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UNITARY DIVISOR PROBLEM FOR ARITHMETIC PROGRESSIONS

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Abstract. In this paper, we establish an asymptotic formula for the sum

$$\sum_{\substack{n\leq x \\ n\equiv \ell \pmod{m}}} \tau^*(n),$$

where $\tau^*(n)$ is the number of unitary divisors of n, m a positive integer and ℓ any integer. Only the special case of this sum corresponding to $\ell = m = 1$ has been considered before in the literature by Mertens, Eckford Cohen and others.

1. Introduction

The classical Dirichlet divisor problem concerns the best estimate of α in

$$\sum_{n\leq x} \tau(n) = x(\log x + 2\gamma - 1) + O(x^{\alpha}), \tag{1.1}$$

where $\tau(n)$ is the number of divisors of n and γ is the Euler constant. It is well known that $1/4 < \alpha < 1/3$ (cf. [4], p. 272). There is a conjecture that $\alpha = (1/4) + \varepsilon$ for every $\varepsilon > 0$. The best result that is presently known in this direction is that the error term in (1.1) is $O(x^{23/73}(\log x)^{315/146})$, which is due to M.N. Huxely [5].

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When one studies variations of asymptotic estimates of divisor sums as $\sum\limits_{n\leq x} \tau(n)$, restrictions can be placed either on the range of values of n or on the nature of the divisors of n (or on both). The case when n is restricted to belong to a given arithmetic progression seems to have been first considered by S. Ramanujan ([8], result H). In 1916, he stated (without proof of course!) that

$$(1.2) \qquad \sum_{\substack{n\leq x\\ n\equiv \ell (\mathrm{mod}\ m)}} \tau(n) = \alpha x(\log x + 2\gamma - 1) + \beta x + O(x^{1/3}\log x),$$

where α and β are constants depending only on ℓ and m, and he further stated that the order of the remainder term is likely to be the same as that of the remainder term occurring in the Dirichlet divisor problem. In 1929, A. Walfisz [12] improved the order of the remainder in (1.2) to $O(x^{27/82}\log^{11/41} x)$; unfortunately this result on the remainder term in (1.2) is not much noticed probably because of the rather difficult accessability of the journal where it appeared. The result of Walfisz mentioned above has further been improved by Werner Georg Nowak [13] in 1984; he established (1.2) with error term $O(x^{35/108+\varepsilon})$. This appears to be the best result on the divisor problem in arithmetic progressions.

In 1970, Kopetzky [5] gave a simple proof for the asymptotic result

$$(1.3) \qquad \sum_{\substack{n\leq x\\ n\equiv b (\mathrm{mod}\ a)}} \tau(n) = \xi_1(a,b)x\log x + \xi_2(a,b)x + O(\sqrt{x})$$

and gave explicit evaluations of ξ_1 and ξ_2 in terms of the prime divisors of a and of the g.c.d. (a,b) of a and b. However a careful examination of the error term $O(\sqrt{x})$ shows that the constant involved in $O(\sqrt{x})$ is not absolute, but depends on a and b. The error term in (1.3) has to be displayed as $O\left(\frac{\sqrt{x}}{a}\sum\limits_{q|(a,b)} q\varphi(a/q)\right) + O(\sqrt{x})$. See §2, Remark 2.1 in the sequel. Here and elsewhere φ is the Euler totient function.

If a and b are relatively prime, the above O-term in (1.3) has been improved to $O(a^{7/3}x^{1/3}\log x)$ in 1988 by D.I. Tolev ([11]). It may be mentioned that in all the results mentioned above except that of Tolev, the error estimates are independent of the variable x only; they do depend on the other parameters.

In this paper, we study the unitary divisor problem for arithmetic progressions. A divisor d of n is called a unitary divisor if $(d, n/d) = 1$ (cf. [1]). We recall that a positive integer n is called square-free if it is not divisible by the square of any prime. A divisor d of n is called a square-free divisor if d

is square-free. Let $\tau^*(n)$ denote the number of unitary divisors of n. Clearly, $\tau^*(n) = \theta(n) = 2^{\omega(n)}$, where $\theta(n)$ is the number of square-free divisors of n and $\omega(n)$ is the number of distinct prime factors of n with $\omega(1) = 0$.

In 1874, Mertens [7] established the following asymptotic formula (with $\theta(n)$ in place of $\tau^*(n)$):

$$\sum_{n\le x} \tau^*(n) = \frac{x}{\zeta(2)}\left(\log x - 2\gamma - 1 - 2\frac{\zeta'(2)}{\zeta(2)}\right) + O(x^{1/2}\log x), \tag{1.2}$$

where γ is Euler's constant, $\zeta(s)$ the Riemann-zeta function and $\zeta'(s)$ its derivative. In 1960, E. Cohen [2] gave an alternate proof of (1.2). If $\Delta(x)$ denotes the error term in (1.2), then in 1966 Gioia and Vaidya [3] improved the error term in (1.2) to $\Delta(x) = O(x^{1/2})$. The best result known on $\Delta(x)$ is due to D. Suryanarayana and V. Siva Rama Prasad (cf. [10], Theorem 3.1, $k = 2$) who obtained that $\Delta(x) = O(x^{1/2}\delta(x))$, where $\delta(x) = \exp\left(-A\log^{3/5} x(\log\log x)^{-1/5}\right)$, where A is a positive constant. Also, on the assumption of the Riemann hypothesis, they (cf. [10], Theorem 3.2, $k = 2$) further improved the order estimate of $\Delta(x)$ to $\Delta(x) = O\left(x^{\frac{2-\alpha}{5-4\alpha}}\omega(x)\right)$, where α is the number which appears in (1.1) and $\omega(x) = \exp\left(B\log x\cdot(\log\log x)^{-1}\right)$, where B is a positive constant.

The object of the present paper is to establish an asymptotic formula for the sum

$$\sum_{\substack{n\le x\\ n\equiv \ell (\bmod m)}} \tau^*(n)$$

(see §3, Theorem) where ℓ is any integer and m is a positive integer.

Such a formula does not appear to have been established so far. Incidentally we also obtain a minor improvement in the error term of the asymptotic formula of the sum

$$\sum_{\substack{n\le x\\ n\equiv b (\bmod a)}} \tau(n)$$

obtained by H.G. Kopetzky (see §2, Remark 2.1).

In §2 we prove certain preliminary results and in §3 we prove the main result.

2. Preliminaries

Lemma 2.1. (cf. [1], Lemma 3.4) *We have*

$$\sum_{\substack{m\le x\\(m,n)=1}} 1 = \frac{x\varphi(n)}{n} + O(\theta(n)),$$

uniformly in x and n, where φ is the Euler-totient function and $\theta(n) = 2^{\omega(n)}$, $\omega(n)$ being the number of distinct prime factors of n with $\omega(1) = 0$. As usual, (m,n) denotes the greatest common divisor of m and n.

Lemma 2.2. (cf. [9], Lemma 2.1) *We have*

$$\sum_{\substack{m\le x\\(m,n)=1}} \frac{1}{m} = \frac{\varphi(n)}{n}(\log x + \gamma + \alpha(n)) + O\left(\frac{\theta(n)}{x}\right),$$

uniformly in x and n, where

$$\alpha(n) = -\frac{n}{\varphi(n)}\sum_{d|n}\frac{\mu(d)\log d}{d} = \sum_{p|n}\frac{\log p}{p-1}, \tag{2.1}$$

μ being the Möbius function, and γ is Euler's constant (here and throughout this paper the letter p is reserved for primes).

Lemma 2.3. *For any positive integer a and any integer b, we have for $x \ge 2$,*

$$S \equiv \sum_{\substack{n\le x\\ n\equiv b(\mathrm{mod}\, a)}} \tau(n) =$$

$$\begin{aligned} &= \frac{x}{a^2}(\log x + 2\gamma - 1)\sum_{q|(a,b)} q\varphi(a/q) - \frac{2x}{a^2}\sum_{q|(a,b)} q\varphi(a/q)\log q + \\ &+ \frac{2x}{a^2}\sum_{q|(a,b)} q\varphi(a/q)\alpha(a/q) + O\left(\frac{\sqrt{x}}{a}\sum_{q|(a,b)} q\theta(a/q)\right) + O(\sqrt{x}), \end{aligned} \tag{2.2}$$

where $\alpha(n)$ is as given in (2.1) and the constants implied by the O-terms are independent of x, a and b.

Proof. We adopt a standard method. Clearly we have

$$S = \sum_{\substack{d\delta \le x \\ d\delta \equiv b (\mathrm{mod} a)}} 1 = 2S_1 - S_2, \tag{2.3}$$

where

$$S_1 = \sum_{\substack{d\delta \le x \\ d \le \sqrt{x} \\ d\delta \equiv b (\mathrm{mod} a)}} 1 \quad \text{and} \quad S_2 = \sum_{\substack{d\delta \le x \\ d \le \sqrt{x} \\ \delta \le \sqrt{x} \\ d\delta \equiv b (\mathrm{mod} a)}} 1.$$

We have

$$S_1 = \sum_{\substack{d \le \sqrt{x} \\ (d,a)|b}} \sum_{\substack{\delta \le x/d \\ d\delta \equiv b (\mathrm{mod} a)}} 1 = \sum_{\substack{d \le \sqrt{x} \\ (d,a)|b}} \sum_{\substack{\delta \le x/d \\ \frac{d}{(a,d)}\delta \equiv \frac{b}{(a,d)} \left(\mathrm{mod} \frac{a}{(a,d)}\right)}} 1 =$$

$$\begin{aligned} &= \sum_{\substack{d \le \sqrt{x} \\ (d,a)|b}} \left\{ \frac{x(a,d)}{ad} + O(1) \right\} = \\ &= \frac{x}{a} \sum_{\substack{d \le \sqrt{x} \\ (d,a)|b}} \frac{(a,d)}{d} + O(\sqrt{x}). \end{aligned} \tag{2.4}$$

By Lemma 2.2, we have

$$\sum_{\substack{d \le \sqrt{x} \\ (d,a)|b}} \frac{(a,d)}{d} = \tag{2.5}$$

$$\begin{aligned} &= \sum_{q|(a,b)} q \sum_{\substack{d \le \sqrt{x} \\ (d,a)=q}} \frac{1}{d} = \\ &= \sum_{q|(a,b)} \sum_{\substack{t \le \sqrt{x}/q \\ \left(t, \frac{a}{q}\right)=1}} \frac{1}{t} = \\ &= \sum_{q|(a,b)} \left\{ \frac{\varphi(a/q)}{(a/q)} \left(\frac{1}{2}\log x - \log q + \gamma + \alpha(a/q) \right) + O\left(\frac{q}{\sqrt{x}} \theta(a/q) \right) \right\} = \\ &= \left(\frac{1}{2}\log x + \gamma \right) \sum_{q|(a,b)} \frac{\varphi(a/q)}{(a/q)} - \frac{1}{a} \sum_{q|(a,b)} q\varphi(a/q)\log q + \\ &\quad + \sum_{q|(a,b)} \frac{\varphi(a/q)}{(a/q)} \alpha(a/q) + O\left(\frac{1}{\sqrt{x}} \sum_{q|(a,b)} q\theta(a/q) \right). \end{aligned}$$

Substituting (2.5) into (2.4) we obtain,

$$\begin{aligned} S_1 = \frac{x}{a^2}\left(\frac{1}{2}\log x+\gamma\right)\sum_{q|(a,b)} q\varphi(a/q) - \frac{x}{a^2}\sum_{q|(a,b)} q\varphi(a/q)\log q + \\ + \frac{x}{a^2}\sum_{q|(a,b)} q\varphi(a/q)\alpha(a/q) + O\left(\frac{\sqrt{x}}{a}\sum_{q|(a,b)} q\theta(a/q)\right) + O(\sqrt{x}). \end{aligned} \tag{2.6}$$

We have

$$\begin{aligned} S_2 &= \sum_{d\le\sqrt{x}}\ \sum_{\substack{\delta\le\sqrt{x}\\ d\delta\equiv b(\mathrm{mod}\,a)}} 1 = \sum_{\substack{d\le\sqrt{x}\\ (d,a)|b}}\ \sum_{\substack{\delta\le\sqrt{x}\\ d\delta\equiv b(\mathrm{mod}\,a)}} 1 = \\ &= \sum_{\substack{d\le\sqrt{x}\\ (d,a)|b}} \left\{\frac{\sqrt{x}(d,a)}{a} + O(1)\right\} = \\ &= \frac{\sqrt{x}}{a}\sum_{\substack{d\le\sqrt{x}\\ (d,a)|b}} (d,a) + O(\sqrt{x}). \end{aligned} \tag{2.7}$$

By Lemma 2.1 we have,

$$\begin{aligned} \sum_{\substack{d\le\sqrt{x}\\ (d,a)|b}} (d,a) &= \sum_{q|(a,b)} q \sum_{\substack{t\le\sqrt{x}/q\\ \left(t,\frac{a}{q}\right)=1}} 1 = \\ &= \sum_{q|(a,b)} q\left\{\frac{\sqrt{x}}{q}\frac{\varphi(a/q)}{(a/q)} + O\left(\theta(a/q)\right)\right\} = \\ &= \frac{\sqrt{x}}{a}\sum_{q|(a,b)} q\varphi(a/q) + O\left(\sum_{q|(a,b)} q\theta(a/q)\right). \end{aligned} \tag{2.8}$$

Putting (2.8) into (2.7), we obtain

$$S_2 = \frac{x}{a^2}\sum_{q|(a,b)} q\varphi(a/q) + O\left(\frac{\sqrt{x}}{a}\sum_{q|(a,b)} q\theta(a/q)\right) + O(\sqrt{x}). \tag{2.9}$$

Lemma 2.3 follows from (2.9), (2.6) and (2.3).

Remark 2.1. In [6], H.G. Kopetzky established formula (2.2) with error term $O(\sqrt{x})$. However, this error term is not uniform in a and b; if we carefully follow his method in estimating the sum $\sum_{1\le i\le\omega}\delta_i(x)$ in his paper (cf. [6], page 289) we see that the error term that can be obtained from his method is $O\left(\frac{\sqrt{x}}{a}\sum_{q|(a,b)} q\varphi(a/q)\right)+O(\sqrt{x})$, which is weaker than the one in (2.2) since $\sum_{q|(a,b)} q\theta(a/q) = O\left(\sum_{q|(a,b)} q\varphi(a/q)\right)$. It may be noted that formula (2.2) was established by D.I. Tolev [11] in case a and b are relatively prime, with a better error term namely $O(a^{7/3}x^{1/3}\log x)$.

Remark 2.2. It is easy to see that formula (2.2) is also valid for the sum

$$\sum_{\substack{n\le x\\ hn\equiv b(\mathrm{mod}\, a)}} \tau(n), \qquad \text{where} \quad (h,a)=1.$$

Lemma 2.4. *We have for any divisor M of m,*

$$\Sigma_1 = \sum_{\substack{d=1\\ (d^2,m)|M}}^{\infty} \frac{\mu(d)(d^2,m)}{d^2} = \frac{\mu^2(M)m^2\varphi(M)(m_1,M)}{\zeta(2)J_2(m)M\varphi((m_1,M))},$$

where

$$J_2(m) = m^2\prod_{p|m}\left(1-\frac{1}{p^2}\right),$$

and

$$m_1 = \prod_{\substack{p^\alpha\|m\\ \alpha\ge 2}} p^\alpha.$$

Proof. The series is absolutely convergent and the general term of the series is a multiplicative function of d. Hence we can expand the given series as an Euler-infinite product (cf. [4], Theorem 285, page 249). We obtain

$$\Sigma_1 = \prod_{(p^2,m)|M}\left(1-\frac{(p^2,m)}{p^2}\right) = \prod_{p|m}\left(1-\frac{1}{p^2}\right)\prod_{\substack{p^2\nmid m\\ p|M}}\left(1-\frac{1}{p}\right)\prod_{p^2|M}(1-1) =$$

$$= \frac{1}{\zeta(2)}\cdot\frac{m^2}{J_2(m)}\cdot\frac{\prod_{p|M}\left(1-\frac{1}{p}\right)}{\prod_{\substack{p^2|m\\ p|M}}\left(1-\frac{1}{p}\right)}\cdot\prod_{p^2|M}(1-1) =$$

$$= \frac{1}{\zeta(2)} \cdot \frac{m^2}{J_2(m)} \cdot \frac{\varphi(M)}{M} \cdot \prod_{\substack{p|m_1 \\ p|M}} \left(1 - \frac{1}{p}\right)^{-1} \prod_{p^2|M} \left(1 - \frac{1}{p}\right)^{-1} =$$

$$= \begin{cases} \frac{1}{\zeta(2)} \frac{m^2}{J_2(m)} \frac{\varphi(M)}{M} \frac{(m_1,M)}{\varphi(m_1,M)} & \text{if } M \text{ is square-free,} \\ 0 & \text{otherwise.} \end{cases}$$

Hence Lemma 2.4 follows.

Lemma 2.5. *We have*

$$\Sigma_2 \equiv \sum_{\substack{d \le x \\ (d^2,m)|\ell}} \frac{\mu(d)(d^2,m)^2}{d^2} \sum_{q | \frac{(m,\ell)}{(d^2,m)}} q\varphi\left(\frac{m}{q(d^2,m)}\right) =$$

$$= \frac{m^3}{\zeta(2)J_2(m)} \sum_{q|(m,\ell)} \sum_{t|\frac{m}{q}} \frac{\mu(t)}{t} f(m,\ell,t,q) +$$

$$+ O\left(\frac{m(m,\ell)}{x} \sum_{q|(m,\ell)} \theta\left(\frac{m}{q}\right)\right),$$

where

$$f(m,\ell,t,q) = \frac{\mu^2\left((\frac{m}{tq}, \frac{\ell}{q})\right) \varphi\left((\frac{m}{tq}, \frac{\ell}{q})\right) \left(m_1, (\frac{m}{tq}, \frac{\ell}{q})\right)}{\left(\frac{m}{tq}, \frac{\ell}{q}\right) \varphi\left((m_1, (m_1, (\frac{m}{tq}, \frac{\ell}{q})))\right)}.$$

Proof. We have

$$\Sigma_2 = \sum_{q|(m,\ell)} q \sum_{\substack{d \le x \\ (d^2,m) \left| \frac{(m,\ell)}{q} \right.}} \frac{\mu(d)(d^2,m)^2}{d^2} \varphi\left(\frac{m}{q(d^2,m)}\right) =$$

$$= m \sum_{q|(m,\ell)} \sum_{\substack{d \le x \\ (d^2,m) \left| \frac{(m,\ell)}{q} \right.}} \frac{\mu(d)(d^2,m)^2}{d^2} \sum_{t \left| \frac{m}{q(d^2,m)} \right.} \frac{\mu(t)}{t} =$$

$$= m \sum_{q|(m,\ell)} \sum_{t|\frac{m}{q}} \frac{\mu(t)}{t} \sum_{\substack{d \le x \\ (d^2,m) \left| (\frac{m}{tq}, \frac{\ell}{q}) \right.}} \frac{\mu(d)(d^2,m)}{d^2} =$$

$$= m \sum_{q|(m,\ell)} \sum_{t|\frac{m}{q}} \frac{\mu(t)}{t} \left\{ \sum_{\substack{d=1 \\ (d^2,m)|(\frac{m}{tq},\frac{\ell}{q})}} \frac{\mu(d)(d^2,m)}{d^2} + O\left(\frac{(m,\ell)}{q} \sum_{d\geq x} \frac{1}{d^2} \right) \right\} =$$

$$= m \sum_{q|(m,\ell)} \sum_{t|\frac{m}{q}} \frac{\mu(t)}{t} \frac{m^2}{\zeta(2)J_2(m)} \frac{\mu^2\left((\frac{m}{tq},\frac{\ell}{q})\right)\varphi\left((\frac{m}{tq},\frac{\ell}{q})\right)\left(m_1,(\frac{m}{tq},\frac{\ell}{q})\right)}{\left(\frac{m}{tq},\frac{\ell}{q}\right)\varphi\left(\left(m_1,\left(\frac{m}{tq},\frac{\ell}{q}\right)\right)\right)} +$$

$$+ O\left(\frac{m(m,\ell)}{x} \sum_{q|(m,\ell)} \frac{1}{q} \sum_{t|\frac{m}{q}} \frac{\mu^2(t)}{t} \right) =$$

$$= \frac{m^3}{\zeta(2)J_2(m)} \sum_{q|(m,\ell)} \sum_{t|\frac{m}{q}} \frac{\mu(t)}{t} f(m,\ell,t,q) + O\left(\frac{m(m,\ell)}{x} \sum_{q|(m,\ell)} \theta\left(\frac{m}{q}\right) \right).$$

Hence Lemma 2.5 follows.

Lemma 2.6. *Let $M|m$. We have*

$$\Sigma_3 = \sum_{\substack{n\leq x \\ (n^2,m)|M}} \frac{\mu(n)(n^2,m)\log n}{n^2} = \frac{-m^2}{M\zeta(2)J_2(m)} \times$$

$$\times \left\{ \sum_{\substack{p|M \\ p^2\nmid m}} \frac{p\log p}{p-1} G(M/p)\mu^2(M/p) + G(M) \sum_{p\nmid m} \frac{p\log p}{(p^2-1)(p-1)} + \right.$$

$$\left. + \sum_{p^2|M} p^2 \log p G(M/p^2) H(M/p^2) \right\} + O\left(\frac{M\log x}{x}\right),$$

where

$$G(t) = \frac{\varphi(t)(m_1,t)}{\varphi((m_1,t))},$$

and $H(M/p^2)$ as given in (2.13).

Proof. If $\Lambda(n)$ denotes Vongoldt's function defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k, \ \ k\geq 1, \\ 0 & \text{otherwise,} \end{cases}$$

it is well-known (cf. [4], Theorem 296) that

$$\sum_{d|n} \Lambda(d) = \log n.$$

Hence we have

$$\begin{aligned} \Sigma_3 &= - \sum_{\substack{p\delta \le x \\ (p^2,m)(\delta^2,m)|M \\ (p,\delta)=1}} \frac{\mu(\delta)(p^2,m)(\delta^2,m)\log p}{p^2\delta^2} = \\ &= - \sum_{\substack{p\le x \\ (p^2,m)|M}} \frac{(p^2,m)\log p}{p^2} \sum_{\substack{\delta \le x/p \\ (\delta,p)=1 \\ (\delta^2,m) \left| \frac{M}{(p^2,m)} \right.}} \frac{\mu(\delta)(\delta^2,m)}{\delta^2}. \end{aligned} \tag{2.10}$$

Letting Σ_{31} denote the inner sum in (2.10), we have

$$\begin{aligned} \Sigma_{31} &= \sum_{\substack{\delta=1 \\ (\delta,p)=1 \\ (\delta^2,m) \left| \frac{M}{(p^2,m)} \right.}}^{\infty} \frac{\mu(\delta)(\delta^2,m)}{\delta^2} + O\left(\frac{pM}{x(p^2,m)}\right) = \\ &= \Sigma_{32} + O\left(\frac{pM}{x(p^2,m)}\right), \end{aligned} \tag{2.11}$$

say. Expanding Σ_{32} as an Euler-infinite product, we obtain (in what follows we temporarily designate q to be a prime number)
(2.12)

$$\begin{aligned} \Sigma_{32} &= \prod_{\substack{q\ne p \\ (q^2,m) \left| \frac{M}{(p^2,m)} \right.}} \left(1 - \frac{(q^2,m)}{q^2}\right) = \\ &= \prod_{\substack{q\ne p \\ q \nmid m}} \left(1-\frac{1}{q^2}\right) \prod_{\substack{q\ne p \\ q^2 \nmid m \\ q \left| \frac{M}{(p^2,m)} \right.}} \left(1-\frac{1}{q}\right) \prod_{\substack{q \ne p \\ q^2 \left| \frac{M}{(p^2,m)} \right.}} (1-1) = \\ &= \begin{cases} \dfrac{m^2p^2\varphi(M/p)(m_1,M/p)\mu^2(M/p)}{M\zeta(2)J_2(m)(p-1)\varphi((m_1,M/p))} & \text{if } p|M \text{ and } p^2 \nmid m, \\ \dfrac{m^2p^3\varphi(M)(m_1,M)\mu^2(M)}{M\zeta(2)J_2(m)\varphi((m_1,M))(p^2-1)(p-1)} & \text{if } p \nmid m, \\ \dfrac{m^2p^2\varphi(M/p^2)(m_1,M/p^2)}{M\zeta(2)\varphi((m_1,M/p^2))} H(M/p^2), & \text{if } p^2|M, \end{cases} \end{aligned}$$

where m_1 is as given in Lemma 2.4 and

$$H\left(\frac{M}{p^2}\right)=\begin{cases}0 & \text{if } M/p^2 \text{ is divisible by a prime} \neq p,\\ 1 & \text{otherwise.}\end{cases} \tag{2.13}$$

Substituting (2.11) into (2.10), we obtain
(2.14)

$$\begin{aligned}\Sigma_3 &= -\sum_{\substack{p\le x\\ (p^2,m)\mid M}} \frac{(p^2,m)\log p}{p^2}\Sigma_{32} + O\left(\frac{M}{x}\sum_{p\le x}\frac{\log p}{p}\right) =\\ &= -\sum_{\substack{p\le x\\ (p^2,m)\mid M}} \frac{(p^2,m)\log p}{p^2}\Sigma_{32} + O\left(\frac{M\log x}{x}\right) =\\ &= -\sum_{(p^2,m)\mid M} \frac{(p^2,m)\log p}{p^2}\Sigma_{32} + O\left(M\sum_{p>x}\frac{\log p}{p^2}\right) + O\left(\frac{M\log x}{x}\right) =\\ &= -\sum_{(p^2,m)\mid M} \frac{(p^2,m)\log p}{p^2}\Sigma_{32} + O\left(\frac{M\log x}{x}\right),\end{aligned}$$

where we used the results $\sum\limits_{p\le x}\frac{\log p}{p}=O(\log x)$ and $\sum\limits_{p>x}\frac{\log p}{p^2}=O\left(\frac{\log x}{x}\right)$. From (2.12), we have

$$\sum_{(p^2,m)\mid M} \frac{(p^2,m)\log p}{p^2}\Sigma_{32} = \tag{2.15}$$

$$\begin{aligned}&= \sum_{\substack{p\mid M\\ p^2\nmid m}} \frac{\log p}{p}\left(\frac{m^2p^2\varphi(M/p)(m_1,M/p)\mu^2(M/p)}{M\zeta(2)J_2(m)(p-1)\varphi((m_1,M/p))}\right)+\\ &\quad+\sum_{p\nmid m}\frac{\log p}{p}\left(\frac{m^2p^3\varphi(M)(m_1,M)\mu^2(M)}{M\zeta(2)J_2(m)\varphi((m_1,M))(p^2-1)(p-1)}\right)+\\ &\quad+\sum_{p^2\mid M}\log p\left(\frac{m^2p^2\varphi(M/p^2)(m_1,M/p^2)H(M/p^2)}{M\zeta(2)\varphi((m_1,M/p^2))}\right)=\end{aligned}$$

$$
\begin{aligned}
=&\frac{m^2}{M\zeta(2)J_2(m)}\left\{\sum_{\substack{p\mid M\\ p^2\nmid m}}\frac{p\ \log p}{(p-1)}\cdot\frac{\varphi(M/p)(m_1,m/p)}{\varphi((m_1,M/p))}\mu^2(M/p)+\right.\\
&+\frac{\varphi(M)(m_1,M)\mu^2(M)}{\varphi((m_1,M))}\sum_{p\nmid m}\frac{p\log p}{(p^2-1)(p-1)}+\\
&\left.+\sum_{p^2\mid M}\frac{p^2\log p\varphi(M/p^2)(m_1,M/p^2)H(M/p^2)}{\varphi((m_1,M/p^2))}\right\}.
\end{aligned}
$$

Substituting (2.15) into (2.14), we obtain Lemma 2.6.

Lemma 2.7. *We have*

$$
\begin{aligned}
\Sigma_4 =& \sum_{\substack{d\le x\\ (d^2,m)\mid \ell}}\frac{\mu(d)(d^2,m)^2\log d}{d^2}\sum_{q\left|\frac{(m,\ell)}{(d^2,m)}\right.} q\varphi\left(\frac{m}{q(d^2,m)}\right)=\\
=&-\frac{m^3A(m,\ell)}{\zeta(2)J_2(m)}+O\left(\frac{m(m,\ell)\log x}{x}\sum_{q\mid(m,\ell)}\theta\left(\frac{m}{q}\right)\right),
\end{aligned}
$$

where

$$
\begin{aligned}
A(m,\ell)=&\sum_{q\mid(m,\ell)}\sum_{t\mid\frac{m}{q}}\frac{\mu(t)}{t}\cdot\frac{1}{\left(\frac{m}{tq},\frac{\ell}{q}\right)}\times\\
&\times\left\{\sum_{\substack{p\mid M\\ p^2\nmid m}}\frac{p\ \log p}{p-1}G(M/p)\mu^2(M/p)+G(M)\sum_{p\nmid m}\frac{p\ \log p}{(p^2-1)(p-1)}+\right.\\
&\left.+\sum_{p^2\mid M}p^2\log pG(M/p^2)H(M/p^2)\right\},
\end{aligned}
$$

and

$$
M=\left(\frac{m}{tq},\frac{\ell}{q}\right).
$$

Proof. We have

$$
\Sigma_4 = m\sum_{q\mid(m,\ell)}\sum_{t\mid\frac{m}{q}}\frac{\mu(t)}{t}\sum_{\substack{d\le x\\ (d^2,m)\left|\left(\frac{m}{tq},\frac{\ell}{q}\right)\right.}}\frac{\mu(d)(d^2,m)\log d}{d^2}.
$$

Now Lemma 2.7 follows from Lemma 2.6.

Lemma 2.8. *We have*

$$\Sigma_5 = \sum_{\substack{d\le x\\(d^2,m)\mid\ell}} \frac{\mu(d)(d^2,m)^2}{d^2} \sum_{q\left|\frac{(m,\ell)}{(d^2,m)}\right.} q\varphi\left(\frac{m}{q(d^2,m)}\right)\alpha\left(\frac{m}{q(d^2,m)}\right) =$$
$$= \frac{-m^3}{\zeta(2)J_2(m)} \sum_{q\mid(m,\ell)} \sum_{t\mid\frac{m}{q}} \frac{\mu(t)}{t}\log t\ f(m,\ell,t,q) +$$
$$+ O\left(\frac{m(m,\ell)}{x}\sum_{q\mid(m,\ell)}\theta\left(\frac{m}{q}\right)\right),$$

where $f(m,\ell,t,q)$ is as given in Lemma 2.5.

Proof. From (2.1), we have

$$-\frac{\alpha(n)\varphi(n)}{n} = \sum_{t\mid n}\frac{\mu(t)\log t}{t}.$$

Using this, it can be shown that

$$\Sigma_5 = -m\sum_{q\mid(m,\ell)}\sum_{t\mid\frac{m}{q}}\frac{\mu(t)\log t}{t} \sum_{\substack{d\le x\\(d^2,m)\left|\left(\frac{m}{tq},\frac{\ell}{q}\right)\right.}} \frac{\mu(d)(d^2,m)}{d^2} =$$
$$= -m\sum_{q\mid(m,\ell)}\sum_{t\mid\frac{m}{q}}\frac{\mu(t)\log t}{t} \sum_{\substack{d=1\\(d^2,m)\left|\left(\frac{m}{tq},\frac{\ell}{q}\right)\right.}}^{\infty} \frac{\mu(d)(d^2,m)}{d^2} +$$
$$+ O\left(\frac{m(m,\ell)}{x}\sum_{q\mid(m,\ell)}\theta\left(\frac{m}{q}\right)\right).$$

Now, Lemma 2.8 follows from Lemma 2.4 and the definition of $f(m,\ell,t,q)$ is given in Lemma 2.4.

Lemma 2.9. *We have*

$$\Sigma_6 = \sum_{\substack{d\le x\\(d^2,m)\mid\ell}} \frac{\mu(d)(d^2,m)^2}{d^2} \sum_{q\left|\frac{(m,\ell)}{(d^2,m)}\right.} q\varphi\left(\frac{m}{q(d^2,m)}\right)\log q =$$
$$= \frac{m^3}{\zeta(2)J_2(m)} \sum_{q\mid(m,\ell)} \log q \sum_{t\mid\frac{m}{q}} \frac{\mu(t)}{t} f(m,\ell,t,q) +$$

$$+O\left(\frac{m(m,\ell)}{x}\cdot\sum_{q|(m,\ell)}\theta\left(\frac{m}{q}\right)\right).$$

Proof. We have

$$\Sigma_6 = m\sum_{q|(m,\ell)}\log q\sum_{t|\frac{m}{q}}\frac{\mu(t)}{t}\sum_{\substack{d\le x\\ (d^2,m)\left|\left(\frac{m}{tq},\frac{\ell}{q}\right)\right.}}\frac{\mu(d)(d^2,m)}{d^2}=$$

$$= m\sum_{q|(m,\ell)}\log q\sum_{t|\frac{m}{q}}\frac{\mu(t)}{t}\sum_{\substack{d=1\\ (d^2,m)\left|\left(\frac{m}{tq},\frac{\ell}{q}\right)\right.}}^{\infty}\frac{\mu(d)(d^2,m)}{d^2}+$$

$$+\,O\left(\frac{m(m,\ell)}{x}\cdot\sum_{q|(m,\ell)}\theta\left(\frac{m}{q}\right)\right).$$

Lemma 2.9 follows from Lemma 2.4 and the definition of $f(m,\ell,t,q)$ is given in Lemma 2.5.

Lemma 2.10. *We have*

$$\Sigma_7 \equiv \sum_{d\le x}\frac{(d^2,m)}{d}\sum_{q\left|\frac{(m,\ell)}{(d^2,m)}\right.}q\theta\left(\frac{m}{q(d^2,m)}\right)=$$

$$= O\left((m,\ell)\sum_{q|(m,\ell)}\theta\left(\frac{m}{q}\right)\cdot\log x\right).$$

Proof. We have

$$\Sigma_7 = \sum_{q|(m,\ell)}q\sum_{\substack{d\le x\\ (d^2,m)\left|\frac{(m,\ell)}{q}\right.}}\frac{1}{d}\theta\left(\frac{m}{q(d^2,m)}\right)(d^2,m)\le$$

$$\le (m,\ell)\sum_{q|(m,\ell)}\theta\left(\frac{m}{q}\right)\sum_{d\le x}\frac{1}{d}=$$

$$= O\left((m,\ell)\sum_{q|(m,\ell)}\theta\left(\frac{m}{q}\right)\cdot\log x\right).$$

3. The main result

We are now in a position to prove the following

Theorem. *We have for any positive integers m and ℓ,*

(3.1)

$$T^* = \sum_{\substack{n\le x\\ n\equiv \ell (\mathrm{mod}\, m)}} \tau^*(n) =$$

$$= \frac{m}{\zeta(2)J_2(m)}\left\{x\ \log x \sum_{q|(m,\ell)}\sum_{t|\frac{m}{q}}\frac{\mu(t)}{t}f(m,\ell,t,q)+\right.$$

$$\left. +x\left(2A(m,\ell)+\sum_{q|(m,\ell)}\sum_{t|\frac{m}{q}}\frac{\mu(t)}{t}f(m,\ell,t,q)(2\gamma-1-2\log t-2\log q)\right)\right\}+$$

$$+O\left(\sqrt{x}\log x\frac{(m,\ell)}{m}\sum_{q|(m,\ell)}\theta\left(\frac{m}{q}\right)\right)+O(\sqrt{x}\log x),$$

uniformly in x, m and ℓ, where $f(m,\ell,t,q)$ is as given in Lemma 2.5 and $A(m,\ell)$ as in Lemma 2.7.

Proof. Since $\tau^*(n) = \sum_{d^2\delta=n}\mu(d)\tau(\delta)$, we have by Lemma 2.3 and Remark 2.2,

$$T^* = \sum_{d\le\sqrt{x}}\mu(d)\sum_{\substack{\delta\le x/d^2\\ d^2\delta\equiv\ell(\mathrm{mod}\, m)}}\tau(\delta) =$$

(3.2)

$$= \sum_{\substack{d\le\sqrt{x}\\ (d^2,m)|\ell}}\mu(d)\sum_{\substack{\delta\le x/d^2\\ \frac{d^2}{(d^2,m)}\delta\equiv\frac{\ell}{(d^2,m)}\mathrm{mod}\left(\frac{m}{(d^2,m)}\right)}}\tau(\delta) =$$

$$= \sum_{\substack{d\le\sqrt{x}\\ (d^2,m)|\ell}}\mu(d)\left\{\frac{x}{d^2}\frac{(d^2,m)^2}{m^2}(\log x-2\log d+2\gamma-1)\sum_{q\left|\frac{(m,\ell)}{(d^2,m)}\right.}q\varphi\left(\frac{m}{q(d^2,m)}\right)-\right.$$

$$-\frac{2x}{d^2}\cdot\frac{(d^2,m)^2}{m^2}\sum_{q\left|\frac{(m,\ell)}{(d^2,m)}\right.}q\varphi\left(\frac{m}{q(d^2,m)}\right)\log q+$$

$$+\frac{2x}{d^2}\cdot\frac{(d^2,m)^2}{m^2}\sum_{q\left|\frac{(m,\ell)}{(d^2,m)}\right.} q\varphi\left(\frac{m}{q(d^2,m)}\right)\alpha\left(\frac{m}{q(d^2,m)}\right)+$$

$$\left.+O\left(\frac{\sqrt{x}}{d}\frac{(d^2,m)}{m}\sum_{q\left|\frac{(m,\ell)}{(d^2,m)}\right.} q\theta\left(\frac{m}{q(d^2,m)}\right)\right)+O\left(\frac{\sqrt{x}}{d}\right)\right\}=$$

$$=\frac{x}{m^2}(\log x+2\gamma-1)\sum_{\substack{d\le\sqrt{x}\\(d^2,m)|\ell}}\frac{\mu(d)(d^2,m)^2}{d^2}\sum_{q\left|\frac{(m,\ell)}{(d^2,m)}\right.} q\varphi\left(\frac{m}{q(d^2,m)}\right)-$$

$$-\frac{2x}{m^2}\sum_{\substack{d\le\sqrt{x}\\(d^2,m)|\ell}}\frac{\mu(d)(d^2,m)^2\log d}{d^2}\sum_{q\left|\frac{(m,\ell)}{(d^2,m)}\right.} q\varphi\left(\frac{m}{q(d^2,m)}\right)-$$

$$-\frac{2x}{m^2}\sum_{\substack{d\le\sqrt{x}\\(d^2,m)|\ell}}\frac{\mu(d)(d^2,m)^2}{d^2}\sum_{q\left|\frac{(m,\ell)}{(d^2,m)}\right.} q\varphi\left(\frac{m}{q(d^2,m)}\right)\log q+$$

$$+\frac{2x}{m^2}\sum_{\substack{d\le\sqrt{x}\\(d^2,m)|\ell}}\frac{\mu(d)(d^2,m)^2}{d^2}\sum_{q\left|\frac{(m,\ell)}{(d^2,m)}\right.} q\varphi\left(\frac{m}{q(d^2,m)}\right)\alpha\left(\frac{m}{q(d^2,m)}\right)+$$

$$+O\left(\frac{\sqrt{x}}{m}\sum_{\substack{d\le\sqrt{x}\\(d^2,m)|\ell}}\frac{(d^2,m)}{m}\sum_{q\left|\frac{(m,\ell)}{(d^2,m)}\right.} q\theta\left(\frac{m}{q(d^2,m)}\right)\right)+$$

$$+O(\sqrt{x}\log x).$$

Now the formula (3.1) follows from (3.2), Lemma 2.5, 2.7, 2.8, 2.9 and 2.10.

Remark 3.1. We believe that the order of the remainder term in (3.1) can be improved.

References

[1] **Cohen E.,** Arithmetical functions associated with the unitary divisors of an integer, *Math.Z.,* **74** (1960), 66-80.

[2] **Cohen E.,** The number of unitary divisors of an integer, *Amer. Math. Monthly,* **67** (1960), 879-880.

[3] **Gioia A.A. and Vaidya A.M.,** The number of square-free divisors of an integer, *Duke Math. J.,* **33** (1966), 797-799.

[4] **Hardy G.H. and Wright E.M.,** *An introduction to the theory of numbers,* fifth edition, Oxford University Press, 1981.

[5] **Huxley M.N.,** Exponential sums and lattice points II., *Proc. London Math. Soc.,* **66** (1993), 279-301.

[6] **Kopetzky H.G.,** Über die Grössenordnung der Teilerfunktion in Restklassen, *Monatsh. Math.,* **82** (1976), 287-295.

[7] **Mertens F.,** Über einige asymptotische Gesetze der Zahlentheorie, *J. Reine Angew. Math.,* **77** (1874), 289-338.

[8] **Ramanujan S.,** Some formulae in the analytic theory of numbers, *Messenger of Mathematics,* **XLV** (1916), 11-15. Collected papers of Ramanujan, Chelsea Publishing Co., New York, 1962.

[9] **Suryanarayana D.,** The greatest divisor of n which is prime to K, *Math. Student,* **37** (1969), 147-157.

[10] **Suryanarayana D., and V. Siva Rama Prasad,** The number of k-free divisors of an integer, *Acta Arith.,* **17** (1971), 345-354.

[11] **Tolev D.I.,** On the division problem in arithmetic progressions, *Comptes rendus de l'Académie Bulgare des Sciences,* **41** (1988), 33-36.

[12] **Walfisz A.,** Opewnym zagadnieniu dzielników Ramanujana (On a problem of Ramanujan divisors), *Prace Mat. Fys.,* **35** (1928-1929), 101-126. (in Polish)

[13] **Nowak W.G.**, On the divisor problem in an arithmetic progression, *Comment. Math. Univ. St. Pauli,* **33** (1984), 209-217.

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