PACKING A FOREST WITH A GRAPH

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Let F be a forest of order n and G a graph of order n. Suppose that $\Delta(G)(\Delta(F)+1) \leq n$. Then, except for three pairs of graphs (G, F), there is a packing of G and F.

1 Introduction

We discuss only finite simple graphs and use standard terminology and notation from [1] except as indicated. For any graph G, we use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of G, respectively. We denote the complement of G by G^c. Let G and H be two graphs of order n. We say that there is a packing of G and H if the complement G^c contains a subgraph isomorphic to H. In this case, we also say that G and H are packable. There are many papers concerning the packing of two graphs which have a small number of edges. For example, Sauer and Spencer [6] proved that if $|E(G)| \leq n-2$ and $|E(H)| \leq n-2$, then there is a packing of G and H. Bollobás and Eldridge [2] found all the forbidden pairs (G, H) of graphs with $\Delta(G) < n-1$, $\Delta(H) < n-1$, $|E(G)|+|E(H)| \leq 2n-3$ for which there are no packings of G and H. Slater, Teo and Yap [7] proved that if $n \geq 5$, G is a tree, H has $n-1$ edges and neither G nor H is a star, then there is a packing of G and H. Sauer and Spencer [6] also proved that if $2\Delta(G)\Delta(H) < n$, then there is a packing of G and H. For more results, see [1, Chapter 8] and [9]. Bollobás and Eldridge [2] conjectured that if $(\Delta(G)+1)(\Delta(H)+1) \leq n+1$, then there is a packing of G and H. This conjecture is still open. Hajnal and Szemerédi [4] proved that if $n = sk (s \geq 3$ and $k \geq 1)$ and G is the vertex-disjoint union of k copies of K_s and $\Delta(H) \leq k-1$, i.e., $(\Delta(G)+1)(\Delta(H)+1) \leq n$, then there is a packing of G and H. The result in the case $s = 3$ was first obtained by Corrádi and Hajnal [3].

In this paper, we consider the case that one of G and H is a forest, i.e., a graph with no cycles. To state our result, we define kG to be the vertex-disjoint union of k copies of G for any positive integer k and graph G. For even positive integer n, there is no packing of the two graphs in each of the following three pairs of graphs: $((n/2)K_2, K_{1,n-1})$, $(K_{(n/2)+1} \cup H, (n/2)K_2)$ where H is any graph of order $n/2-1$ and '$\cup$' means 'vertex-disjoint union', and $(K_{n/2,n/2}, (n/2)K_2)$ with $n/2$ odd. To see this, we observe that in each pair, the complement of the graph which is not $(n/2)K_2$ does not have a perfect matching. We especially name these three pairs as three forbidden pairs of graphs. We prove the following.

Theorem *Let F be a forest of order n and G a graph of order n. Suppose that $\Delta(G)(\Delta(F)+1) \le n$. Then there is a packing of G and F unless the pair (G, F) is one of the three forbidden pairs of graphs.*

For the proof of the theorem, we recall some terminology and notation.

Let G be a graph, U a subset of $V(G)$ and u a vertex of G. As usual, $N_G(u)$ is the set of neighbors of u, $d_G(u)$ is the degree of u in G and $N_G(U)$ is the union of all $N_G(u)$ for $u \in U$. We define $N_G(u, U)$ to be $N_G(u) \cap U$ and let $d_G(u, U) = |N_G(u, U)|$. If H is a subgraph of G, we define $d_G(u, H)$ to be $d_G(u, V(H))$. Then $d_G(u, G)$ is just the degree of u in G.

Let σ be a bijection on $V(G)$. We define a graph G_σ with $V(G_\sigma) = V(G)$ and $E(G_\sigma) = \{\sigma(u)\sigma(v) | uv \in E(G)\}$. Clearly, G_σ is isomorphic to G under σ. Let $x_1, x_2, \ldots, x_k$ be distinct vertices of G. Then $G_{(x_1, x_2, \ldots, x_k)}$ stands for G_σ where $\sigma(x_i) = x_{i+1} (1 \le i \le k-1)$, $\sigma(x_k) = x_1$ and $\sigma(x) = x$ for all $x \in V(G) - \{x_1, x_2, \ldots, x_k\}$.

2 Proof of the Theorem

Let F be a forest of order n and G a graph of order n such that $\Delta(G)(\Delta(F)+1) \le n$. We use induction on $|E(F)|$ to prove the theorem. The theorem is trivial if $|E(F)| = 0$. Assume that the theorem holds at $|E(F)| = m - 1$. We shall prove the theorem for $|E(F)| = m$. We may assume that G and F are not packable and then prove that (G, F) is one of the pairs mentioned in the theorem.

We distinguish three cases: $\Delta(F) = 1$, $\Delta(F) = 2$ or $\Delta(F) \ge 3$.

Case 1. $\Delta(F) = 1$.

In this case, $\Delta(G) \le n/2$ and $\delta(G^c) \ge n - 1 - n/2 = n/2 - 1$. As F consists of independent edges and isolated vertices, G^c doesn't contain $\lceil (n-1)/2 \rceil$ independent edges. Let b be the edge independence number of G^c and $d = n - 2b$. Then $d \ge 2$ if n is even, and $d \ge 3$ if n is odd. By the well known standard proof of Tutte's Theorem [1, pp. 55–57], there exists a maximal subset $S_0 \subseteq V(G^c)$ such that $o(G^c - S_0) = |S_0| + d$, where $o(G^c - S_0)$ is the number of odd components of $G^c - S_0$. Furthermore, $o(G^c - S) \le |S| + d$ for all subsets $S \subseteq V(G^c)$. If $G^c - S_0$ has an even component D, let $x \in V(D)$. Then $|S_0 \cup \{x\}| + d \ge o(G^c - S_0 - x) \ge o(G^c - S_0) + 1 = |S_0 \cup \{x\}| + d$, contradicting the maximality of S_0. Hence $G^c - S_0$ contains no even components. Let $D_1, D_2, \ldots, D_{k+d}$ be a list of all odd components of $G^c - S_0$, where $k = |S_0|$. We may assume that $|V(D_1)| \le |V(D_2)| \le \cdots \le |V(D_{k+d})|$. Let $x \in V(D_1)$. Then

$$\begin{aligned} n/2 - 1 &\le d_{G^c}(x) \le |S_0| + |V(D_1)| - 1 && (1) \\ &\le \frac{1}{2}(|S_0| + |V(D_1)| + |V(D_2)| + \cdots + |V(D_{k+d})|) - 1 && (2) \\ &= n/2 - 1. && (3) \end{aligned}$$

Hence equality holds in (1), (2) and (3). This implies that $d = 2$ and n is even. Moreover, if $S_0 = \emptyset$, then $|V(D_1)| = |V(D_2)| = n/2$, $n/2$ is odd and G^c is $2K_{n/2}$.

Hence F is $(n/2)K_2$ and G is $K_{n/2,n/2}$. If $S_0 \neq \emptyset$, then $k = n/2 - 1$, $|V(D_1)| = |V(D_i)| = 1 (1 \leq i \leq n/2+1)$. Furthermore, $V(G^c) - S_0$ is an independent set of vertices of G^c and $yz \in E(G^c)$ for all $y \in S_0$ and all $z \in V(G^c) - S_0$. Hence F is $(n/2)K_2$ and G is $K_{(n/2)+1} \cup H$ where H is a graph of order $n/2 - 1$.

Case 2. $\Delta(F) = 2$.

In this case, $\Delta(G) \leq n/3$ and $\delta(G^c) \geq n - 1 - n/3 \geq (n-1)/2$. From this, we can easily deduce that G^c is connected. Let $P = x_1 x_2 \ldots x_k$ be a longest path of G^c. Then $k \geq 3$. Moreover, $d_{G^c}(x_1, P) + d_{G^c}(x_k, P) = d_{G^c}(x_1) + d_{G^c}(x_k) \geq n - 1$. If $k \leq n-1$, then by the well-known Ore's condition [5], G^c contains a cycle C with $V(C) = V(P)$. This implies that G^c contains a longer path than P as G^c is connected. Hence $k = n$ and therefore P contains F as F consists of vertex-disjoint paths.

Case 3. $\Delta(F) \geq 3$.

Let $x_0 y_0$ be an edge of F with $d_F(x_0) = 1$. By the induction hypothesis, we may assume that $F - x_0 y_0$ is a subgraph of G^c. Then $x_0 y_0$ is an edge of G. Let

$$C = N_G(x_0) \cap N_G(y_0) \quad (4)$$
$$A = N_G(x_0) - C \cup \{y_0\} \quad (5)$$
$$B = N_G(y_0) - C \cup \{x_0\} \quad (6)$$
$$Y_0 = N_F(y_0) - \{x_0\} \quad (7)$$
$$V_1 = V(G) - A \cup B \cup C \cup Y_0 \cup \{x_0, y_0\}. \quad (8)$$

As there is no packing of G and F, we have the following four claims.

Claim 1. For every $u \in A \cup V_1$, there exists $v \in N_G(x_0)$ such that uv is an edge of F, i.e., $uv \in E(F)$.

Suppose, for a contradiction, that there exists $u_0 \in A \cup V_1$ such that $u_0 v \notin E(F)$ for all $v \in N_G(x_0)$. Then $u_0 y_0 \notin E(G)$ and $x_0 w \notin E(G)$ for all $w \in N_F(u_0)$. Therefore $F_{(u_0, x_0)}$ is a subgraph of G^c, a contradiction. This proves the claim.

By Claim 1, we have that

$$\begin{aligned}
|V_1| &\leq |A|(\Delta(F) - 1) + |C|\Delta(F). && (9)\\
n &= |\{x_0, y_0\}| + |A| + |B| + |C| + |Y_0| + |V_1| && (10)\\
&\leq 2 + |A| + |B| + |C| + \Delta(F) - 1 + |A|(\Delta(F) - 1) + |C|\Delta(F) && (11)\\
&= 1 + (|A| + |C| + 1)\Delta(F) + |B| + |C| && (12)\\
&\leq 1 + \Delta(G)\Delta(F) + \Delta(G) - 1 && (13)\\
&= \Delta(G)(\Delta(F) + 1) \leq n. && (14)
\end{aligned}$$

Hence equality holds in (9) through (14). This implies the following.

$$d_G(x_0) = d_G(y_0) = \Delta(G); \quad (15)$$
$$d_F(y_0) = \Delta(F) = d_F(u) \textit{ for all } u \in A \cup C; \quad (16)$$
$$d_F(u, V_1) = \Delta(F) - 1 \textit{ for all } u \in A; \quad (17)$$
$$d_F(u, V_1) = \Delta(F) \textit{ for all } u \in C; \quad (18)$$

$$N_F(u, V_1) \cap N_F(v, V_1) = \emptyset \text{ for all } u, v \in A \cup C \text{ with } u \neq v; \quad (19)$$
$$N_F(u, V_1) \cap Y_0 = \emptyset \text{ for all } u \in A \cup C. \quad (20)$$

Claim 2. $|A| = |B| = 0$.

From (15), we see that $|A| = |B|$. Suppose, for a contradiction, that $A \neq \emptyset$. Choose an arbitrary vertex $u \in N_F(A) \cap V_1$. By (19), $N_F(u, C) = \emptyset$. Suppose that $N_F(u, B) = \emptyset$. Then it is clear that $N_G(y_0, N_F(u)) = \emptyset$. It is also clear that $N_G(x_0, Y_0) = \emptyset$ and $ux_0 \notin E(G)$. This implies that $F_{(x_0,u,y_0)}$ has no edges in common with G, a contradiction. Hence, for all $u \in N_F(A) \cap V_1$, there exists $v \in B$ such that $uv \in E(F)$. Since $\Delta(F) \geq 3$ and by (17), $d_F(y, V_1) \geq 2$ for all $y \in A$. This implies that $|B| \geq 1 + |A|$ as F doesn't contain cycles, a contradiction. This proves the claim.

Claim 2 says that $N_G(x_0) - \{y_0\} = N_G(y_0) - \{x_0\}$. Let $C = \{y_1, y_2, \ldots, y_{k-1}\}$, where $k = \Delta(G)$. If $k = 1$, then F is $K_{1,n-1}$ and therefore n must be even and G must be $(n/2)K_2$ for otherwise G and F are packable. So we may assume that $k \geq 2$ in the following.

For every $y_i \in C$, it is easy to see that $F_{(x_0,y_i)} - y_0y_i$ is a subgraph of G^c and the degree of y_i in $F_{(x_0,y_i)}$ is one. Hence by the similarity, we may assume that $N_G(y_0) - \{y_i\} = N_G(y_i) - \{y_0\}$. This implies that the subgraph G_1 of G induced by $N_G(y_0) \cup \{y_0\}$ is K_{k+1}. Obviously, G_1 is a component of G. Let $Y_i = N_F(y_i)$ for $1 \leq i \leq k-1$. Set $t = \Delta(F)$. Then by (17) and (18), $|Y_0| = t - 1$ and $|Y_i| = t$ for $1 \leq i \leq k-1$. Note that Y_i is an independent set of vertices of F for all $i \in \{0, 1, \ldots k-1\}$ since F contains no cycles.

Claim 3. For all $i \in \{0, 1, \ldots, k-1\}$, $d_G(z, Y_i) \geq 2$ for all $z \in Y_i$.

Suppose, for a contradiction, that there exist $i \in \{0, 1, \ldots, k-1\}$ and a vertex $z_i \in Y_i$ such that $d_G(z_i, Y_i) \leq 1$. Choose a vertex $w_i \in Y_i$ such that $w_i \neq z_i$ and if $d_G(z_i, Y_i) = 1$ then $w_iz_i \in E(G)$.

We assume first that $i \neq 0$. Without loss of generality, say $i = 1$. It is clear that $N_G(y_1, N_F(z_1)) = \emptyset$ and $N_G(z_1, N_F(y_1)) \subseteq \{w_1\}$. Hence $F' = F_{(y_1,z_1)}$ has at most two edges x_0y_0 and w_1z_1 in common with G. Obviously, $w_1y_1 \notin E(F')$ as $w_1z_1 \notin E(F)$. As above, it is easy to see that $N_G(x_0, N_{F'}(w_1)) = \emptyset$. Hence $F'_{(x_0,w_1)}$ has no edges in common with G, a contradiction.

Next, we assume that $i = 0$. As in the above, it is easy to see that $F^1 = F_{(y_0,z_0)}$ has at most one edge w_0z_0 in common with G. As G and F are not packable, w_0z_0 must be an edge of G. As before, since $w_0y_0 \notin E(F^1)$, we have that $N_G(x_0, N_{F^1}(w_0)) = \emptyset$. Then w_0z_0 is still the only common edge of $F^2 = F^1_{(w_0,x_0)}$ and G. But the degree of w_0 in F^2 is one. By the argument of Claim 1 and Claim 2, we may assume that G has a component G_2 which is K_{k+1} and contains w_0z_0. As Claim 3 is true for all i, $1 \leq i \leq k-1$ and $\Delta(G) = k$, we see that there exists $i \in \{1, 2, \ldots, k-1\}$ such that $Y_i \cap V(G_2) = \emptyset$. This implies that $F^2_{(w_0,y_i)}$ has no edges in common with G, a contradiction. This proves the claim.

Since F doesn't contain cycles, we see that there is at most one edge of F between Y_i and Y_j for any $i, j \in \{0, 1, \ldots, k-1\}$ with $i \neq j$. Construct a graph H such that $V(H) = \{Y_0, Y_1, \ldots, Y_{k-1}\}$ and $Y_iY_j \in E(H)$ if and only if there is an edge of F between Y_i and Y_j. Then H is a forest as F is a forest. Hence there exist

$i, j \in \{0, 1, \ldots, k-1\}$ with $i \neq j$ such that $d_H(Y_i) \leq 1$ and $d_H(Y_j) \leq 1$. We may assume without loss of generality that $d_H(Y_{k-1}) \leq 1$. If $d_H(Y_{k-1}) = 1$, let Y_p denote the neighbor of Y_{k-1} in H and $z_1 z_2$ denote the edge of F with $z_1 \in Y_{k-1}$ and $z_2 \in Y_p$. Let

$$I = \{i | 0 \leq i \leq k-2 \text{ } and \text{ } d_G(u, Y_i) = 0 \text{ } for \text{ } some \text{ } u \in Y_{k-1}\}. \qquad (21)$$

Set $S = \cup_{i \in I} Y_i$. Clearly, $|S| \geq |I|t - 1$.

Claim 4. There exist $i \in I$ and a vertex $v \in Y_i$ such that $d_G(v, Y_{k-1}) = 0$. Furthermore, if $Y_i = Y_p$, then $v \neq z_2$.

Suppose, for a contradiction, that for each $i \in I$, $d_G(v, Y_{k-1}) \geq 1$ for all $v \in Y_i - \{z_2\}$. Then $\sum_{u \in Y_{k-1}} d_G(u, Y_{k-1} \cup S) \geq 2|Y_{k-1}| + |S| - 1 = (|I| + 2)t - 2$. Since $t \geq 3$, we have that $\lceil ((|I| + 2)t - 2)/t \rceil = |I| + 2$. Hence there exists $u \in Y_{k-1}$ such that $d_G(u, Y_{k-1} \cup S) \geq |I| + 2$. On the other hand, $d_G(u, Y_j) \geq 1$ for all $j \in \{0, 1, \ldots, k-2\} - I$. Therefore $d_G(u, \cup_{i=0}^{k-1} Y_i) \geq k+1$, a contradiction as $\Delta(G) = k$. This proves the claim.

By Claim 4, let $i_0 \in I$ and $u_{i_0} \in Y_{i_0}$ be such that $d_G(u_{i_0}, Y_{k-1}) = 0$. Furthermore, if $i_0 = p$, then $u_{i_0} \neq z_2$. Let $u_{k-1} \in Y_{k-1}$ be such that $d_G(u_{k-1}, Y_{i_0}) = 0$. Note that $u_{i_0} u_{k-1}$ is not an edge of F by the choice of Y_{k-1} and u_{i_0}. We conclude our proof of the theorem as follows.

First, we assume that $i_0 = 0$. Then it is easy to see that $N_G(u_{k-1}, N_F(y_0)) = \emptyset$, $N_G(y_{k-1}, N_F(u_{k-1})) = \emptyset$, $N_G(u_0, N_F(y_{k-1})) = \emptyset$ and $N_G(y_0, N_F(u_0)) = \emptyset$. Hence $F_{(y_0, u_{k-1}, y_{k-1}, u_0)}$ has no edges in common with G unless $y_0 y_{k-1}$ is an edge of $F_{(y_0, u_{k-1}, y_{k-1}, u_0)}$. But in that case, $u_0 u_{k-1}$ must be an edge of F, contradicting the choice of u_0.

Next, we assume that $i_0 \neq 0$. Then as in the above, it is easy to see that $F^1 = F_{(y_{i_0}, u_{k-1}, y_{k-1}, u_{i_0})}$ has only the edge $x_0 y_0$ in common with G. Since $t \geq 3$ and by the choice of Y_{k-1}, we can choose a vertex $v_{k-1} \in Y_{k-1} - \{u_{k-1}\}$ such that $d_F(v_{k-1}) = 1$. Obviously, both $u_{i_0} x_0$ and $v_{k-1} y_0$ are not edges of G. Hence $F^1_{(v_{k-1}, x_0)}$ has no edges in common with G.

In summary, we have proved the theorem.

3 References

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