On Hamilton Cycles And Contractible Edges In 3-Connected Graphs

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ABSTRACT

It has previously been shown that all 3-connected graphs having a longest cycle containing less than six contractible edges of the parent graph are hamiltonian; moreover, those having longest cycles containing less than four contractible edges have been characterized. In this paper we characterize the remaining ones.

INTRODUCTION

An edge in a 3-connected graph is called *contractible* if the graph resulting from contracting that edge is also 3-connected. In this paper we continue the study of contractible edges contained in longest cycles. For $k \geq 0$, let C_k denote the class of 3-connected graphs defined as follows:

> $H \in C_k$ iff all longest cycles in H contain at least k contractible edges of H and moreover, H has at least one longest cycle that contains only k contractible edges of H.

Thus C_k is the class of graphs that are extremal with respect to the property that all longest cycles contain at least k contractible edges of the parent graph.

There are several general results about these classes and they have been characterized for small values of k as follows: $C_0 = \{K_4\}$ [DHT87], $C_1 = \emptyset$ [DHT87], $C_2 = \{K_2 \times K_3\}$ [DH089] and C_3 consists of three easily described (and we will

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do so shortly) infinite classes plus one sporadic graph [AH092]. Moreover, Ellingham, Hemminger and Johnson [EHJ92] recently proved a conjecture, due to K. Ota, which was circulating at the Second Japan Conference of Graph Theory and Combinatorics, 1990; namely, that longest cycles in a nonhamiltonian 3-connected graph H must contain at least six contractible edges of H. Thus, as was already known from the characterizations of C_k for $k \leq 3$, the graphs in C_4 and C_5 are also hamiltonian. Consequently, we feel that it is of interest to characterize these classes as well.

That is the purpose of this paper. Along the way we will give some useful tools that apply more generally to $\mathcal{H}_k$; the class of hamiltonian members of C_k. However, since their use in the development of the characterization of C_5 is quite like that in C_4, we will only sketch their use in C_5. Our procedure will be to describe (constructively) four classes of C_4 graphs and then prove that we have all of them. But first we need some definitions.

DEFINITIONS

Let G be a 3-connected graph and let H be any subgraph of G. We set $\nu(H) = |V(H)|$, $\epsilon(H) = |E(H)|$, and let $G[H]$ denote the subgraph induced by $V(H)$. If x and y are any two vertices of G, a path from x to y will be called an *xy-path.* We denote the set of contractible edges in G by $E_c(G)$ and the set of noncontractible edges in G by $E_n(G)$. We write $E_c(G,H)$ for $E_c(G) \cap E(H)$, and $E_n(G,H)$ for $E_n(G) \cap E(H)$. The cardinalities of these sets are denoted by $\epsilon_c(G,H)$ and $\epsilon_n(G,H)$, respectively.

Notational convention: Since contractibility will always be relative to G, we will usually suppress the G in this notation.

The set of neighbors in H of a vertex x is denoted by $N_H(x)$. We write $x \in H$ to mean $x \in V(H)$ and $xy \in H$ to mean $xy \in E(H)$. $N(x)$ means $N_G(x)$.

Throughout the paper G will denote a 3-connected hamiltonian graph and C will denote a fixed hamilton cycle in G with the property that no other hamilton cycle contains fewer contractible edges than C. We arbitrarily designate one of the directions around C as the positive direction and label the vertices in that direction by $1, 2, \ldots, \nu \pmod{\nu}$; thus for any $x \in V(C)$, $x+1,\ x+2, \ldots \pmod{\nu}$ will be the vertices *following x in C*, while $x-1,\ x-2, \ldots \pmod{\nu}$ are the vertices *preceding x in C*. We hasten to emphasize that we are not involved with directed graphs here; however, the orientation of C, and the induced orientation of other objects relative to C, will simplify our presentation. Thus, for x and y in $V(C)$, we denote by $[x,y]$ the path from x to y obtained by traversing C in the positive direction; we also define (x,y) to be the path $[x,y] - \{x,y\}$, $[x,y)$ to be $[x,y] - \{y\}$, and $(x,y]$ to be

$[x, y] - \{x\}$. If $x = y$, then $[x, y] = \{x\}$ while (x, y), $[x, y)$ and $(x, y]$ are all empty. For $n \geq 3$, the orientation on C naturally induces an n-ary "less than" relation $a_1 < a_2 < \cdots < a_n$ on the vertices of the path $[a_1, a_n]$, meaning that the vertices $a_1, a_2, \cdots, a_n$ occur in that order and that $a_i \neq a_{i+1}$. In this context, for $x \neq y$, "$x + 2 < y$" is used to mean $x < x + 2 < y$.

If $f = xy \in E_n(G)$, $G \not\cong K_4$, then there is at least one vertex s such that $S = \{x, y, s\}$ is a 3-cut of G; we say that f and s are *associates* (or that either one is *associated* with the other) and denote the set of all such s by $A(f)$. Similarly, we say that f and S are *associates.* We will only be interested in 3-cuts of G of this form when $f \in E(C)$ and we simply call them *cuts*; that is, a 3-cut T of G is a cut iff $G[T]$ contains an edge of C. Two cuts $S = \{x, x + 1, s\}$ and $T = \{y, y + 1, t\}$, with $x \neq y$, are *crossed* if either $t \in (s, x] \subseteq (y + 1, x]$ or $s \in (t, y] \subseteq (x + 1, y]$.

A *head* is a path $P = [a, b]$ in C with $a \neq b$ and such that $\{a(a+1), (b-1)b\} \subseteq E_c(C)$; it is an *$i$-head* if $|E_c(C) \cap [a, b]| = i$. We will be especially interested in 2-heads. A 2-head $P = [a, b]$ is called a *a small 2-head* if $b = a + 2$; it is called a *spacer* if $b = a + 4$ and $(a + 1)(b - 1) \in E(G)$. A spacer $P = [a, b]$ is said to be *restricted at* $a + 1$ if $d(a + 1) = 3$. In this case, $N(a + 1) = \{a, a + 2, a + 3 = b - 1\}$ is a cut associated with $(a + 2)(a + 3)$. If $N(a + 1)$ is the unique cut associated with $(a + 2)(a + 3)$, we say that P is *totally restricted at* $a + 1$. Similarly for $b - 1$. If P is (totally) restricted at both $a + 1$ and $b - 1$ we say that P is *(totally) restricted.* To show that P is totally restricted, we shall see that it suffices to show that P is totally restricted on one side. If P is a head that doesn't contain all the contractible edges of G that are within C and if Q is the largest head contained in $[b, a]$, then P and Q are called *coheads.*

Let $P = [a, b]$ and $Q = [c, d]$ be coheads. If $f \in E_n(C)$ and $s \in A(f)$ with either $f \in [b, c]$ and $s \in [d, a]$ or $s \in [b, c]$ and $f \in [d, a]$, then we say that the cut $S = V(f) \cup \{s\}$ *separates* P and Q. We will also say that S *traps* P (of course it also traps Q). If such a cut S separates a pair of coheads, neither of which is a 1-head, then s is called a *pivot* and S is called a *pivot cut.* The set of all f associated with a pivot s, whether the associated cut is a pivot cut or not, is called the *spread associated with* s, and is denoted by Σ_s. We note that $f \in E_n(C)$ can be in more than one spread.

A *jumper* is an edge $g \in E(G) - E(C)$. A jumper of the form $x(x+2)$ is called a *squaring edge*, while a jumper of the form $x(x + 3)$ is called a *2-jumper.* If $P = [a, b]$ and $Q = [c, d]$ are coheads with neither a 1-head and if $g = xy$ is a jumper with $b \leq x \leq c < d \leq y \leq a$, then g is called a *(P, Q)-splitter.*

CONSTRUCTION OF C_4 GRAPHS

Our characterization will show that the class C_4 is intimately related to the class C_3. So we will first describe the class C_3, then give some "natural" ways of constructing members of C_4, and finally show that except for one sporadic class on twelve vertices or less, we in fact have all of them.

The members of C_3 on twelve vertices are displayed in Figure 1. We will follow two conventions in *all* figures: (1) we will use bold lines to indicate the members of $E_c(C)$ and (2) dotted lines to indicate optional edges in G where an edge not in C is called *optional* if its absence does not effect the 3-connectedness of G and the same edges of C are contractible in G when that edge is absent as when it is present.

To describe C_3 we start with the cycle $C = (1, 2, 3, \ldots, \nu, 1)$. Now, for $\nu \geq 5$, let $G_{\nu,2}$ be C plus the squaring edges to the path $[1, \nu]$, that is, the edges $x(x+2)$ for $1 \leq x \leq \nu - 2$. Then $G \in C_3$ and $E_c(G_{\nu,2}, C) = \{(\nu-1)\nu, \nu(1), 1(2)\}$. For $\nu \geq 6$, the graph $G_{\nu,1}$ is obtained from $G_{\nu,2}$ by dropping the edge $3(5)$, adding the 2-jumper $2(5)$, and if $\nu > 6$, making the jumper $2(4)$ an optional edge. If instead, we do the symmetric modification on the ν-side, that is, drop $(\nu-4)(\nu-2)$, add $(\nu-4)(\nu-1)$ and make $(\nu-3)(\nu-1)$ optional, then we get what we call the *reverse* of $G_{\nu,1}$ and which we denote by $G^r_{\nu,1}$. Note that in $G_{\nu,1}$, $[\nu, 4]$ is a totally restricted spacer and $G \in C_3$ with $E_c(G_{\nu,1}, C) = \{(\nu-1)\nu, \nu(1), 3(4)\}$. If $\nu \geq 8$ and if we make both of these "spacer modifications," that is, on the ν side as well as the 1 side, then we get the member of C_3 that we denote by $G_{\nu,0}$ and which has $E_c(G_{\nu,0}, C) = \{(\nu-3)(\nu-2), \nu(1), 3(4)\}$. The subscript i in $G_{\nu,i}$ refers to the number of pairs of members of $E_c(C)$ that are adjacent. Finally, the one sporadic member of C_3 is denoted by G_9 and is obtained from $G_{9,0}$ by adding the edge $(\nu-1)2$ and eliminating $2(4)$ and $(\nu-3)(\nu-1)$ as optional edges. It is not included in Figure 1. Note that G_9 is the only member of C_3 having *three* totally restricted spacers.

That completes the description of the members of C_3 in that all members of C_3 are isomorphic to one of these. Of course, $G^r_{\nu,1} \simeq G_{\nu,1}$ as well, but it is convenient to have this concept available in describing C_4. Note that in all members of C_3 except for $G_{5,2}$ and G_9, the edge $\nu(1)$ has the unique characteristic that $N(\nu) \cap \{2, 3\} = \phi$, $N(1) \cap \{\nu-1, \nu-2\} = \phi$ and $2 \notin N(\nu-1)$. In general, we will refer to such a contractible edge as an *isolated contractible edge of* C. We will now give three ways of constructing C_4 graphs; two of which consist of modifying C_3 graphs.

Adding a 2-jumper. The 2-jumper $z(z+3)$ is added to $G_{\nu,i}$ and the squaring edges $z(z+2)$ and $(z+1)(z+3)$ are made optional, all subject to one of the following conditions.

(0) $i = 0, \nu \geq 10$ and $4 \leq z \leq \nu - 6$, or

(1) $i = 1, \nu \geq 7$, $4 \leq z \leq \nu - 3$ and $z(z+2)$ must be an edge if $z = \nu - 4$ or $\nu - 3$ (so that $(\nu-1)\nu$, respectively $\nu(1) \in E_c$), or

(2) $i = 2, \nu \geq 5,\ 1 \leq z \leq \nu - 3$ and $z(z+2)$ must be an edge if $z = \nu - 4$ or $\nu - 3$ (so that $(\nu-1)\nu$, respectively $\nu(1) \in E_c$) and $(z+1)(z+3)$ must be an edge if $z = 1$ or 2 (so that $\nu(1)$, respectively $1(2) \in E_c$). Thus *both* optional edges are required if $\nu = 5$ or if $\nu = 6$ and $z = 2$.

Each of these graphs is in C_4 and the four members of $E_c(C)$ in each case are those of $G_{\nu,1}$ plus the edge $(z+1)(z+2)$. Of course a modification to $G^r_{\nu,1}$ symmetric to that in (1) would give a "reverse" to that in (1), but it would also be isomorphic to that in (1).

Barriers. Let C be a cycle of length at least five and pick vertices a, b, c and d around C (in the positive direction) so that $\nu(a,b) = 1$ or 3 and $\nu(c,d) = 1$ or 3 ($a = d$ is allowed, but not $b = c$). We will now construct a graph $G \in C_4$ in which C is a hamilton cycle having $P = [a,b]$ and $Q = [c,d]$ as 2-heads; moreover, each will be a restricted 2-head if it isn't a small 2-head. In drawings of G we will depict C as an oval with P on the left and Q on the right (e.g. see Figure 2). We will also denote the middle vertex of P by p ($p = a+1$ if P is a short head and $a+2$ otherwise) and that of Q by q. First pick integers $k \geq 2$ and $\epsilon = 0$ or 1 with $k \leq \nu[b,c]$ and $k - \epsilon \leq \nu[d,a]$; then choose distinct vertices $x_1, x_2, \cdots, x_k$ from $[b,c]$ and in positive order with $b = x_1$ and $c = x_k$ and distinct vertices $y_1, y_2, \ldots, y_{k-\epsilon}$ from $[d,a]$ but in *reverse* order (that is, with subscripts increasing as you proceed in the negative direction on $[d,a]$) and with $y_1 = a$ and $y_{k-\epsilon} = d$. Now the edges of G are as follows.

Required edges:

(1) Those in C.

(2) The squaring edges on $[x_i, x_{i+1}], 1 \leq i \leq k-1$ and those in $[y_{j+1}, y_j],\ 1 \leq j \leq k - \epsilon - 1$.

(3) $p(x_1+1)$ and $(y_i+1)(x_i+1)$ for $2 \leq i \leq k-1$.

(4) $(y_{i-1}-1)(x_i-1)$ for $2 \leq i \leq k-\epsilon$.

(5) $(x_k-1)q$ if $\epsilon = 1$, $(y_k+1)q$ if $\epsilon = 0$.

(6) $(p-1)(p+1)$ if $\nu(P) = 5$ and $(q-1)(q+1)$ if $\nu(Q) = 5$. [Results in restricted 2-heads in those cases.]

Optional edges:

(1) $y_i x,\ \forall x \in [x_i, x_{i+1}],\ 1 \leq i \leq k-1$.

(2) $x_i y,\ \forall y \in [y_i, y_{i-1}],\ 2 \leq i \leq k-\epsilon$.

(3) pa and pb if $\nu(P) = 5$ and qc and qd if $\nu(Q) = 5$.

(4) If $k = 2,\ \epsilon = 1$ and $P(Q)$ is a small 2-head, then at least one optional edge other than y_1x_1 (y_1x_2) must be selected so that $pb \in E_c(C)$ ($cq \in E_c(C)$).

One easily sees that G is in fact in C_4 with P and Q being 2-heads and where $y_1, y_2, \ldots, y_{k-1}$ and $x_2, x_3, \ldots, x_{k-\epsilon}$ are all pivots with $[x_i, x_{i+1}] \subseteq \sum_{y_i}$ and $[y_i, y_{i-1}] \subseteq \sum_{x_i}$. The induced subgraph $G[[b,c] \cup [d,a]]$ is referred to as the *barrier*. If $\epsilon = 0$ ($\epsilon = 1$), then G is said to be of the *even* (*odd*) *barrier type*. An example of a graph having an even barrier is given in Figure 2. In order to avoid a cluttered look, only a few of the optional edges are included. The simplest odd barrier type has $k = 2$ and $\epsilon = 1$ (so $a = d = y_1$ is the only pivot). The following graphs are intimately related to the $k = 3$ odd barrier types by adding the squaring edge $(x_2 - 1)(x_2 + 1)$.

Splitting an isolated edge with a spacer/3-fan. Let $G \in C_3$, $G \not\cong G_9$. Form a new graph G_z from G by replacing the isolated edge $\nu(1)$ by a restricted spacer on $\{\nu, z-1, z, z+1, 1\}$ where $z-1, z, z+1$ are new vertices. Thus, if we use N_z for the neighborhood function in G_z, then $N_z(z-1) = \{\nu, z, z+1\}$, $N_z(z+1) = \{z-1, z, 1\}$, $N_z(\nu) - \{z-1\} = N(\nu) - \{1\}$, and $N_z(1) - \{z+1\} = N(1) - \{\nu\}$. All other edges (optional edges) of G are also edges (optional edges) of G_z. Finally, if we let $[c, b]$ be the 3-head in G with $\nu(1)$ as the middle contractible edge (e.g. in $G_{\nu,1}$, $c = \nu - 1$ and $b = 4$), then G_z has optional jumper edges $z\alpha, z\beta$ and $z\gamma$, at least one of which must be selected, subject to the following conditions.

(1) We always have β in $[b, c]$. Moreover, $\alpha = \beta - 1$ and $\gamma = \beta + 1$ except when $\beta = 4$ in $G_{\nu,1}$ or $G_{\nu,0}$ in which case $\alpha = \beta - 2$ or, by symmetry, when $\beta = \nu - 3$ in $G^r_{\nu,1}$ or $G_{\nu,0}$ in which case $\gamma = \beta + 2$; that is each of α and γ is taken to be the middle vertex of a restricted spacer when the general rule would have selected one of the restricted vertices.

There are no further restrictions if $b + 2 \leq \beta \leq c - 2$. The remaining conditions refer to limiting situations and because of the symmetry involved we only give them relative to the "left" side, i.e. when $\beta < b + 2$ and depending on whether $[\nu, 4]$ is a restricted spacer in G or not.

(2) If $\beta = b = 2$, then $z\gamma = z(3)$ must be selected so that $1(2) \in E_c(G_z)$.

(3) If $\beta = b + 1 = 3$, then at least one of $z\beta$ or $z\gamma$ must be selected so that $1(2) \in E_c(G_z)$. Note that if G_z came from $G_{5,2}$ the symmetric condition would also require that at least one of $z(3)$ or $z(4)$ be selected.

(4) If $\beta = b = 4$ (so $\alpha = 2$) and if neither $z\beta$ nor $z\gamma$ is selected, then $3(5)$, explicitly excluded from G, becomes an optional edge as it no longer affects the contractibility of $1(2)$ or any other edge in C. Also in this case, if G_z came from $G_{6,1}$, at least one of $z\alpha = z(2)$ or $z\beta = z(4)$ and at least one of $z\beta = z(4)$ and $\alpha\beta = 2(4)$, must be selected so that $5(6) \in E_c(G_z)$. Furthermore, if $z\beta$ is chosen, then $2(4)$, which was required in $G = G_{6,1}$, becomes an optional edge.

(5) The graph in Figure 3a is not obtained by the above but is considered to

be of this type and could be loosely viewed as coming from K_4 by this construction.

One now easily sees that $G_z \in C_4$ via the cycle C_z which is obtained from C by replacing the edge $\nu(1)$ by the path $[\nu, z-1, z, z+1]$ for then $E_c(C_z) = (E_c(C) - \{\nu(1)\}) \cup \{\nu(z-1),\ (z+1)1\}$. Also note that at least one of $z-1$ and $z+1$ are pivots and that $[b, c] \subseteq \Sigma_{z-1} \cup \Sigma_{z+1}$. Examples of spacer/3-fan graphs are given in Figure 3b and 3c.

We let $\mathcal{J}_4$ denote the graphs constructed by the 2-jumper method, $\mathcal{B}_4$ those constructed by barrier method and $\mathcal{F}_4$ those constructed by the 3-fan method.

PREVIOUS LEMMAS

Lemma A [DHO89]. If $S = \{x, x+1, s\}$ is a cut and if A is a component of $G - S$, then

(1) A contains an endvertex of an edge from $E_c(C)$ and
(2) if $s = x+3$, then $(x+2)(x+3) \in E_c(C)$. (In fact, (2) is a simple consequence of the lemma stated in [DHO 89].)

Lemma B [AHO92]. If $S = \{x, x+1, s\}$ is a cut, then there exist xw- and $(x+1)z$-jumpers with $w \in [x+2, s-1]$ and $z \in [s+1, x-1]$. Moreover, if $G[v, u]$ is 2-connected ($[u, v]$ a path in C) with $x \in (u, v-1)$, then $s \in [u, x-2] \cup [x+3, v]$.

Lemma C [AHO92]. If $G[v, u]$ is 2-connected (in particular if uv is a jumper), then

(1) if $v = u+3$, that is $V(u, v) = \{u+1, u+2\}$, then $(u+1)(u+2) \in E_c(C)$,
(2) if $\nu(u, v) \geq 3$ and $\nu(v, u) \geq 1$, then $\epsilon_c[u, v] \geq 2$.

Note that since $V(G) = V(C)$, the converse of (1) is also true; that is, $z(z+1) \in E_c(C)$ if and only if $G[z+2, z-1]$ is 2-connected.

NEW LEMMAS

Our main goal is the characterization of C_4. However, much of our development holds in $\mathcal{H}_k$, $k \geq 4$, so we pull those results out into this separate section.

Lemma [Crossed Cuts]. If $S = \{x, x+1, s\}$ and $T = \{y, y+1, t\}$ are crossed cuts with $t \in (s, x] \subseteq (y+1, x]$, then

(1) $s \neq y+1$ and $t \neq x$,
(2) $\{x, x+1, t\}$ is a cut if $(t, x) \neq \emptyset$, and $\{y, y+1, s\}$ is a cut if $(y+1, s) \neq \emptyset$, and
(3) $st \in E_c(C)$ if $y \neq x+1$.

Proof. (1) If $s = y+1$, then $y \neq x+1$ and so y cannot have a neighbor in $(y+1, t)$ as required.

(2) Since (t, x) has no neighbors in $(x+1, s)$ because of the cut S and none in $(y+1, t) \supseteq [s, t)$ because of the cut T, it has none in $(x+1, t)$; that is, $\{x, x+1, t\}$ is a cut (assuming that $(t, x) \neq \emptyset$).

(3) Suppose that $(s, t) \neq \emptyset$. Now (s, t) has no neighbors in (t, y) because of the cut T and none in $(x+1, s)$ because of the cut S. Hence, (s, t) has no neighbors in (t, s) since $y \neq x+1$; that is, (s, t) is a 2-cut, a contradiction. Thus $t = s+1$. That $st \in E_c(C)$ now follows from (1) of this lemma, since an associated cut must cross either S or T. ∎

$K_2 \times K_3$ shows that (3) need not hold when $y = x+1$.

Corollary 1 [Crossed Cuts]. If $S = \{x, x+1, s\}$ and $T = \{s, s+1, t\}$ are cuts, then $t \in [s+3, x]$. Moreover, if $t = x$, then $(s+1)(x+1) \in E(G)$.

Proof. The first statement is just the contrapositive of (1); we have stated it since this is the form in which we will frequently use it. The latter statement is immediate from Lemma B. ∎

Corollary 2 [Crossed Cuts]. If $S = \{x, x+1, x+3\}$ is the unique cut associated with $x(x+1) \in E_n(C)$ and if $(x+1)(x+2) \in E_n(C)$, then $[x-1, x+3]$ is a totally restricted spacer.

Proof. Using Lemma A and applying the Crossed Cuts Lemma to S and the cuts associated with the edge $(x+1)(x+2)$ we immediately get the corollary. ∎

The import of this corollary is that we can conclude that a spacer is totally restricted by just showing that it is totally restricted on one side. The role of Lemma A in the proof was to show that $[x-1, x+3]$ was a spacer. More generally, Lemma A gives that a cut $\{x, x+1, s\}$ "traps" at least one contractible edge in each of the two segments $[x+1, s]$ and $[s, x]$. This is used so frequently, that we will seldom mention it explicitly. Equally important in our development, as it was in the characterization of C_3 [AH092], is the following lemma which describes $[x, s]$ if $\epsilon_c[x, s] = 1$. (Clearly, there is a symmetric version when $\epsilon_c[s, x] = 1$.)

Lemma [Trap One]. If $S = \{x, x+1, s\}$ is a cut (so $s \geq x+3$) and if $K = [x, s]$ has $\epsilon_c(K) = 1$, then

(1) if $s = x+3$, then $E_c(K) = \{(s-1)s\}$ and $N(s-1) = \{x, x+1, s\}$.

(2) if $s > x+3$ and $E_c(K) = \{(s-1)s\}$, then $G[K] = K^2, dg(s-1) = dg(s) = 3$ and $s(s+1) \in E_c(C)$,

(3) if $s > x+3$ and $(s-1)s \in E_n(C)$, then $E_c(K) = \{(s-3)(s-2)\}$, $G[K] = K^2 \cup \{(s-4)(s-1)\} - \{(s-4)(s-2)\}$ with the edge $(s-3)(s-1)$ being optional,

and $dg(s) = dg(s-2) = 3$ with $s(s+1) \in E_c(C)$; in particular, $[s-3, s+1]$ is a restricted spacer.

Note that in both (2) and (3), G has no other edges at vertices within $[x+2, s)$ since S is a cut. Also note that in (2) and (3) $\{x, x+1, s+1\}$ is a cut if $s+1 \neq x-1$, as is certainly the case in $\mathcal{C}_k$, for $k \geq 4$. The three situations in the lemma are pictured in Figure 4.

Proof. (1) This is just (2) of Lemma A.

(2) Let $g = (s-2)(s-1) \in E_n(C)$ and let $T = \{s-2, s-1, t\}$ be an associated cut. Thus $t \in (s, x)$ by Lemma A. But since $s-2 > x+1$, we conclude, by the Crossed Cuts Lemma, that $t = s+1$, $st \in E_c(C)$, and $N(s) = \{s-2, s-1, s+1\}$. By a similar application of Lemma A and the Crossed Cuts Lemma to each edge of C between $x(x+1)$ and g, we see that $G[K] = K^2$.

(3) The proof of this is in [AH092] and holds in $\mathcal{C}_k$, $k \geq 2$. We especially note that the $\nu = 6$ case gives the prism $K_2 \times K_3$. ∎

Our characterization theorem will divide $\mathcal{C}_4$ into two classes: those having pivot cuts and those having no pivot cuts. Since we can say more about the latter type in general, we let $\mathcal{S}_k$, $k \geq 4$, denote the members of $\mathcal{H}_k$ (the hamiltonian members of $\mathcal{C}_k$) without pivot cuts. Note that if G has no pivot cuts with respect to a qualifying cycle, C, then each cut of G containing both ends of a noncontractible edge in C, must "trap" a single contractible edge as in (1) of the Trap One Lemma (or its symmetric version).

Lemma [No Pivots]. Let $G \in \mathcal{S}_k$, $k \geq 4$. If $f \in E_n(C)$, then f is one of the noncontractible edges of a totally restricted spacer. Consequently $\nu(G) \leq 3k$.

Proof. Let $S = \{x, x+1, s\}$ be a cut associated with $f = x(x+1) \in E_n(C)$. Since G has no pivot cuts, we can assume by the Trap One Lemma and symmetry that S is as in (1) of that Lemma. Let $g = (x+1)(x+2)$, and let $t \in A(g)$. Then, by the Trap One Lemma, $\epsilon_c[x+2, t+1] > 1$. Thus since G has no pivot cuts and $k \geq 4$, $A(g) = \{x-1\}$. Hence $A(f) = \{x+3\}$ by the Crossed Cuts Lemma. ∎

Corollary $[\mathcal{S}_4]$. $G \in \mathcal{S}_4$ if and only if G is one of the graphs in Figure 5 where the dotted edges are optional.

Proof. First note that all graphs in Figure 5 are in $\mathcal{S}_4$. So let $G \in \mathcal{S}_4$, let w, x, y, and z be the middle vertices of the four 2-heads in order around C and let h be the number of those that are spacers. We divide on the possible values of h, which in turn gives the values of $\nu(G)$ as 12, 10, 8 or 6. Thus the structure of G

is known except for $G[w, x, y, z]$. In getting that we repeatedly use the following observation: if $[a, b]$ is a spacer, then $G[b, a]$ is 2-connected since $[a, b]$ is totally restricted.

(1) $h = 4$; and so $\nu(G) = 12$. Since $G[x + 2, x - 2]$ is 2-connected, if the splitter $wy \notin E(G)$, then wz and $zy \in E(G)$. Similar statements hold for w, y and z of course. Thus either both splitters wy and xz are there, with the other four edges being optional, or else the other four edges are there with the splitters being optional. Note that this division has a nonempty overlap, for example, when all six edges are present.

(2) $h = 3$; and so $\nu(G) = 10$. Let w be the vertex incident with two members of $E_c(C)$. The splitter xz must be there since $G[z - 1, x + 1]$ is 2-connected. If the splitter $wy \in E(G)$, then all four nonsplitters are optional; otherwise they must be there.

(3) $h = 2$; and so $\nu(G) = 8$. This can happen in two nonsymmetric ways.

(3.1) Let w and z be the two vertices incident with two members of $E_c(C)$. First note that the nonsplitter xy must be there so that $wz \in E_c(C)$. The only other requirement is that $dg(z), dg(w) \geq 3$.

(3.2) Let y and w be the two vertices incident with two members of $E_c(C)$. We must now have the splitter xz or $\{y, w\}$ is a 2-cut. If $wy \in E(G)$, then $G \in \mathcal{S}_4$, so the four jumpers wx, xy, yz and zw are optional; otherwise they are all required.

(4) $h = 1$; and so $\nu(G) = 6$. If we let x be the middle vertex of the single spacer, then we easily see that wy, xw, xz, and xy must all be edges. ∎

In $\mathcal{C}_3$ a cut always traps one contractible edge on one side or the other. This needs no longer be true in $\mathcal{C}_k$ for $k \geq 4$, so a "Trap Two Lemma" would likely be useful in $\mathcal{C}_4$ and $\mathcal{C}_5$. We in fact have four, corresponding to characteristics of the cuts. The terminology used in their names is as follows: for coheads $P = [a, b]$ and $Q = [c, d]$, a cut $S = \{x, x + 1, s\}$ with $x \in [b, c)$ and $s \in [d, a]$ is called a *half tight trap of P* if $x = b$ and $s \neq a$, while it is called a *tight trap of P* if $x = b$ and $s = a$. As usual there is a symmetric version having $x \in [d, a)$ and $s \in [b, c]$.

See the first row of Figure 6 for a "typical" P in each of the three cases in the Half Tight Trap Two Lemma; the second row illustrates the three cases for the Tight Trap Two Lemma.

Our interest in half tight and tight cuts that trap a 2-head evolves from the idea of a "closest" cut trapping it. So let $S = \{x, x + 1, s\}$ be a cut separating a 2-head $P = [a, b]$ from its cohead $Q = [c, d]$. By symmetry, we take $x \in [b, c]$ and $s \in [d, a]$. Then S is said to be a cut *closest* to P if there is no $w \in [b, x]$ and $t \in [s, a]$ such that $\{w, w + 1, t\}$ is a cut different from S, nor is there a $z \in [s, a - 1]$ and $u \in [b, x]$ such that $\{z, z + 1, u\}$ is a cut. (Note that, by the Crossed Cuts Lemma, for such

w or z there is no such $t \in [d, s)$ nor $u \in (x, c]$ either.) Thus, by applying the Trap One and Crossed Cuts Lemmas to $b(b+1) \in E_n(C)$, we see that $b = x$.

That is, if there is a cut trapping a 2-head P, then such a cut closest to P is unique and is either tight or half tight.

Lemma [Half Tight Trap Two]. Let $G \in \mathcal{C}_k, k \geq 4$. If $P = [a, b]$ is a 2-head with cohead $Q = [c, d]$, if $S = \{b, b+1, s\}$ is a cut closest to P, and if $s \in [d, a)$, then $[d-1, a+1]$ is a totally restricted spacer and one of the following accounts for all remaining edges incident with $[d, b-1]$.

(1) P is a small 2-head with $N(a+1) - \{a, b\} = \{a-1, b+1\}$ and with $b+1 \in N(a-1)$ while $(a-1)b$ is an optional edge unless $c = b+1$ and $N(b) \cap [c+2, d] = \emptyset$, in which case it must be present so that the edge $c(c+1)$ is contractible.

(2) P is a totally restricted 2-head with $\{a-1, b+1\} \subset N(a+2)$ while $(a-1)b$, $(a-1)(b+1)$ and $(a+2)b$ are all optional edges except that at least one of $(a-1)b$ and $(a+2)b$ must be present if $c = b+1$ and $N(b) \cap [c+2, d] = \emptyset$, so that the edge $c(c+1)$ is contractible.

(3) $b > a+2$ and the following hold.
   (a) $(a-1)(a+2) \in E(G)$,
   (b) $G[a+1, b-1] = [a+1, b-1]^2$,
   (c) $(b-2)(b+1) \in E(G)$, and
   (d) $(a-1)(a+1)$, $(b-2)b$ and $(b-1)(b+1)$ are all optional edges except that $(b-2)b$ is required if $c = b+1$ and $N(b) \cap [c+2, d] = \emptyset$.

Proof. Since $s \neq a$, let $T = \{a-1, a, t\}$ be a cut associated with $(a-1)a \in E_n(C)$. Now $t \notin [d, b-1]$ by the Trap One Lemma and $t \neq b$ since S was closest to P. Thus $t \in [b+1, d-1]$ which implies that $s = d = t+1$ by (3) of the Crossed Cuts Lemma. That $[d-1, a+1]$ is a totally restricted spacer now follows from the second Crossed Cuts Corollary. Since S is a "closest" cut to P, neither $\{b, b+1, a\}$ nor $\{a-2, a-2, b\}$ is a cut, and therefore there are vertices $x, y \in [a+1, b-1]$ such that $x \in N(a-1)$ and $y \in N(b+1)$.

To gain further information about P we divide into cases, first assuming that P is a small 2-head. Thus $x = y = a+1$, that is $(a+1)(b+1)$ and $(a+1)(a-1)$ are required edges since S is a closest cut to P and $(a-1)(b+1)$ is required so that $(b-1)b \in E_c(C)$. No other edges incident with $[d, b-1]$ are possible because of the cut S. Whether $b(a-1)$ is an edge of G or not has no effect on the contractibility of edges in $[b+2, d]$ because of the edge $(a-1)(b+1)$. If $c > b+1$ or $N(b) \cap [c+2, d] \neq \emptyset$, then P is as in (1). Otherwise, $c = b+1$ and $N(b) \cap [c+2, d] = \emptyset$, in which case $(a-1)b$ is required to ensure that $c(c+1) \in E_c(C)$.

So we assume that P is not a small 2-head and let $f = (b-2)(b-1)$ and

$g = (a+1)(a+2)$. Of course $f = g$ is possible, but in any case, the Crossed Cuts Lemma gives that $A(g) \subseteq \{b, d-1, d\}$ with $d \in A(g)$ if $d-1 \in A(g)$. Likewise $A(f) \subseteq \{d-1, d, a\}$ with $d \in A(f)$ if $d-1 \in A(f)$.

Suppose first that $d \notin A(f)$, and hence that $A(f) = \{a\}$. Clearly $b \neq a+3$, and if $b > a+4$, then f and g are not adjacent and we get a contradiction to the Crossed Cuts Lemma. So $b = a+4$ and P is a totally restricted spacer by the second Crossed Cuts Corollary. Thus $x = y = a+2$, that is, $(a-1)(a+2)$ and $(a+2)(b+1)$ are required edges. The inclusion or omission of edges $(a-1)b$, $(b-2)b$ can have no effect on the contractibility of edges in $[b+2, d]$. If $c > b+1$ or $N(b) \cap [c+2, d] \neq \emptyset$, then P is as in (2). Otherwise, $c = b+1$ and $N(b) \cap [c+2, d] = \emptyset$, in which case one of $(a-1)b$ and $(b-2)b$ must be present to ensure that $c(c+1) \in E_c(C)$, the other is optional.

So we may assume that $d \in A(f)$. Conditions (a) and (b) of (3) now follow from the Trap One Lemma as does the optionality of $(a-1)(a+1)$. And (c) must hold so that $(b-1)b \in E_c(C)$. Likewise, in (d), $(b-2)b$ is required if $c = b+1$ and $N(b) \cap [c+2, d] = \emptyset$ so that $c(c+1) \in E_c(C)$. There can be no further edges incident with $[d, b-1]$ other than the optional ones listed in (d) due to the cuts existing at this stage. Thus P is as in (3) in this case. ∎

Note that in all cases of the Half Tight Trap Two Lemma, if $c = b+1$ and $d = c+2$, the graph G is completely determined and lies in C_4; otherwise, $\{b, b+1, d-1\}$ is a tight cut trapping the three contractible edges in $[d-1, b]$. That will also be the case in (3) of the next lemma.

Lemma [Tight Trap Two]. Let $G \in C_k, k \geq 4$. If $P = [a, b]$ is a 2-head with cohead $Q = [c, d]$ and if $S = \{b, b+1, a\}$ is a cut, then one of the following accounts for all edges incident with $[a+1, b-1]$.

(1) P is a small 2-head with $N(a+1) = \{a, b, b+1\}$.

(2) P is a restricted spacer with $(a+2)(b+1) \in E(G)$. The edges $(a+2)a$, $(a+2)b$ are optional except in the following circumstances.

  (a) When $N(a) \backslash \{a+2\} = \{a-1, a+1\}$, the edge $(a+2)a$ is required for G to be 3-connected.

  (b) When $N(a) \cap V([b, a-1]) = \{a-1, a-2\}$, the edge $(a+2)a$ is required if $(a-1)(a-2) \in E_c(C)$, while it is forbidden if $(a-1)(a-2) \in E_n(C)$ and $G - \{a, a-1, a-2\}$ is 2-connected .

  (c) When $N(b) \cap V([b+1, a]) = \{b+1, b+2\}$, the edge $(a+2)b$ is required if $(b+1)(b+2) \in E_c(C)$, while it is forbidden if $(b+1)(b+2) \in E_n(C)$ and $G - \{b, b+1, b+2\}$ is 2-connected .

(3) $b > a+2$ and the following hold:

(a) $N(a) = \{a-1, a+1, a+2\}$,
(b) $G[a, b-1] = [a, b-1]^2$,
(c) $(a-1)a \in E_c(C)$,
(d) $(b-2)(b+1) \in E(G)$, and
(e) $(b-2)b$ and $(b-1)(b+1)$ are optional except that the former is required when $a = d = c+2 = b+3$ and the latter is required if $b = a+3$.

Proof. If $b = a+2$, then P must be a small 2-head and condition (1) follows immediately since G is 3-connected and S is a cut.

So let us assume that $b > a+2$ and let f and g denote the edges $(b-2)(b-1)$ and $(a+1)(a+2)$ respectively. Of course, $f = g$ is possible, but in any case, the Crossed Cuts Lemma gives that $A(f) \subseteq \{a, a-1\}$ and $A(g) \subseteq \{b, a-1\}$. Note that if $a-1 \in A(f) \cup A(g)$, then $(a-1)a \in E_c(C)$.

If $b \in A(g)$, then, by the Trap One Lemma, P is a spacer with $dg(b-1) = 3$. Furthermore, since $a \in A(f)$, $dg(a+1) = 3$ and P is a restricted spacer. So far we have $N(b-1) = \{b-3, b-2, b\}$, $N(a+1) = \{a+2, a+3, a\}$ and $N(a+2) \subseteq \{a, a+1, a+3, b, b+1\}$. Since $(b-1)b \in E_c(C)$, $(a+2)(b+1) \in E(G)$. Thus to show that G is as in (2), we consider $G' = G - \{(a+2)a, (a+2)b\}$ in order to decide if $(a+2)a$, respectively $(a+2)b$ is a required edge, is excluded as an edge or is an optional edge.

First note that if G' is not 3-connected, then there must be a 2-cut $\{x, y\}$ in G' with $x \in [b+1, a-1]$ and $y \in \{a+1, b-1\}$. In fact, if $y = a+1$, then $x = a-1$ (or $\{x, a\}$ is a 2-cut in G) and $N(a) \cap [b, a-2] = \emptyset$, so that the edge $a(a+2)$ is required for G to be 3-connected, as in (2)(a). Since S is a 3-cut in G, $N(b) \cap [b+2, a-1] \neq \emptyset$ and thus there can be no 2-cut in G' using $b-1$.

So assume that G' is 3-connected and consider an edge $e \in [b+1, a]$. If e is contractible in G but $V(e) \cup \{a+1\}$ $(V(e) \cup \{b-1\})$ is a 3-cut in G', then the edge $(a+2)a$ $((a+2)b)$ is required. By the Crossed Cuts Lemma, this can occur only when $e = (a-1)(a-2)$ $(e = (b+1)(b+2))$, that is when e is subtended by a (b) which is of degree 3 in G', as in (2)(b) ((2)(c)).

On the other hand, if e is not contractible in G but contractible in $G' + (a+2)a$ $(G' + (a+2)b)$, then $V(e) \cup \{a+1\}$ $(V(e) \cup \{b-1\})$ is the unique 3-cut associated with e. By the Crossed Cuts Lemma, this occurs only when $e = (a-1)(a-2)$ $(e = (b+1)(b+2))$ and $G - \{a, a-1, a-2\}$ $(G - \{b, b+1, b+2\})$ is 2-connected, as in the remaining part of (2)(b) ((2)(c)).

Thus we may assume that $b \notin A(g)$, i.e. $A(g) = \{a-1\}$. Consequently, $N(a) = \{a-1, a+1, a+2\}$ and $(a-1)a \in E_c(C)$. Condition 3(b) is immediate when $f = g$ and follows from the Trap One Lemma, applied to $V(f) \cup \{a\}$ when $f \neq g$. Thus the edge $(b-2)(b+1)$ is required so that $(b-1)b \in E_c(C)$, giving condition 3(d). It is now clear that the inclusion or omission of the edges $(b-2)b$ and $(b-1)(b+1)$ can have no effect on the contractibility of edges in C other than $(b-3)(b-2)$ and $(b+1)(b+2)$. If $b > a+3$, then $\{b-3, b-2, a-1\}$ is always a cut. If $b = a+3$, then $(b-1)(b+1)$ is required so that $(b-3)(b-2) = a(a+1)$ is contractible.

Considering the edge $(b+1)(b+2)$, an argument similar to the above when $b \in A(g)$ establishes the remainder of 3(d). ∎

THE THEOREM

If we let $\mathcal{M}_4 = \mathcal{J}_4 \cup \mathcal{B}_4 \cup \mathcal{F}_4 \cup \mathcal{S}_4$ we are now ready to state and prove our theorem.

Theorem. $C_4 = \mathcal{M}_4$.

Proof. Since we clearly have $\mathcal{M}_4 \subseteq C_4$, we will complete the proof by starting with $G \in C_4$ and showing that $G \in \mathcal{M}_4$. We easily see that if $\nu = 5$, then G is of the barrier type in $\mathcal{M}_4$, so we hereafter assume that $\nu \geq 6$.

By the definition of $\mathcal{S}_4$ and symmetry we can assume that $S = \{x, x+1, s\}$ is a pivot cut with $x \in [b, c-1]$ and $s \in [d, a]$ where $P = [a, b]$ is a 2-head and $Q = [c, d]$ is its cohead. Moreover, we assume that S is a pivot cut closest to P. Thus, by the Half Tight Trap Two Lemma and the observation preceding it, we have $x = b$ and either $s = d = a-2$ (Half Tight) or $s = a$ (Tight).

(A) $s = a-2$ as in the Half Tight Trap Two Lemma. Thus $\{b, b+1, d-1\}$ is a cut trapping one contractible edge in $[c, d-1]$ unless $c = b+1 = d-2$.

Consider the latter case first. If we have case (1) of the Half Tight Trap Two Lemma, then G is obtained from K_4 by a spacer/3-fan modification as described in Note (5) of that construction with $z = a-1$. If we have case (2) of the Half Tight Trap Two Lemma, then G is obtained from $G_{6,1}$ by a spacer/3-fan modification as in note (4) of that construction with $z = a-1$. If we have case (3) of the Half Tight Trap Two Lemma, then G is obtained from $G_{\nu,1}, \nu \geq 8$ by adding the 2-jumper $(b-2)(b+1)$.

When $\{b, b+1, d-1\}$ is a cut trapping one contractible edge in $[c, d-1]$, we apply the Trap One Lemma to $[c, d-1]$ to determine the structure of G not already determined by the Half Tight Trap Two Lemma. Note that when case (1) of the Trap One Lemma applies, the cut $\{b, b+1, d\}$ means that $d-1$ can only be adjacent to $d, c = d-2$ and b or $b+1$. Since $(b+1)c \in E_n(C)$, $d-1$ is not adjacent to b, so $N(d-1) = \{d, d-2, b+1\}$. Furthermore, $(d-1)d \in E_n(C)$. Thus, all the conclusions of case (2) of the Trap One Lemma are valid here in case (1) also, and we therefore treat (1) and (2) together. There are six possibilities to consider which we label by ordered pairs (i, j) when we are in case (i) of the Half Tight Trap Two Lemma and part (j) of the Trap One Lemma.

In case (1,1) ((1,2)), G is obtained from $G_{\nu,2}, \nu = 5 (\nu \geq 6)$ by a spacer/3-fan modification with $z = a - 1$.

In case (1,3), G is obtained from $G^r_{\nu,1}, \nu \geq 6$ by a spacer/3-fan modification with $z = a - 1$.

In case (2,1) ((2,2)), G is obtained from $G_{\nu,1}, \nu = 7 (\nu \geq 8)$ by a spacer/3-fan modification with $z = a - 1$.

In case (2, 3), G is obtained from $G_{\nu,0}, \nu \geq 8$ by a spacer/3-fan modification with $z = a - 1$.

In case (3,1) ((3,2)), G is obtained from $G_{\nu,1}, \nu \geq 9 (\nu \geq 10)$ by adding a 2-jumper over $b(b-1)$.

In case (3,3), G is obtained from $G_{\nu,0}, \nu \geq 10$ by adding a 2-jumper over $b(b-1)$.

That completes the possibilities with $s \neq a$. Thus, by symmetry, we hereafter assume that $\{b, b+1, a\}$ is a cut and that either $\{c-1, c, d\}$ or $\{d, d+1, c\}$ is a cut. Of course $a = d$ and $c = b+1$ are possible, but if $a \neq d$, then, by 3(c) of the Tight Trap Two Lemma, P and Q are each either small 2-heads or totally restricted spacers; moreover, even when $a = d$, at least one of them is so (except that the qualifier "totally" need not necessarily apply) because of the degree restriction in condition (3a). Accordingly, we continue, by symmetry, with the following cases.

(B) $a = d$, $[b, c] \subseteq \Sigma_a$ and Q is either a small 2-head or a restricted spacer. Now $G[b, c] = [b, c]^2$, for a missing jumper will put $a(a+1) \in E_n(C)$. Thus, by the Tight Trap Two Lemma, the only remaining structure in question concerns the edges at a. But if P is a small 2-head or a restricted spacer, then all of them are allowed (other than to restricted vertices) and we have the simplest of the odd barrier type ($x_1 = b, x_2 = c, y_1 = a$). If that isn't the case, then again by the Tight Trap Two Lemma, $N(a) = \{a-1, a+1, a+2\}$ and G is a member of C_3 modified by the 2-jumper $(b-2)(b+1)$.

Thus we assume hereafter that $a \neq d$ and that P and Q are each either small 2-heads or totally restricted spacers.

There are three nonsymmetric possibilities depending on whether $0, 1$ or 2 of P and Q are small 2-heads. The three cases are similar so, in order to be more specific and yet representative, we assume that P is a totally restricted spacer and that Q is a small 2-head.

(C) $\{b, b+1, a\}$ and $\{c-1, c, d\}$ are cuts with $a \neq d$. Thus $b = a+4, d = c+2, N(c+1) = \{c-1, c, c+2\}$ and $\{b+1\} \subseteq N(b-2) - \{b-3, b-1\} \subseteq \{b+1, a, b\}$.

Let $f = (a-1)a$ and $g = d(d+1)$. Of course $f = g$ is possible, but in any case, the Crossed Cuts Lemma gives $A(f) \subseteq [b+1, c-1] \cup \{d-1\}$ and $A(g) \subseteq [b+1, c-1] \cup \{a+1\}$. Moreover, by the Trap One Lemma, $d-1$ can only be in $A(f)$

if $a = d+2$ and $N(d) = \{d-1, d+1, a\}$. (The converse does not hold.) Likewise, if $a+1 \in A(g)$, then $a = d+2$ and $N(a) = \{d, a-1, a+1\}$.

Now there exists $y \in [b,c]$ such that $\Sigma_a \cap [b,c] = [b,y]$, that is $\{x, x+1, a\}$ is a cut for all $x \in [b, y-1]$, but $\{y, y+1, a\}$ is not a cut. First suppose that $y = c$. Thus $G[b,c] = [b,c]^2$ and, as above, $A(f) = \{d-1\}, a = d+2, N(d) = \{a, d+1, d-1\}$. But now it is clear that the only cut associated with $g = d(d+1)$ is $\{d, d+1, a+1\}$, that is, $[d-1, a+1]$ is a totally restricted spacer. Moreover, because of the existing cuts, $N(a-1) - \{a, d\} \subseteq \{c-1, c\}$ and $c-1 \in N(a-1)$ in order to have $c(c+1) \in E_c(C)$. Since $G[b,c] = [b,c]^2, G$ is a special case of a 3-fan from the middle vertex of a spacer.

Thus we can assume that $y < c$. Likewise, by symmetry, we can assume that $x > b$ where $x \in [b,c]$ such that $\Sigma_d \cap [b,c] = [x,c]$. The Crossed Cuts Lemma also gives that $x \geq y-2$ as well as the existence of a $t \in [d, a-1]$ such that $\{y, y+1, t\}$ is a cut, while $\{y, y+1, s\}$ is not a cut for any $s \in [t+1, a]$. Because t was picked "closest" to a we also have that $\Sigma_s \cap [y,c] = \emptyset$ for all $s \in [t+1, a]$.

Let $h = t(t+1)$ and recall that $f = (a-1)a$. Now, by the Crossed Cuts Lemma with respect to $\{y, y+1, t\}$ we have $A(f) \subseteq \{y, d-1\}$ and $A(h) \subseteq \{y, a+1\}$.

Suppose first that $y \notin A(f) \cap A(h)$. Then, by the Trap One Lemma, $[d-1, a+1]$ is a spacer, and hence a totally restricted spacer by the second Crossed Cuts Corollary. And since $y \notin A(f) \cap A(h)$ we have $G[b,c] = [b,c]^2$ and that $N(a-1) - \{a-2, a\} \subseteq \{y-1, y, y+1\}$ and must include at least one of the latter. That is, G is obtained from $G_{\nu,1}$ (because of the choice of P and Q) by a spacer/3-fan modification.

So we now assume that $y \in A(f) \cap A(h)$. Thus $[t,a] \subseteq \Sigma_y$. Moreover, because of the paired cuts $\{y-1, y, a\}$ and $\{a-1, a, y\}$ we must have $(y-1)(a-1) \in E(G)$. Similarly, we must have $(y+1)(t+1) \in E(G)$. Finally, we see that $G[b,y] = [b,y]^2, G[t,a] = [t,a]^2, y+1 \notin N(y-1),\ t+1 \notin N(t-1)$ and the sets $N(a) \cap [b,y]$ and $N(y) \cap [t,a]$ are optional. By iteration of this process we see that G is of the odd barrier type.

(D) $\{b, b+1, a\}$ and $\{d, d+1, c\}$ are cuts with $a \neq d$. As before, there exists $y \in [b,c]$ such that $\Sigma_a \cap [b,c] = [b,y]$. And we continue to let f denote the edge $(a-1)a \in E_n(C)$. Now $A(f) \cap [y,c] \neq \emptyset$ by the Crossed Cuts Lemma and we let u be the member of this set that is "closest" to y, that is $u \in A(f) \cap [y,c]$ while $A(f) \cap [y,u) = \emptyset$.

In fact $u = y$. For if $u \neq y$, we consider a cut $S = \{y, y+1, s\}$. The associate s is in $[d-1, a-1]$ by the Crossed Cuts Lemma applied to S and $\{b, b+1, a\}$. By Crossed Cuts Corollary 1 we now have $u > y+1$. Thus S crosses $\{a-1, a, u\}$ and hence we have $y+2 = u = c = s-1$. But now, by Crossed Cuts Corollary 1, any associate of $(c-1)c$ must be in $[a, b-1]$ and hence by the Crossed Cuts Lemma, a is an associate of $(c-1)c$. Consequently, a is an associate of $y(y+1)$ which contradicts the choice of y.

Thus $u = y$, including the possibility that $y = c$. As before, there is a vertex $t \in [d, a]$ such that $\Sigma_y \cap [d, a] = [t, a]$. Moreover, $(a-1)(y-1) \in E(G)$, $G[b, y] = [b, y]^2$, $G[t, a] = [t, a]^2$, there is an edge $(t+1)w'$ with $w' \in [y+1, t-1]$, and the sets $N(a) \cap [b, y]$ and $N(y) \cap [t, a]$ are optional. If $y = c$, then $t = d$ and G is the simplest of the even barrier type; if $y \neq c$, we iterate the process and again conclude that G is of the even barrier type. ∎

BEYOND C_4

In the characterization of C_4 graphs on more than 12 vertices, the existence of pivot cuts was critical in that they separated 2-heads which, by use of the Trapping Lemmas, were found to be the basic building blocks. However, in $\mathcal{H}_k$, $k \geq 5$, a pivot cut separates heads, not both of which are 2-heads. To overcome this problem we introduce the following definition which does provide appropriate building blocks (at least in C_5).

Definition. A k-head $P = [a, b]$, with $k \geq 2$, is *irreducible* iff there exists a pivot cut $S = \{x, x+1, s\}$ (assume $x \in [b, c]$ and $s \in [d, a]$ by symmetry) separating P from its cohead $Q = [c, d]$ such that S is the only pivot cut with all of its vertices in $[s, x+1]$.

Note that this S is "closest" to P in the same sense as for 2-heads and, as in that case, we easily see that S is either tight or half tight. And again, in the half tight case, $[s-1, a+1]$ is a totally restricted spacer. Also note that the Tight and Half Tight Trap Two Lemmas describe the structure of irreducible 2-heads. We now describe the structure of an irreducible k-head for $k \geq 3$, noting the strong similarity with that of an irreducible 2-head.

We consider the tight case first. By symmetry, we let $x = b$ and $s = a$ where $P = [a, b]$ is an irreducible k-head, $k \geq 3$.

Either $[a, b]$ contains a noncontractible edge or it does not. Suppose first that there is a noncontractible edge in $[a+1, b-1]$. Let $u, v \in [a, b]$ be such that all edges in $[a, u]$ are contractible while all edges in $[u, v]$ are noncontractible and $v(v+1) \in E_c(C)$. Let $f = (v-1)v$ and let $T = V(f) \cup \{t\}$ be an associated cut. Since $v(v+1) \in E_c(C)$ and P is irreducible, $t \in [b+1, a] \cup \{u-1\}$. By the Crossed Cuts Lemma, if $A(f) \cap [b+1, a] \neq \emptyset$, then $V(f) \cup \{a\}$ is a cut unless $v = u+1 = a+2$. If $V(f) \cup \{a\}$ is a cut, then it must be a cut trapping just one contractible edge from P, namely, $a(a+1)$, and thus the structure of $G[a, v]$ is determined by the Trap One Lemma. In the exceptional case when $v = u+1 = a+2$, we must have $dg(a) = 3$ and $(a-1)a \in E_c(C)$ by the Crossed Cuts Lemma. (And again we have the structure of $G[a, v]$.)

Whether $a = u - 1$ or not, if there is a noncontractible edge $g \in [a, b]$ with $A(g) \cap \{a, a-1\} = \emptyset$, then by the irreducibility of P and the Crossed Cuts Lemma, the only cut associated with g is as in (1) of the Trap One Lemma, and hence, by the second Crossed Cuts Corollary, is an edge of a totally restricted spacer. Moreover, unless $g = (a+1)(a+2)$, there must be a jumper within $[a+1, b+1]$, over g so that $A(g) \cap \{a, a-1\} = \emptyset$.

Thus if P contains a noncontractible edge, P consists of strings of contractible edges separated by totally restricted spacers except when $(a+1)(a+2) \in E_n(C)$, in which case the initial segment, $[a, v]$ is as determined by (1) or (2) of the Trap One Lemma. All remaining edges from $[a, b+1]$ to P are optional, subject only to the following.

(1) G must be 3-connected; in particular all vertices must be of degree at least three,
(2) for each contractible edge and each pair of noncontractible edges in a spacer in $[a+1, b]$ there must be at least one corresponding jumper <u>over</u> it that is within $[a+1, b+1]$,
(3) all restricted vertices in the above must remain so, and
(4) those vertices in $[a, v]$ whose neighborhoods are determined by the Trap One Lemma can have no other neighbors.

If all edges in P are contractible, then all edges in $[a, b+1]$ are optional subject only to Conditions (1) and (2) above.

Likewise, if S is a half tight cut, the only modification to the above is that the segment $[a-2, v]$ is determined by (3) of the Trap One Lemma.

Note that the above conditions in themselves guarantee the desired contractibility of all edges within P and has no effect (by the Crossed Cut Lemma) on the contractibility of edges outside of P. That is, we know the structure of P.

If $k = 3$ in the above and $G \in C_5$, then Q is a 2-head trapped by S and we have the remaining structure of G as before.

Claim. All other members of C_5 are obtained from C_4 graphs by either adding a 2-jumper over a noncontractible edge (with restrictions like those in going from a C_3 graph to a C_4 graph) or by one of the following two operations.

Expanding a spacer/3-fan into a spacer/4-fan. Let $G \in \mathcal{F}_4$ and let all vertices in G be labelled as in the definition of a spacer/3-fan graph; in particular, z is the middle vertex of the spacer, β is the middle vertex of the fan with $b \le \beta \le c$ and where $(b-1)b,\ c(c+1) \in E_c(C)$ and $[c, b]$ contains all four contractible edges of G that are in C. Now "split" β into vertices β' and β'' by replacing the path

$(\beta-1,\beta,\beta+1)$ with the path $(\beta-1,\beta',\beta'',\beta+1)$ where β' has the same neighbors (and optional neighbors) within $[1,\beta-1]$ as β did in G, where β'' has the same neighbors (and optional neighbors) within $[\beta+1,\nu]$ as β did in G, and where all edges on $\{\alpha,\beta',\beta'',\gamma,z\}$ are optional subject to the following conditions.

(i) z,β' and β'' must all be of degree at least three,

(ii) if $\alpha\gamma$ is not selected, then both $z\alpha$ and $z\gamma$ must be selected, and

(iii) if $\beta=b=2$ $(\beta=c=\nu-1)$, then at least one of $z\gamma$ or $z\beta''$ $(z\alpha$ or $z\beta')$ must be selected so that $\alpha\beta'$ $(\beta''\gamma)$ is contractible.

Splitting a barrier at a pivot. Let $G\in\mathcal{B}_4$ and let x be one of the pivot vertices in the barrier where all vertices are labelled as in the definition of the $\mathcal{B}_4$ graphs. We consider three nonsymmetric cases: (1) $x=x_i\neq b,c$ $(x=y_j\neq d,a$ is similar), (2) $x=x_2=c$ and $y_2=d\neq a$ $(x=y_2=d\neq a$ is similar) and (3) $x=y_1=a=d$.

In all cases the idea is the same and only varies due to limiting situations. The common theme is that x is "split" into two vertices x' and x'' with the path $(x-1,x,x+1)$ replaced by the path $(x-1,x',x'',x+1)$ and with the neighbors (including optional neighbors) of x divided between x' and x'' so that we don't have a neighbor of x'' more than two vertices around C (in the positive direction) from a neighbor of x'. The result is that $x'x''$ is a fifth contractible edge in C_x. We now make this more precise.

(1) $x=x_i\neq b,c$. Thus $1<i<k$. We form G_x from G by deleting x and adding new vertices x' and x''. Letting N_x be the neighborhood function on G_x, we take $x-2,x-1,x''\in N_x(x')$ and $x',x+1,x+2\in N_x(x'')$. For the optional edges (that were incident with x in G) we pick y' and $y''\in[y_i,y_{i-1}]$ with $y'\geq y''-2$ and make $x'y$ optional for all $y\in[y',y_{i-1}]$ and we make $x''y$ optional for all $y\in[y_i,y'']$. The remaining edges and optional edges of G_x (that is, that are not incident with x' or x'') are the same as those in G that are not incident with x.

(2) $x=x_2=c$ and $y_2=d\neq a$. Thus, as is true in (1) as well, $(x-1)(a-1)$ is a (P,Q)-splitter in G. In fact, the only difference from (1) is that, if Q is a restricted spacer, then we can choose $y',y''\in[q,a]-\{d-1\}$ with $y''=d+1$ allowed when $y'=q$. Of course at least one of the optional edges at x'' must be selected so that G is 3-connected.

(3) $x=y_1=a=d$. Thus $x_2=c$ here as well as in (2) and the similar modification holds for the choice of y' and y'' if Q is a restricted spacer. The new feature is that the symmetric modification is imposed if P is a restricted spacer and the optional edges xp and xq in G are replaced by the optional edges $x''p$ and $x'q$ in G_x.

Proof of the Claim (Outline). First assume that G contains no pivot cuts. Then, by the No Pivots Lemma we know that $\nu \leq 15$ and that G is one of the graphs in $\mathcal{S}_5$.

Thus we may assume that G contains a pivot cut and let S be a pivot cut in G closest to a 2-head (i.e. from all pivot cuts in G, S is closest to its associated 2-head). There we are faced with two possibilities; S is half tight or S is tight against the 2-head $P = [a, b]$.

1. $S = \{b, b+1, a-2\}$ is half tight against P. Using the Half Tight Trap Two Lemma we are able to determine all edges incident with vertices in $[a-2, b-1]$ and that $[a-3, a+1]$ is a totally restricted spacer. Consequently $S' = \{b, b+1, a-3\}$ is also a pivot cut trapping a 2-head containing the remaining two contractible edges in C. Let $Q = [c, d]$ be the 2-head so trapped. Thus there is a closest cut to Q trapping it which is either tight or half tight. The possible structures for Q can now be determined from the appropriate Trap Two Lemma, while the structure of G between the heads $[a-3, b]$ and $[c, d]$ can be determined by similar arguments to those used in the proof of the C_4 theorem. Graphs arising in this case are clearly obtained as in the claim.

2. $S = \{b, b+1, a\}$ is a tight cut trapping P. Using the Tight Trap Two Lemma we are able to determine all edges incident with vertices in $[a+1, b-1]$. We also note that S traps $Q = [c, e]$, the cohead of P, where Q is either an irreducible 3-head, a case we have already dealt with in the discussion preceding the claim, or else we have a pivot cut within Q which separates off an irreducible 2-head Q' as in the Trap Two Lemmas. Now we may proceed as in the C_4 case to determine the structure of G between the 2-heads P and Q' and one can easily see that G is obtained as in the claim. ∎

A similar approach can be taken in $\mathcal{H}_k$ for $k \geq 6$, but the new difficulty is in determining how the irreducible heads "meld together". An example of an $\mathcal{H}_6$ graph is given in Figure 7 where three irreducible 2-heads meld together at a common barrier.

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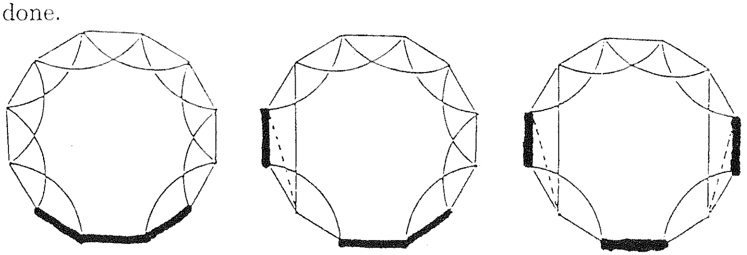


Figure 1

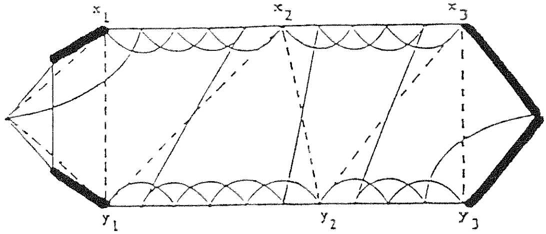
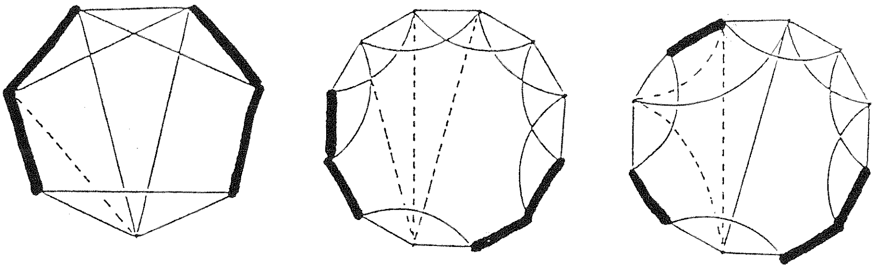


Figure 2



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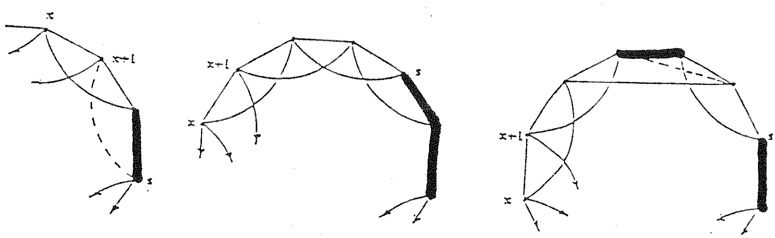

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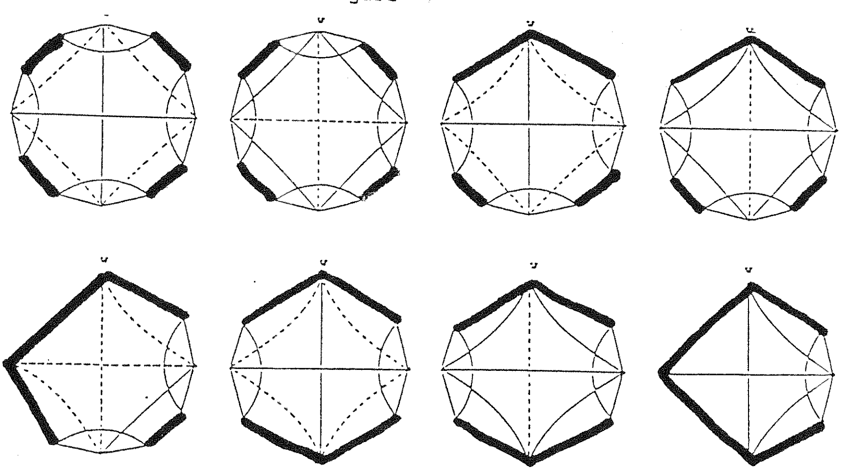

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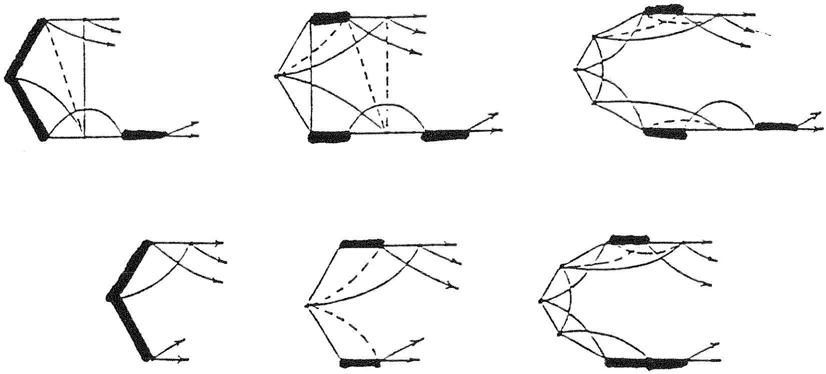

Figure 6

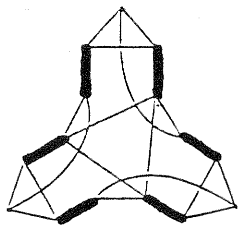

Figure 7

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