := \zeta(1/2) \prod_p \left(1 + \sum_{a=4}^{\infty} \frac{2^{\omega(a)} - 2^{\omega(a-1)} - 2^{\omega(a-2)} + 2^{\omega(a-4)}}{p^{a/2}} \right).$$

Theorem 5.

$$\limsup_{n \to \infty} \frac{\log t^{(e)}(n) \log \log n}{\log n} = \frac{1}{2} \log 2.$$

The function $\kappa^{(e)}$ is multiplicative and $\kappa^{(e)}(p^a) = p^{\kappa(a)}$ for every prime power p^a. Hence for every prime p, $\kappa^{(e)}(p) = p$, $\kappa^{(e)}(p^2) = p^2$, $\kappa^{(e)}(p^3) = p^3$, $\kappa^{(e)}(p^4) = p^2,\dots$.

To obtain an asymptotic formula for $\kappa^{(e)}$ we use the following general theorem, of which parts (i) and (ii) are a variant of a result given in [6] and cited in the first part [11] of this paper.

Theorem 6. *Let $k \geq 2$ be a fixed integer and f be a complex valued multiplicative arithmetic function satisfying*

(a) $f(p) = f(p^2) = \dots = f(p^{k-1}) = 1$ for every prime p,

(b) there exists $K > 0$ such that $|f(p^a)| \leq K$ for every prime power p^a with $a \geq k + 1$,

(c) there exist $M > 0$ and $\beta \geq 1/(k+1)$ such that $|f(p^k)| \leq Mp^{-\beta}$ for every prime p.

Then

(i)

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \frac{\zeta(s)}{\zeta(ks)} W(s), \quad \text{Re } s > 1,$$

where the Dirichlet series $W(s) := \sum_{n=1}^{\infty} \frac{w(n)}{n^s}$ is absolutely convergent for $\text{Re } s > 1/(k+1)$.

(ii)

$$\sum_{n \leq x} f(n) = C_f x + O(x^{1/k} \delta(x)),$$

where

$$C_f := \prod_p \left(1 + \sum_{a=k}^{\infty} \frac{f(p^a) - f(p^{a-1})}{p^a} \right)$$

and

$$\delta(x) = \delta_A(x) := \exp(-A(\log x)^{3/5} (\log \log x)^{-1/5}),$$

A being a positive constant.

(iii) If RH is true, then the error term is $O(x^{1/(k+1)+\varepsilon})$ for every $\varepsilon > 0$.

Theorem 7.

$$\sum_{n \leq x} \kappa^{(e)}(n) = \frac{1}{2} \prod_p \left(1 + \sum_{a=4}^{\infty} \frac{p^{\kappa(a)} - p^{1+\kappa(a-1)}}{p^a} \right) x^2 + O(x^{5/4} \delta(x)).$$

If RH is true, then the error term is $O(x^{6/5+\varepsilon})$ for every $\varepsilon > 0$.

3. Proofs

Proof of Theorem 1. Let $g = f * \mu$ in terms of the Dirichlet convolution. Then g is multiplicative, $g(p) = f(p) - 1 = 0$, $g(p^a) = f(p^a) - f(p^{a-1})$ and $|g(p^a)| \le |f(p^a)| + |f(p^{a-1})| \le 2$ for every prime p and every $a \ge 2$. Therefore $|g(n)| \le \ell(n)2^{\omega(n)}$ for every $n \ge 1$, where $\ell(n)$ is the characteristic function of the squarefull integers and we have

$$\sum_{n \le x} f(n) = \sum_{de \le x} g(d) = \sum_{d \le x} g(d)\left(\frac{x}{d} + O(1)\right) = x\sum_{d \le x}\frac{g(d)}{d} + O\left(\sum_{d \le x}|g(d)|\right) =$$

$$= x\sum_{d=1}^{\infty}\frac{g(d)}{d} + O\left(x\sum_{d > x}\frac{\ell(d)2^{\omega(d)}}{d}\right) + O\left(\sum_{d \le x}\ell(d)2^{\omega(d)}\right).$$

Here

$$\ell(n)2^{\omega(n)} = \sum_{d^2 e = n}\tau(d)h(e),$$

where τ is the divisor function and h is given by

$$\sum_{n=1}^{\infty}\frac{h(n)}{n^s} = \prod_{p}\left(1 + \frac{2}{p^{3s}} - \frac{1}{p^{4s}} - \frac{2}{p^{5s}}\right),$$

absolutely convergent for Re $s > 1/3$, cf. [7]. We obtain

$$\sum_{n \le x}\ell(n)2^{\omega(n)} = \sum_{e \le x}h(e)\sum_{d \le (x/e)^{1/2}}\tau(d) = \sum_{e \le x}h(e)\, O\left((x/e)^{1/2}\log(x/e)\right) =$$

$$= O\left(x^{1/2}\log x\sum_{e \le x}|h(e)|e^{-1/2}\right) = O\left(x^{1/2}\log x\right),$$

and by partial summation,

$$\sum_{n > x}\frac{\ell(n)2^{\omega(n)}}{n} = O\left(x^{-1/2}\log x\right),$$

which finishes the proof.

Proof of Theorem 2. (i) Let $\mu_2(n) = \mu(m)$ or 0, according as $n = m^2$ or not, and let $E_2(n) = m^2$ or 0, according as $n = m^2$ or not. The given equality is verified for $\mu^{(e)} = \mu^2 * \mu_2 * u$, equivalent to $u = \mu^{(e)} * \lambda * E_2$, in terms of the Dirichlet convolution, where λ is the Liouville function. It is easy to check that $u(p) = u(p^2) = u(p^3) = u(p^4) = 0$, $|(\lambda * E_2)(p^a)| \le a$ for every prime power p^a with $a \ge 1$, hence $|u(p^b)| \le 1 + \sum_{a=1}^{b} |(\lambda * E_2)(p^a)| < b^2$ for every prime power p^b with $b \ge 5$. We obtain that the Dirichlet series of the function u is absolutely convergent for Re $s > 1/5$.

(ii) According to (i), $\sum_{n \le x} \mu^{(e)}(n) = \sum_{n \le x} u(n) S(x/n)$, where

$$S(x) := \sum_{nd^2 \le x} \mu^2(n)\mu(d).$$

We first estimate the sum $S(x)$. Let $\varrho = \varrho(x)$ such that $0 < \varrho < 1$ to be defined later. If $nd^2 \le x$, then both $n > \varrho^{-2}$ and $d > \varrho\sqrt{x}$ can not hold good in the same time, therefore

$$S(x) = \sum_{\substack{nd^2 \le x \\ d \le \varrho\sqrt{x}}} \mu^2(n)\mu(d) + \sum_{\substack{nd^2 \le x \\ n \le \varrho^{-2}}} \mu^2(n)\mu(d) - \sum_{\substack{d \le \varrho\sqrt{x} \\ n \le \varrho^{-2}}} \mu^2(n)\mu(d) =$$

$$= S_1(x) + S_2(x) - S_3(x),$$

say. We use the following estimates of A. Walfisz [13]:

$$M(x) := \sum_{n \le x} \mu(n) = O(x\delta(x)), \quad E(x) := \sum_{n \le x} \mu^2(n) = \frac{x}{\zeta(2)} + O(x^{1/2}\delta(x)).$$

Note that $\delta(x)$, defined in Section 2, is decreasing and $x^\varepsilon \delta(x)$ is increasing for every $\varepsilon > 0$. By partial summation,

$$R(x) := \sum_{n > x} \frac{\mu(n)}{n^2} = O(x^{-1}\delta(x)).$$

Here

$$S_1(x) = \sum_{d \le \varrho\sqrt{x}} \mu(d) E(x/d^2) = \frac{x}{\zeta(2)} \sum_{d \le \varrho\sqrt{x}} \frac{\mu(d)}{d^2} + O\left(x^{1/2} \sum_{d \le \varrho\sqrt{x}} \frac{\delta(x/d^2)}{d}\right) =$$

$$= \frac{x}{\zeta(2)} \left(\frac{1}{\zeta(2)} - R(\varrho\sqrt{x}) \right) + O \left(x^{1/2}\delta(\varrho^{-2}) \sum_{d \le \sqrt{x}} \frac{1}{d} \right) =$$

$$= \frac{x}{\zeta^2(2)} + O \left(\varrho^{-1}x^{1/2}\delta(\varrho\sqrt{x}) \right) + O \left(x^{1/2}\delta(\varrho^{-2}) \log x \right),$$

$$S_2(x) = \sum_{n \le \varrho^{-2}} \mu^2(n)M((x/n)^{1/2}) = O \left(\sum_{n \le \varrho^{-2}} (x/n)^{1/2}\delta((x/n)^{1/2}) \right) =$$

$$= O \left(\delta(\varrho\sqrt{x})x^{1/2} \sum_{n \le \varrho^{-2}} \frac{1}{\sqrt{n}} \right) = O \left(\varrho^{-1}x^{1/2}\delta(\varrho\sqrt{x}) \right),$$

$$S_3(x) = M(\varrho\sqrt{x})E(\varrho^{-2}) = O \left(\varrho^{-1}x^{1/2}\delta(\varrho\sqrt{x}) \right).$$

We obtain that

$$S(x) = \frac{x}{\zeta^2(2)} + O \left(\varrho^{-1}x^{1/2}\delta(\varrho\sqrt{x}) \right) + O \left(x^{1/2}\delta(1/\varrho^2) \log x \right).$$

Take $\varrho = \exp(-(\log x)^\beta)$, where $0 < \beta < 1$. Then

$$\varrho\sqrt{x} = \exp \left(\frac{1}{2}(\log x) - (\log x)^\beta \right) \ge \exp \left(\frac{1}{4}(\log x) \right) = x^{1/4}$$

for sufficiently large x. Hence $\delta(\varrho\sqrt{x}) \le \delta(x^{1/4}) \ll \delta_B(x)$ with a suitable constant $B > 0$. For $\beta < 3/5$ we obtain $\varrho^{-1}\delta(\varrho\sqrt{x}) \ll \exp((\log x)^\beta - B(\log x)^{3/5}(\log \log x)^{-1/5}) \ll \delta_C(x)$ with a suitable constant $C > 0$.

If $\eta < 3/5$, then $\delta_A(x) \ll \exp(-A(\log x)^\eta)$ and obtain that $\delta(\varrho^{-2}) \ll \exp(-A(2(\log x)^\beta)^\eta) = \exp(-D(\log x)^{\beta\eta})$ with a suitable $D > 0$, where $\beta\eta < 9/25$.

Therefore,

$$S(x) = \frac{x}{\zeta^2(2)} + O \left(x^{1/2}\exp(-c(\log x)^\Delta) \right),$$

where $\Delta < 9/25$ and $c > 0$ are constants. Now,

$$\sum_{n \le x} \mu^{(e)}(n) = \sum_{n \le x} u(n)S(x/n) =$$

$$= \sum_{n \leq x} u(n) \left(\frac{x}{\zeta^2(2)n} + O\left((x/n)^{1/2} \exp(-c(\log(x/n))^\Delta) \right) \right) =$$

$$= \frac{x}{\zeta^2(2)} \sum_{n \leq x} \frac{u(n)}{n} + O\left(x^{1/2} \sum_{n \leq x} \frac{|u(n)|}{n^{1/2}} \exp(-c(\log(x/n))^\Delta) \right),$$

where, using that $x^\varepsilon \exp(-c(\log x)^\Delta)$ is increasing for any $\varepsilon > 0$, the O-term is

$$O\left(x^{1/2} \sum_{n \leq x} \frac{|u(n)|}{n^{1/2}} \left(\frac{x}{n} \right)^{-\varepsilon} \left(\frac{x}{n} \right)^\varepsilon \exp(-c(\log(x/n)^\Delta)) \right) =$$

$$= O\left(x^{1/2} x^\varepsilon \exp(-c(\log x)^\Delta) x^{-\varepsilon} \sum_{d \leq x} \frac{|u(n)|}{n^{1/2-\varepsilon}} \right) =$$

$$= O\left(x^{1/2} \exp(-c(\log x)^\Delta) \right)$$

for $1/2 - \varepsilon > 1/5$. Furthermore,

$$\sum_{n \leq x} \frac{u(n)}{n} = U(1) + O\left(\sum_{n > x} \frac{u(n)}{n} \right),$$

with $U(1) = \zeta^{-2}(2)m(\mu^{(e)})$ and $\sum_{n > x} \frac{u(n)}{n} = O\left(x^{-3/5} \sum_{n > x} \frac{u(n)}{n^{2/5}} \right) = O(x^{-3/5})$,

which finishes the proof of (ii).

(iii) Assume RH. We use that, see [10],

$$M(x) := \sum_{n \leq x} \mu(n) = O\left(x^{1/2} \omega(x) \right),$$

where $\omega(x) := \exp(A(\log x)(\log \log x)^{-1})$, A being a positive constant, which gives by partial summation,

$$R(x) := \sum_{n > x} \frac{\mu(n)}{n^2} = O(x^{-3/2} \omega(x)).$$

Suppose that $D(x) := \sum_{n \leq x} \mu^2(n) - x/\zeta(2) = O(x^{r+\varepsilon})$ for every $\varepsilon > 0$, where $1/4 < r < 1/3$. Then we obtain by similar computations that

$$S_1(x) = \frac{x}{\zeta^2(2)} + O\left(\varrho^{-3/2} x^{1/4} \omega(\varrho\sqrt{x}) \right) + O\left(x^{1/2} \varrho^{1-2(r+\varepsilon)} \right),$$

$$S_2(x) = O\left(\varrho^{-3/2}x^{1/4}\omega(\varrho\sqrt{x})\right), \qquad S_3(x) = O\left(\varrho^{-3/2}x^{1/4}\omega(\varrho\sqrt{x})\right).$$

Therefore

$$S(x) = \frac{x}{\zeta^2(2)} + O\left(\varrho^{-3/2}x^{1/4}\omega(\varrho\sqrt{x})\right) + O\left(x^{1/2}\varrho^{1-2(r+\varepsilon)}\right).$$

Choose $\varrho = x^{-t}$, $t > 0$. Then $\varrho^{-3/2}x^{1/4} = x^{(6t+1)/4}$, $\varrho\sqrt{x} = x^{1/2-t} < x$, hence $\omega(\varrho\sqrt{x}) < \omega(x) \ll x^\varepsilon$ for every $\varepsilon > 0$ and obtain

$$S(x) = \frac{x}{\zeta^2(2)} + O\left(x^{(6t+1)/4+\varepsilon}\right) + O\left(x^{1/2-t(1-2r)+\varepsilon}\right).$$

Take $(6t+1)/4 = 1/2 - t(1-2r)$, this gives $t = 1/(10-8r)$ leading to the common value $(2-r)/(5-4r) + \varepsilon$ of the exponents.

Proof of Theorem 3. Apply Theorem 6 for $f(n) = |\mu^{(e)}(n)|$, $k = 4$ on the assumption of RH.

Proof of Theorem 4. The proof is similar to the proof of Theorem 1 of [11], see also [12] for a more general result of this type.

(i) To obtain the given equality let $f = \mu_2 * \mu$, where μ_2 is defined in the Proof of Theorem 2, and let $v = t^{(e)} * f$. Here both f and v are multiplicative and it is easy to check that $f(p) = f(p^2) = -1, f(p^3) = 1, f(p^a) = 0$ for each $a \geq 4$, and $v(p) = v(p^2) = v(p^3) = 0, v(p^a) = 2^{\omega(a)} - 2^{\omega(a-1)} - 2^{\omega(a-2)} + 2^{\omega(a-4)}$ for $a \geq 4$.

(ii) According to (i), $t^{(e)} = v * \tau(1, 2, \cdot)$, where $\tau(1, 2, n) = \sum\limits_{ab^2=n} 1$ for which

$$\sum_{n \leq x} \tau(1, 2, n) = \zeta(2)x + \zeta(1/2)x^{1/2} + O(x^{1/4}),$$

cf. [4], pp. 196-199. Therefore,

$$\sum_{n \leq x} t^{(e)}(n) = \sum_{d \leq x} v(d) \sum_{e \leq x/d} \tau(1, 2, e)$$

and we obtain the above result by usual estimates.

Proof of Theorem 5. We use the following general result given in [9]: Let F be a multiplicative function with $F(p^a) = f(a)$ for every prime power p^a, where f is positive and satisfying $f(n) = O(n^\beta)$ for some fixed $\beta > 0$. Then

$$\limsup_{n \to \infty} \frac{\log F(n) \log\log n}{\log n} = \sup_m \frac{\log f(m)}{m}.$$

Take $F(n) = t^{(e)}(n)$, $f(a) = 2^{\omega(a)}$. Here $\omega(a) \leq a/2$ and $\frac{\log f(2)}{2} = \frac{1}{2} \log 2$, which proves the result.

Proof of Theorem 6. (i), (ii) Take $f = q_k * w$, in terms of the Dirichlet convolution, where q_k denotes the characteristic function of the k-free integers and use the estimate of A. Walfisz [13],

$$\sum_{n \leq x} q_k(n) = \frac{x}{\zeta(k)} + O(x^{1/k}\delta(x)).$$

For details cf. [6], [12].

(iii) If RH is true, then the error term of above is $O(x^{1/(k+1)+\varepsilon})$, according to the result of H.L. Montgomery and R.C. Vaughan [5], and take into account that $W(s)$ is absolutely convergent for Re $s > 1/(k+1)$.

Proof of Theorem 7. Apply Theorem 6 for $f(n) = \kappa^{(e)}(n)/n$, $k = 4$, $\beta = 2$, where $f(p^4) = 1/p^2$. Then by partial summation we obtain the result.

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