SINGLE CHANGE NEIGHBOR DESIGNS

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Abstract. A single change neighbor design $SCN(v,k)$ is a way of selecting a sequence of cycles of length k from a complete graph on v vertices with the properties: any cycle is derived from the preceding one by changing one vertex; and every edge is covered in at least one cycle. Some properties of these designs are investigated. In particular, designs with the smallest possible number of cycles are constructed for all v when $k = 4$.

1. Introduction.

We shall define a general *block design* (V, B) with parameters (v, k) to consist of a v-set V together with a family B of subsets of size k (called *blocks*) of V. A covering design of index λ is a block design such that, given two members x and y of V, there are at least λ blocks in B which contain $\{x, y\}$. A general discussion of covering designs is found in [1].

Single-change covering designs are discussed, for example in [3]. A *single-change covering design* $SC(v, k)$ consists of a covering design of index 1 with parameters (v, k), together with an ordering $B = (B_1, B_2, \ldots, B_b)$ of the block-set B, such that for $1 \le i < b$, $B_{i+1} \backslash B_i$ has exactly one element. For practical purposes it is desirable to minimize the number b of blocks in an $SC(v, k)$, which we shall call

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the *size* of the design.

A block design can be defined in graph-theoretic terms. If the complete graph on n vertices is denoted K_n, then the set V can be interpreted as the vertex-set of a K_v, and the blocks as cliques K_k in that K_v. A covering design of index 1 is a clique covering of K_v with all cliques of size k.

Cliques are the appropriate data structures in experiments where some k objects can be treated simultaneously in one instance of the experiment, and every pair among the k objects interact to the same degree. In other experiments, other data structures are appropriate. We wish to consider neighbor designs, in which the appropriate data structure is a cycle. For example, in bacteriological experiments, cultures are sometimes grown around the rim of a circular plate; adjacent cultures interact, but not others on the same plate. For a discussion of neighbor designs, and further references, see [2].

We define a *single change neighbor design* with parameters (v,k), or $SCN(v,k)$, to be an ordered list of cycles $C_1, C_2, \ldots, C_b$ of length k chosen from a complete graph K_v on v vertices, with the following properties:

(i) every edge of the K_v occurs in at least one of the cycles;

(ii) For $1 \le i < b$, C_{i+1} can be obtained from C_i by replacing one vertex by a different vertex of K_v.

If C_{i+1} is formed by replacing vertex x in C_i by vertex y, we say that x is *introduced*, and y is *dropped*, in C_{i+1}. All elements of C_1 are introduced in C_1.

We can always assume that $k \ge 3$, so that cycles are defined. When $k = 3$, an $SCN(v,k)$ is just an $SC(v,k)$, because a 3-cycle is a K_3. Single-change covering designs are studied in [3], and the spectrum of sizes for which there exists an $SC(v,3)$ is completely determined there. So we shall concentrate on the cases with $k \ge 4$. In that case it is obvious that $v \ge k+1$ is a necessary condition for the existence of an $SCN(v,k)$.

Several examples of SCNs are presented in Table 3.3, and will be used later. For convenience each cycle is represented as a column of an array. In order to better indicate the structure, we only show an element when it is introduced; thus the columns

$$\begin{array}{cc} 1 & \cdot \\ 2 & \cdot \\ 3 & \cdot \\ 4 & 5 \end{array}$$

represent the two cycles $(1,2,3,4)$ and $(1,2,3,5)$.

2. Elementary Lower Bounds.

Without loss of generality, suppose the first two blocks of an $SCN(v,k)$ are $C_1 = (1,2,\ldots,k-1,k)$ and $C_2 = (1,2,\ldots,k-1,k+1)$. Then C_1 contains the k edges $(1,2)$, $(2,3)$, $\ldots$, $(k,1)$, and C_2 contains some edges which are in C_1 and also $(k-1,k+1)$ and $(k+1,1)$. In general, C_{i+1} can cover at most two edges which were not in any of the earlier cycles in the design; so an SCN of size b can cover at most $k+2(b-1)$ edges. So we must have

$$\binom{v}{2} \leq k+2(b-1),$$

or

$$b \geq \left[\binom{v}{2} - k + 2\right] \Big/ 2$$
$$= \frac{v(v-1)-2k+4}{4}.$$

Since b must be an integer, we have:

Lemma 1. *If an $SCN(v,k)$ has size b, then*

$$b \geq \left\lceil \frac{v(v-1)-2k+4}{4} \right\rceil. \tag{2.1}$$

A design which attains the minimum size, so that (2.1) is an equality, is called *economical.* If v and k are such that $v(v-1)-2k+4$ is divisible by 4, we say case (v,k) is *tight.* An economical SCN in a tight case is called a *tight SCN*.

Lemma 2. *If k is even then case (v,k) is tight if and only if $v \equiv 0$ or $1 \pmod 4$. If k is odd then case (v,k) is tight if and only if $v \equiv 2$ or $3 \pmod 4$.*

3. Small Cases.

We have investigated small single-change neighbor designs with $k < v \leq 10$, by hand and by computer. Table 3.1 lists the economical value of b for each v and k, as given by Lemma 1. Tight cases are denoted by asterisks.

In every case where $v = k+1$, it was found that the economical design did not exist. In fact, there was no $SCN(10,9)$ with 20 blocks, and this is the only case in which the smallest existing design was found to have two blocks more than the number calculated for an economical design. So the sizes of possible designs with $k < v \leq 10$ are as shown in Table 3.2. Examples of the designs are given in Table 3.3 (see pages 252, 253).

In view of the non-existence of an economical $SCN(10,9)$, we were interested to know whether there is an economical $SCN(11,9)$. Such a design exists. Three designs on more than ten elements are needed in Section 4 below; they are $SCN(11,4)$, $SCN(12,4)$ and $SCN(13,4)$. Examples of these are given in Table 3.4 (see page 254).

v \ k	4	5	6	7	8	9
5	4*					
6	7	6*				
7	10	9*	9			
8	13*	13	12*	12		
9	17*	17	16*	16	15*	
10	22	21*	21	20*	20	19*

Sizes of economical designs
TABLE 3.1

v \ k	4	5	6	7	8	9
5	5					
6	7	7				
7	10	10	10			
8	13	13	13	13		
9	17	17	16	16	16	
10	22	21	21	20	20	21

Smallest designs existing
TABLE 3.2

4. The Case $SCN(v, 4)$.

We have the following extension on v for an economical $SCN(v, 4)$.

Lemma 3. *An economical $SCN(v,4)$ can be extended to an economical $SCN(v+8q, 4)$ for all $q > 0$.*

Proof. Suppose there exists an $SCN(v, 4)$ with b cycles $C_1, C_2, \ldots, C_b$, based on a v-set S of symbols. We show how to construct an $SCN(v+8, 4)$ based on $S \cup \{1,2,3,4,5,6,7,8\}$ with $b + 4v + 14$ cycles. From (2.1), if the original design was economical, the new design will also be economical. Repeated application of this construction proves the Lemma.

The first b cycles will be $C_1, C_2, \ldots, C_b$. Suppose $C_b = (w, x, y, z)$. Then the next four cycles are $(1, x, y, z)$, $(1, x, 3, z)$, $(1, 2, 3, z)$, $(1, 2, 3, 4)$. There is an economical $SCN(8, 4)$ with first cycle $(1, 2, 3, 4)$ and last (thirteenth) cycle $(8, 7, 3, 6)$–see Table 3.3; cycles $b+4$ to $b+16$ will be the thirteen cycles of that design. Then the sequence from block $b + 16$ onward is

$$\begin{array}{cccccccccccccccc}
8 & y & y & & y & y & y & & y & y & y & & y & y & y & \\
7 & 7 & 7 & & 7 & 2 & 2 & & 2 & 5 & 5 & & 5 & 1 & 1 & \\
3 & 3 & x & P & w & w & w & P & x & x & x & P & w & w & w & Q \\
6 & 6 & 6 & & 6 & 6 & 4 & & 4 & 4 & 8 & & 8 & 8 & 3 &
\end{array}$$

where P denotes the sequence of $v - 3$ cycles obtained by replacing the third entry successively with members of $S \backslash \{w, x, y\}$, and Q is similar but only members of $S \backslash \{w, x, y, z\}$ are used.

It will be seen that the first sixteen new blocks cover all pairs with both members in $\{1,2,3,4,5,6,7,8\}$, and also $(1, x)$, $(3, x)$, $(1, z)$, $(3, z)$. The other blocks cover the remaining pairs with one member in S. The total number of cycles is

$$\begin{aligned} & b + 16 + 11 + 3(v-3) + v - 4 \\ &= b + 4v + 14, \end{aligned}$$

as required.

Theorem. *There exists an economical $SCN(v,4)$ for each $v \geq 6$.*

Proof. This is an immediate consequence of Lemma 3, and the economical cases of $SCN(v,4)$ for $v = 6,7,\ldots,13$ given in Tables 3.3 and 3.4 (which were constructed by computer).

5. Some extensions.

Several ways of extending the results of this paper present themselves. Some are obvious – for example, can one always construct an economical $SCN(v,5)$? We shall elaborate on four topics.

(1) Does there ever exist a tight economical $SCN(k+1,k)$? None has yet been constructed. An exhaustive search shows that no economical $SCN(k+1,k)$ exists for $k < 10$. We recently constructed an economical $SCN(11,10)$ (see Table 3.4), but this case is not tight. This is the limit of our knowledge.

(2) Isomorphism. We would call two different $SCN(v,k)$ *isomorphic* if one can be obtained from the other by a permutation of elements. For given v and k, what can be said about the number of equivalence classes under isomorphism? In particular, one can ask whether a given design is isomorphic to the one derived by reversing the order of its blocks – starting with the last block and proceeding to the first. The $5-(5,4)$ design in Table 3.3 has this property. We expect this to be a rare occurrence; we conjecture that the $5-(5,4)$ design is the only "best-possible" design with this property.

(3) Some of the tabulated designs have a "fixed" element which is never dropped. Examples include the 5-$(5,4)$, 7-$(6,4)$, 10-$(7,4)$, 13-$(8,4)$ and 13-$(8,5)$ designs of Table 3.3. A computation shows that there is a design with a fixed element in all the "best-possible" cases with $v < 11$ except for the case $v = 10, k = 4$ which is still undecided. What are the conditions for a fixed-element design to be possible? The 5-$(5,4)$, 7-$(6,4)$ and 10-$(7,4)$ designs of Table 3.3 all in fact have two fixed elements. This never happens in larger economical designs with $k = 4$: if elements 1 and 2 are fixed, then as we go through all the blocks the remaining pair of elements must run through all pairs from $\{3,4,\ldots,v\}$, and there are $(v-2)(v-3)/2$ such pairs; when $v > 7$ this is greater than $(v(v-1)-2)/4$.

(4) In some designs, such as the 5-$(5,4)$, 7-$(6,4)$, 10-$(7,4)$, 7-$(6,5)$ and 13-$(8,5)$ of Table 3.3, all but one of the elements in the last block are also in the first block. No 13-$(8,4)$ design of this kind exists. What are the conditions for this to be possible? If the conditions are satisfied for a particular parameter-set, computation shows that quite frequently there are at least two designs for that parameter-set, one having first and last block that cover just $k+1$ elements and the other not. What are the precise conditions for this to happen? In the 5-$(5,4)$ and 10-$(7,4)$ designs, transition from the last to the first block can be made by a single change. We conjecture that this never happens in larger economical designs.

TABLE 3.3

Examples of small designs. T stands for 10.

$b-(v,k)$ denotes an $SCN(v,k)$ with b blocks.

1	·	·	·	·
2	·	·	·	·
3	·	4	·	5
4	5	·	3	·

5-(5, 4)

1	·	·	·	·	·	·
2	·	·	·	·	·	·
3	·	4	·	5	·	6
4	5	·	6	·	3	·

7-(6, 4)

1	·	·	·	·	·	·	·	·	·
2	·	·	·	·	·	·	·	·	·
3	·	4	·	5	·	6	·	7	·
4	5	·	6	·	3	·	7	·	4

10-(7, 4)

1	·	4	6	·	·	2	·	1	5	4	·	8
2	·	·	·	1	7	·	·	·	·	·	·	·
3	·	·	·	·	·	·	·	·	·	·	·	·
4	5	·	·	·	·	·	8	·	·	·	6	·

13-(8, 4)

1	·	·	·	·	·	·	9	·	·	·	·	·	·	·	8	·
2	·	·	·	·	·	3	·	1	4	·	·	5	·	·	·	·
3	·	·	·	4	8	·	·	·	·	5	6	·	2	·	·	·
4	5	6	7	·	·	·	·	·	·	·	·	·	·	6	·	9

17-(9, 4)

1	·	·	·	·	9	·	·	·	·	1	7	·	5	·	·	·	·	·	·	·	·
2	·	·	·	8	·	T	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
3	·	6	·	·	·	·	·	·	·	·	·	·	·	·	4	·	·	3	2	·	8
4	5	·	7	·	·	·	1	2	3	·	·	4	·	9	·	2	8	·	·	7	·

22-(10, 4)

1	·	·	·	·	·	·
2	·	·	4	·	·	·
3	·	·	·	2	·	·
4	·	5	·	·	·	6
5	6	·	·	·	3	·

7-(6, 5)

1	·	·	·	·	·	3	·	·	·
2	·	·	·	·	·	·	1	·	·
3	·	·	·	5	4	·	·	·	7
4	·	5	7	·	·	·	·	2	·
5	6	·	·	·	·	·	·	·	·

10-(7, 5)

1	·	·	·	·	·	·	·	·	·	·	·	·
2	·	·	·	·	·	·	·	·	·	4	·	·
3	·	·	·	·	·	5	·	·	6	·	2	·
4	·	·	·	5	6	·	7	·	·	·	·	8
5	6	7	8	·	·	·	·	3	·	·	·	·

13-(8, 5)

1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
2	·	·	·	·	·	·	·	·	·	·	·	3	·	·	·	·
3	·	·	·	·	·	·	5	·	·	·	7	·	9	·	·	8
4	·	·	·	·	5	6	·	7	·	8	·	·	·	2	·	·
5	6	7	8	9	·	·	·	·	6	·	·	·	·	·	4	·

17-(9, 5)

1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
2	·	·	·	·	·	·	·	·	·	·	·	·	·	·	3	·	·	·	·	·
3	·	·	·	·	·	·	6	·	·	8	5	·	·	7	·	8	·	9	·	·
4	·	·	·	·	·	5	·	3	7	·	·	8	9	·	·	·	6	·	2	·
5	6	7	8	9	T	·	·	·	·	·	·	·	·	·	·	·	·	·	·	4

21-(10, 5)

1	·	·	·	4	·	·	·	·	3
2	·	·	·	·	·	·	6	·	·
3	·	6	·	·	·	7	·	·	·
4	·	·	3	·	·	·	·	2	·
5	·	·	·	·	·	·	·	·	·
6	7	·	·	·	1	·	·	·	·

10-(7, 6)

1	·	·	·	·	·	4	·	·	·	·	·	3
2	·	·	·	·	·	·	1	·	·	·	·	·
3	·	·	·	·	·	·	·	·	·	·	5	·
4	·	·	·	·	6	·	·	·	5	7	·	·
5	·	·	6	7	·	·	·	2	·	·	·	·
6	7	8	·	·	·	·	·	·	·	·	·	·

13-(8, 6)

1 . . . . 5 . . . 8
2 . . . . 3 . . . .
3 . . . . . . 6 . . . . . . 9 .
4 . . . . . 7 . . . .
5 . . 6 . 8 . . . 2 4 . . . . .
6 7 8 . 9 . . . . 1 2 . .

16-(9, 6)

1 . . . .
2 . . . . 3 . . 4 . . . . . .
3 . . . . 4 9 . . T . 7 . 9 . . .
4 . . . . . . 7 8 . . . . T . 7 6 .
5 . . . . 6 . . . . . . 2 3
6 7 8 9 T . 5 . . . .

21-(10, 6)

1 . . . . 3 .
2 8 . . 6 1
3 . . . . 8 . . . 4 . . .
4 . . . . 7 . . . .
5 . 2 . . . .
6 . . 5 . . 3 . . . 8 . .
7 . . . . . . 5 . . . . .

13-(8, 7)

1 . . . .
2 . . . . 5 . . . 6
3 . . . . 5 . 4 . . . . .
4 . . . . 5 . 9 . . . .
5 . . . 8 . . . . 2 . . .
6 . . 7 . . . . 6 8 .
7 8 9 . . . 4 . . 3 . . . . . .

16-(9, 7)

1 3
2 . . . . 3 4 . . 6 . . . . .
3 . . . . 8 . . . 2 . 4 . . . .
4 . . . . 7 . . . . T .
5 . . . . . 9 . . . .
6 . . . 7 8 . 5 . . . . 3 . . . 2 . . .
7 8 9 T . . . . 5 . .

20-(10, 7)

1 . . . . 2 .
2 . . . . . . 3 . 4 . . . . . .
3 . . . . . 8 . . . .
4 . . . 5 . . . .
5 . . 7 . . . . 2 . 9 . 1
6 . . . .
7 . 8 . . 4 . . 2 . 3 . . . . .
8 9 . . . . 7 . . .

16-(9, 8)

1 . . . . 2 . . . . .
2 . . . . 4 . . . .
3 . . . . 8 . . . .
4 . . . . . 5 3
5 . . . . 9 . . . . . . 2 7 . 1 . . . .
6 . . . T . . . .
7 . 8 . . . . 4 . . 3 . . . . 6 .
8 9 . 7 . . . . 6 . . . . 7 9 . .

20-(10, 8)

1 . . . . 8 . . . .
2 . . . . 4 . . . . . 1 . . . .
3 . . . . . 9 . T . . . . 5 . .
4 . . . . 9 . . . .
5 . . . 6 3
6 . . 8 . . . . 2 . 1 . 3 . . . T .
7 . . . .
8 . 9 . . 5 . . . . 4 . . .
9 T . . . . . 3 . . . . . . 2 . . . . . .

21-(10, 9)

TABLE 3.4

Five important examples (T, E, D, R stand for ten, eleven, twelve, thirteen).

1 · · · · · · · · · · · 2 · · · · · · 5 · 4 · ·
2 · · · · · · 8 · · · · · · · · · · · · · · · ·
3 · · · · · · · · · · · · · · 6 · · · · · · · ·
4 · · · · · 7 · · · · · · 1 · · 3 T · · · · · E
5 · · · · · · · · · · · · · · · · · 3 · · · · ·
6 · · · · · · · · · 2 9 · · · · · · · · · · · ·
7 · · · · 9 · · 2 4 · · · · · · · · · · 2 · · ·
8 · · 9 T · · · · · · · · · 7 · · · · · · · · ·
9 T E · · · · · · · · · · · · · · · · · · · 1 ·
24-(11, 9)

1 · · · · · · · · · · · · · · · · · · E · · · · 5 7 ·
2 · · · · · · · · · 3 E · · · · · · · · · · · · · · ·
3 · · · · 5 6 7 4 9 · · 2 · 3 8 4 · · · · · 6 · · · ·
4 5 6 7 8 · · · · · · · · E · · · 9 6 · 5 7 · 9 · · 5
27-(11, 4)

1 · · · · · · · · · · · · · · · · · · · D · · 6 · · · · 5 8 · ·
2 · · · · · · · · · · · 3 E · · · · · · · · · · · · · · · · · ·
3 · · · · · 5 6 7 8 4 T · · 2 · 3 9 4 · · · · · · · 7 · · · · ·
4 5 6 7 8 9 · · · · · · · · · D · · · T · 6 5 · 7 8 · T · · D 5
32-(12, 4)

1 · R · · · · · 6 · · 9 · · 7 · · ·
2 · · · · · · · · · · · · · 3 D ·
3 · · · · · · 5 6 7 8 9 4 E · · 2 · 3 T 4 · · · · · · · · 5 · · · · · · 8 ·
4 5 6 7 8 9 T · · · · · · · · · · R · · · E · 6 5 7 8 9 · · E · 7 8 · 6 · E
38-(13, 4)

1 ·
2 · · · · · · · · 3 · · 8 · · 9 · · · · · · · ·
3 · · · · · · · 6 · · · · · · · · · · · · · · 2
4 · · · · · · · · · · · · · · · · E · · · · · ·
5 · · · · · 9 · · · · · · · E · 8 · · · · · · ·
6 · · 9 · E · · · · · · · 3 · · · · · 5 · · · ·
7 · · · · · · · · · · · · · · · · · · · 3 · · ·
8 · · · · · · · · · · T · · · · · · · · · · · ·
9 · T · · · · · · · 2 · · · · · · · · · · · 4 ·
T E · · 6 · · 5 · · · · · · · · · · 4 · · 7 · ·
24–(11, 10)

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