Matrix Constructions of Family (A) Group Divisible Designs

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Abstract

In this note we use matrices to construct group divisible designs (GDDs). The constructions of GDDs of the form $A \otimes D + \overline{A} \otimes \overline{D}$ will be carried out in two cases. The first case uses the incidence matrix D of a GDD with a certain $(0,1)$ matrix A. The second case uses the incidence matrix D of a BIBD with A as in the first case. In both cases necessary and sufficient conditions in terms of parameters of A and D are derived for N to be the incidence matrix of a GDD. This construction yields besides regular also semi-regular and singular family(A) GDDs. Moreover, this construction produces also some known GDDs constructed earlier by several authors.

1 Introduction

A *group divisible design* (GDD) with parameters $(m, n, k, \lambda_1, \lambda_2)$ or sometimes $(m, n, b, r, k, \lambda_1, \lambda_2)$ is an incidence structure with the following properties: it has mn points and blocks of size k. The points are divided into m point classes (sometimes called groups) with n points each. Any two distinct points are covered by λ_1 or λ_2 blocks, depending on whether these points belong to the same class, or belong to distinct classes, respectively. The GDDs are further subdivided into three classes by Bose and Connor [3]:

1. Singular (S) if $r - \lambda_1 = 0$.
2. Semi-regular (SR) if $r - \lambda_1 > 0$ and $rk - v\lambda_2 = 0$.
3. Regular (R) if $r - \lambda_1 > 0$ and $rk - v\lambda_2 > 0$.

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For convenience, the following notation is used: I_v is the $v \times v$ identity matrix, $J_{v,v}$ is the $v \times v$ matrix whose entries are all 1, $J_{s,t}$ is the $s \times t$ matrix whose entries are all 1, and $\mathbf{1}_v$ is the $v \times 1$ matrix whose entries are all 1.

The following easy facts about the parameters of GDDs will be used in the sequel:

$$vr = bk, \qquad r(k-1) = (n-1)\lambda_1 + n(m-1)\lambda_2.$$

General references for GDDs can be found e.g. in the book by A.P. Street and D.J. Street [10].

Crucial in this note is the well-known fact that the incidence matrix N of a GDD which is a $v \times b$ $(0,1)$ matrix with $v = mn$ satisfies

$$N^T \mathbf{1}_{mn} = k\mathbf{1}_b, \tag{1}$$

$$NN^T = (\lambda_2 J_{m,m} + (\lambda_1 - \lambda_2)I_m) \otimes J_{n,n} + (r - \lambda_1)I_{mn}. \tag{2}$$

where $\otimes$ denotes the Kronecker product.

Matrix constructions of GDDs and more generally of PBIBDs have appeared in the recent literature. Street in [11] Theorem 2.1 has constructed PBIBDs of the forms

(a) $X_2 \otimes Y_2 + X_3 \otimes Y_3$,

(b) $X_2 \otimes Y_1 + X_3 \otimes Y_2$,

provided that

(1) $X_1 = X_2 + X_3$ and X_1, X_2 and X_3 are the incidence matrices of PBIBDs,

(2) $Y_1 = Y_2 + Y_3$ and Y_1, Y_2 and Y_3 are the incidence matrices of PBIBDs, and

(3) $X_2 X_3^T = X_3 X_2^T$ or $Y_2 Y_3^T = Y_3 Y_2^T$.

As remarked in [2] p.126 certain GDDs might be constructed using Street's technique; the reason for non-feasibility of this technique for constructing GDDs possibly lies in the too strong conditions (1), (2) and (3), where all $X_i, Y_i, 1 \le i \le 3$, are incidence matrices of PBIBDs. Arasu, Jungnickel, Haemers and Pott [2] gave a matrix construction of GDDs of the form

$$N = A \otimes J + I \otimes D, \tag{3}$$

where A is a certain square $(0,1)$ matrix which is the incidence matrix of a PBIBD. This construction enables Haemers [6] to classify all GDDs with parameters $r - \lambda_1 = 1$. However, Haemers' et al. construction [2], [6] has produced exclusively regular GDDs (see also [4]). Note that the above construction satisfies *all three conditions* of Street's theorem.

In this note we provide a matrix construction of GDDs of the form

$$N = A \otimes D + \overline{A} \otimes \overline{D}, \tag{4}$$

where A is either a BIBD or "almost" a BIBD (see Section 2 for precise statements) and D is either a BIBD or GDD; here, $\overline{A} = J - A$. Our construction is feasible although it does not satisify Street's condition (1) (conditions (2) and (3) are satisified with $X_2 = A$, $X_3 = \overline{A}$, $Y_2 = D$, $Y_3 = \overline{D}$). Moreover, our construction produces besides regular also semi-regular and singular GDDs depending on the parameters of the given A and D. **Note that all these GDDs constructed here satisfy $\mathbf{b = 4(r - \lambda_2)}$.** Following S.S. Shrikhande [9], GDDs with parameters satisfying $b = 4(r - \lambda_2)$ are called *family(A)* GDDs. Using the language of group rings Arasu and Pott [1] have also constructed Menon-type divisible difference sets (DDSs) called DDSs with *property(M)* , i.e. *symmetric* family(A) GDDs.

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2 The constructions

The construction of GDDs of the form (4) will be carried out in two cases. In the first case D is the incidence matrix of a GDD $(m, n, b, r, k, \lambda_1, \lambda_2)$ with $m, n > 1$ and $\lambda_1 \neq \lambda_2$, and in the second case D is the incidence matrix of a BIBD (v, b, r, k, λ), whereas in both cases, A is either a $(0,1)$ matrix which is the incidence matrix of a BIBD or if it satisfies all conditions of the incidence matrix of a BIBD except $A^T\mathbf{1} = k\mathbf{1}$ then $v = 2k$. The matrix calculation NN^T will be done in order to find conditions for N to be the incidence matrix of a GDD and to get its precise and explicit parameters. The GDDs so constructed can be either singular, semi-regular, or regular.

Theorem 1 *(i) Let A be a $(0,1)$ matrix of size $v' \times b'$ that satisifies*

$$AA^T = (r' - \lambda')I_{v'} + \lambda' J_{v',v'},$$

where r' and λ' are integers, and D the incidence matrix of a GDD $(m, n, b, r, k, \lambda_1, \lambda_2)$ with $m, n > 1$ and $\lambda_1 \neq \lambda_2$, and either one of the following conditions

(a) $A^T\mathbf{1}_{v'} = k'\mathbf{1}_{b'}$ i.e. A is a BIBD$(v', b', r', k', \lambda')$ or

(b) if $A^T\mathbf{1}_{v'} \neq k'\mathbf{1}_{b'}$ then $v = mn = 2k$.

Then

$$N = A \otimes D + \overline{A} \otimes \overline{D}$$

is the incidence matrix of a GDD with parameters

$$(m^*, n^*, b^*, r^*, k^*, \lambda_1^*, \lambda_2^*)$$

$$= \begin{cases} (v'm, n, b'b, (b'-r')(b-r) + r'r, (v'-k')(v-k) + k'k, \\ \quad (b'-r')(b-2r+\lambda_1) + r'\lambda_1, (b'-2r'+\lambda')(b-r) + \lambda' r) \\ \quad \text{in case (a)}, \\ (v'm, n, b'b, b'r, v'k, b'\lambda_1, b'\lambda_2) \qquad \text{in case (b)}, \end{cases}$$

if and only if $b' = 4(r' - \lambda')$ *and* $b = 4(r - \lambda_2)$. *Moreover,* N *is singular iff* D *is also singular. Otherwise,* N *is regular iff* $v \neq 2k$.

(ii) *Let* A *be as in (i) and* D *the the incidence matrix of a GDD* $(m, n, b, r, k, \lambda_1, \lambda_2)$ *with* $m = 1$ *or* $n = 1$ *or* $\lambda_1 = \lambda_2 = \lambda$, *i.e. a BIBD*$(v, b, r, k, \lambda)$, *then*

$$N = A \otimes D + \overline{A} \otimes \overline{D}$$

is the incidence matrix of a GDD with parameters

$$(m^*, n^*, b^*, r^*, k^*, \lambda_1^*, \lambda_2^*)$$

$$= \begin{cases} (v', v, b'b, (b' - r')(b - r) + r'r, (v' - k')(v - k) + k'k, \\ \quad (b' - r')(b - 2r + \lambda) + r'\lambda, (b' - 2r' + \lambda')(b - r) + \lambda' r) \\ \quad \text{in case (a)}, \\ (v', v, b'b, b'r, v'k, b'\lambda, \frac{b'r}{2}) \qquad \text{in case (b)}, \end{cases}$$

if and only if $b' = 4(r' - \lambda')$. *Moreover,* N *is regular iff* $v \neq 2k$.

Proof: (i)

$$\begin{aligned} N^T \mathbf{1}_{v'v} &= (A \otimes D + \overline{A} \otimes \overline{D})^T \mathbf{1}_{v'v} \\ &= (A^T \otimes D^T)\mathbf{1}_{v'v} + (\overline{A}^T \otimes \overline{D}^T)\mathbf{1}_{v'v} \\ &= \begin{cases} v'k\mathbf{1}_{b'b} & \text{in case (a)} \\ k'k + (v' - k')(v - k)\mathbf{1}_{b'b} & \text{in case (b).} \end{cases} \end{aligned}$$

Now we calculate NN^T :

$$\begin{aligned} NN^T &= (A \otimes D + \overline{A} \otimes \overline{D})(A \otimes D + \overline{A} \otimes \overline{D})^T \\ &= (AA^T \otimes DD^T) + (\overline{A}A^T \otimes \overline{D}D^T) + (A\overline{A}^T \otimes D\overline{D}^T) + \\ &\quad (\overline{A}\overline{A}^T \otimes \overline{D}\overline{D}^T) \\ &= ((r' - \lambda')I_{v'} + \lambda' J_{v',v'}) \otimes [((\lambda_1 - \lambda_2)I_m + \lambda_2 J_{m,m}) \otimes J_{n,n} + (r - \lambda_1)I_{mn}] + \\ &\quad 2((r' - \lambda')(J_{v',v'} - I_{v'})) \otimes \\ &\quad [((r - \lambda_2)J_{m,m} - (\lambda_1 - \lambda_2)I_m) \otimes J_{n,n} - (r - \lambda_1)I_{mn}] + \\ &\quad ((r' - \lambda')I_{v'} + (b' - 2r' + \lambda')J_{v',v'}) \otimes \\ &\quad [((\lambda_1 - \lambda_2)I_m + (b - 2r + \lambda_2)J_{m,m}) \otimes J_{n,n} + (r - \lambda_1)I_{mn}] \\ &= [((r' - \lambda')I_{v'} + \lambda' J_{v',v'}) \otimes ((\lambda_1 - \lambda_2)I_m + \lambda_2 J_{m,m}) + \\ &\quad 2(r' - \lambda')(J_{v',v'} - I_{v'}) \otimes ((r - \lambda_2)J_{m,m} - (\lambda_1 - \lambda_2)I_m) + \\ &\quad ((r' - \lambda')I_{v'} + (b' - 2r' + \lambda')J_{v',v'}) \otimes ((\lambda_1 - \lambda_2)I_m + (b - 2r + \lambda_2)J_{m,m})] \otimes J_{n,n} \\ &\quad + [\lambda'(r - \lambda_1) - 2(r - \lambda_1)(r' - \lambda') + (r - \lambda_1)(b' - 2r' + \lambda')]J_{v',v'} \otimes I_{v,v} + \\ &\quad [(r - \lambda_1)(r' - \lambda') + 2(r - \lambda_1)(r' - \lambda') + (r - \lambda_1)(r' - \lambda')]I_{v'v} \\ &= [4(r' - \lambda')(\lambda_1 - \lambda_2)I_{v'm} + ((b' - 2r' + \lambda')(b - r) + \lambda' r)J_{v'm,v'm} + \\ (*) \quad &\quad (r' - \lambda')(b - 4r + 4\lambda_2)I_{v'} \otimes J_{m,m} + (b' - 4r' + 4\lambda')(\lambda_1 - \lambda_2)J_{v',v'} \otimes I_m] \otimes J_{n,n} \\ &\quad + (r - \lambda_1)(b' - 4r' + 4\lambda')J_{v',v'} \otimes I_{v,v} + 4(r - \lambda_1)(r' - \lambda')I_{v'v}, \end{aligned}$$

then N is the incidence matrix of a GDD iff the coefficients of $I_{v'} \otimes J_{m,m}$, $J_{v',v'} \otimes I_m$, and $J_{v',v'} \otimes I_{v,v}$ in $(*)$ are zero iff $b' = 4(r' - \lambda')$ and $b = 4(r - \lambda_2)$. So we have

$$\begin{aligned} NN^T &= [b'(\lambda_1 - \lambda_2)I_{v'm} + ((b' - 2r' + \lambda')(b - r) + \lambda' r)J_{v'm,v'm}] \otimes J_{n,n} + \\ &\quad b'(r - \lambda_1)I_{v'v} \qquad \text{in case (a)}, \end{aligned}$$

and

$$NN^T = [b'(\lambda_1 - \lambda_2)I_{v'm} + b'\lambda_2 J_{v'm,v'm}] \otimes J_{n,n} + b'(r - \lambda_1)I_{v'v} \qquad \text{in case (b)},$$

and we get the desired GDDs.

Since

$$\begin{aligned} r^* - \lambda_1^* &= (b' - r')(b - r) + r'r - ((b' - r')(b - 2r + \lambda_1) + r'\lambda_1) \\ &= (b' - r')(r - \lambda_1) + r'(r - \lambda_1) \\ &= b'(r - \lambda_1) \geq 0, \end{aligned}$$

and using the following equations (5), (6)

$$v'r' = b'k' = 4(r' - \lambda')k', \tag{5}$$

$$\begin{aligned} (v' - 1)\lambda' &= r'(k' - 1) \\ \Rightarrow \quad v'\lambda' &= r'k' - (r' - \lambda'), \end{aligned} \tag{6}$$

we have

$$(v' - 1)(r' - \lambda') = 3r'k' - 4\lambda'k'. \tag{7}$$

So we can use the above equations to show

$$\begin{aligned} r^*k^* - v^*\lambda_2^* &= [(b' - r')(b - r) + r'r][(v' - k')(v - k) + k'k] - \\ &\quad v'v[(b' - 2r' + \lambda')(b - r) + \lambda' r] \\ &= (r' - \lambda')v'vb + (3r'b - 4\lambda'b - 2r'r + 4\lambda'r)(2k'k - k'v - v'k) \\ &= (r' - \lambda')v'vb + vb(4k'\lambda' - 3r'k') + \\ &\quad bk(-3v'r' + 8r'k' + 4v'\lambda' - 12k'\lambda') \\ &\quad + rk(2v'r' - 4r'k' - 4v'\lambda' + 8k'\lambda') \\ &= (r' - \lambda')v'vb - vb[(v' - 1)(r' - \lambda')] - bk[4(r' - \lambda')] + \\ &\quad rk[4(r' - \lambda')] \qquad \text{(Using (5),(6), and (7))} \\ &= (r' - \lambda')(vb - 4bk + 4rk) \\ &= (r' - \lambda')\frac{r}{k}(v^2 - 4vk + 4k^2) \\ &= \frac{r}{k}(r' - \lambda')(v - 2k)^2 \geq 0. \end{aligned}$$

Thus if $r = \lambda_1$ we get a singular GDD. Otherwise, if $v = 2k$ we get a semi-regular GDD and if $v \neq 2k$ we get a regular GDD.

(ii) If $n = 1$ or $\lambda_1 = \lambda_2 = \lambda$ (analogously for $m = 1$) then

$$\begin{aligned}(*) \quad = \quad & [(r' - \lambda')(b - 4r + 4\lambda)I_{v'} + ((b' - 2r' + \lambda')(b - r) + \lambda' r)J_{v'.v'}] \otimes J_{v,v} \\ & +(r - \lambda)(b' - 4r' + 4\lambda')J_{v',v'} \otimes I_{v,v} + 4(r' - \lambda')(r - \lambda)I_{v'v},\end{aligned}$$

so N is the incidence matrix of a GDD iff $b' = 4(r' - \lambda')$, then

$$\begin{aligned}NN^T \quad = \quad & [(r' - \lambda')(b - 4r + 4\lambda)I_{v'} + ((b' - 2r' + \lambda')(b - r) + \lambda' r)J_{v'.v'}] \otimes J_{v,v} \\ & +b'(r - \lambda)I_{v'v} \qquad \text{in case (a)},\end{aligned}$$

$$NN^T = [\frac{b'}{4}(4\lambda - 2r)I_{v'} + \frac{b'}{2} r J_{v'.v'}] \otimes J_{v,v} + b'(r - \lambda)I_{v'v} \qquad \text{in case (b)},$$

and we get the desired GDDs. Similarly, if $v = 2k$ we get a semi-regular GDD and if $v \neq 2k$ we get a regular GDD. ∎

In the following two Corollaries and two examples, we show how our Theorem can be applied to produce new family(A) semi-regualr GDDs. To this end, we either extend the matrix A occurring in Theorem 1, or delete some rows from it, in order to satisfy the sufficient conditions of the Theorem.

Corollary 1 *(i) Let A and D be as in Theorem 1 (i) with $b' \leq 4(r' - \lambda')$, $b = 4(r - \lambda_2)$, and $v = mn = 2k$, then A can be extended to A^{ext} so that*

$$N = A^{ext} \otimes D + \overline{A}^{ext} \otimes \overline{D}$$

is the incidence matrix of a semi-regular (singular if $r = \lambda_1$) GDD with parameters

$$(v'm, n, b'^{ext} b, b'^{ext} r, v'k, b'^{ext} \lambda_1, b'^{ext} \lambda_2).$$

(ii) Let A and D be as in Theorem 1 (ii) with $b' \leq 4(r' - \lambda')$ and $v = 2k$, then A can be extended to A^{ext} so that N above is the incidence matrix of a semi-regular GDD with parameters

$$(v', v, b'^{ext} b, b'^{ext} r, v'k, b'^{ext} \lambda, \frac{b'^{ext} r}{2}),$$

where $b'^{ext} := 4(r' - \lambda')$.

Proof: If $b' < 4(r' - \lambda')$ then we can extend A as follows

$$A^{ext} = [A \; 0_{v',4(r'-\lambda')-b'-t} \; J_{v',t}],$$

where $0 \leq t \leq 4(r' - \lambda') - b'$, so we can apply Theorem 1 with A^{ext}. Moreover,

$$r^{ext} - \lambda^{ext} = r' - \lambda',$$

we get the GDDs with parameters as above. ∎

Corollary 2 *(i) Let A' be a BIBD$(v', b', r', k', \lambda')$ with $b' = 4(r' - \lambda')$ and D the incidence matrix of a GDD $(m, n, b, r, k, \lambda_1, \lambda_2)$ with $v = mn = 2k$, $b = 4(r - \lambda_2)$, $m, n > 1$ and $\lambda_1 \neq \lambda_2$, let A be the matrix obtained by deleting i rows, $0 \leq i \leq (v' - 2)$, from A', then there are semi-regular (singular if $r = \lambda_1$) GDDs with parameters*

$$((v' - i)m, n, b'b, b'r, (v' - i)k, b'\lambda_1, b'\lambda_2).$$

(ii) Let A' be a BIBD$(v', b', r', k', \lambda')$ with $b' = 4(r' - \lambda')$ and D is the incidence matrix of a BIBD(v, b, r, k, λ) with $v = 2k$, let A be the matrix obtained by deleting i rows, $0 \leq i \leq (v' - 2)$, from A', then there are semi-regular GDDs with parameters

$$(v' - i, v, b'b, b'r, (v' - i)k, b'\lambda, \frac{b'r}{2}).$$

∎

Example 1 *If a Hadamard matrix of order $4s$ exists, then there is a BIBD$(4s - 1, 2s - 1, s - 1)$. Let A be the incidence matrix of BIBD$(4s - 1, 2s - 1, s - 1)$ and extend A as follows*

$$A^{ext} = [A\ \mathbf{1}_{4s-1}],$$

or

$$A^{ext} = [A\ \mathbf{0}_{4s-1}].$$

Then we can delete i rows, $0 \leq i \leq 4s - 3$, from A^{ext}, and we get the following two cases:

(i) if D is the incidence matrix of a GDD $(m, n, b, r, k, \lambda_1, \lambda_2)$ with $v = mn = 2k$, $b = 4(r - \lambda_2)$, $m, n > 1$ and $\lambda_1 \neq \lambda_2$, then we get GDDs with parameters

$$((4s - 1 - i)m, n, 4sb, 4sr, (4s - 1 - i)k, 4s\lambda_1, 4s\lambda_2).$$

(ii) if D is the incidence matrix of a BIBD(v, b, r, k, λ) with $v = 2k$, then we get GDDs with parameters

$$((4s - 1 - i), v, 4sb, 4sr, (4s - 1 - i)k, 4s\lambda, 2sr).$$

∎

Note that our example is Theorem 2.2 and 2.3 of [7] and it generalizes Theorem 1.2 – 1.6 of [8].

Example 2 *It has been shown by Shrikhande [9] that there are two subfamilies (A_1) and (A_2) of family(A) BIBD with parameters*

$A_1(s, t) : (s^2, 2st, (s - 1)t, \frac{s(s-1)}{2}, \frac{(s-2)t}{2})$ *when s is even, and $2t \geq s$,*
$A_2(s, t) : (s^2, 4st, 2(s - 1)t, \frac{s(s-1)}{2}, (s - 2)t)$ *when s is odd, and $4t \geq s$.*

Then using our Theorem 1 with A as $A_1(s, t)$ and $A_2(s, t)$, we get the following GDDs.

(i) If D is the incidence matrix of a GDD $(m, n, b, r, k, \lambda_1, \lambda_2)$ with $b = 4(r - \lambda_2)$, $m, n > 1$ and $\lambda_1 \neq \lambda_2$, then we get the GDDs with parameters

$$(s^2m, n, 2stb, ((s+1)b - 2r)t, \tfrac{s(s+1)}{2}v - sk, \\ ((s+1)(b-2r) + 2s\lambda_1)t, (\tfrac{s+2}{2}b - 2r)t) \quad \textit{and,} \\ (s^2m, n, 4stb, 2((s+1)b - 2r)t, \tfrac{s(s+1)}{2}v - sk, \\ 2((s+1)(b-2r) + 2s\lambda_1)t, ((s+2)b - 4r)t), \quad \textit{respectively.}$$

(ii) If D is the incidence matrix of a BIBD(v, b, r, k, λ) then we get the GDDs with parameters

$$(s^2, v, 2stb, ((s+1)b - 2r)t, \tfrac{s(s+1)}{2}v - sk, \\ ((s+1)(b-2r) + 2s\lambda)t, (\tfrac{s+2}{2}b - 2r)t) \quad \textit{and,} \\ (s^2, v, 4stb, 2((s+1)b - 2r)t, \tfrac{s(s+1)}{2}v - sk, \\ 2((s+1)(b-2r) + 2s\lambda)t, ((s+2)b - 4r)t), \quad \textit{respectively.}$$

∎

Note that if D is the incidence matrix of a BIBD$(v, v, 1, 1, 0)$ in (ii) then we get Theorem 2 of [5].

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