The Algorithmic Complexity of Signed Domination in Graphs

Johannes H. Hattingh *
Department of Mathematics
Rand Afrikaans University, P.O. Box 524
Auckland Park, South Africa

Michael A. Henning †
Department of Mathematics
University of Natal, Private Bag X01
Pietermaritzburg 3209 South Africa

Peter J. Slater
Mathematical Sciences
The University of Alabama in Huntsville
Huntsville, Alabama 35899

Abstract

A two-valued function f defined on the vertices of a graph $G = (V, E)$, $f : V \to \{-1, 1\}$, is a signed dominating function if the sum of its function values over any closed neighborhood is at least one. That is, for every $v \in V, f(N[v]) \geq 1$, where $N[v]$ consists of v and every vertex adjacent to v. The weight of a signed dominating function is $f(V) = \sum f(v)$, over all vertices $v \in V$. The signed domination number of a graph G, denoted $\gamma_s(G)$, equals the minimum weight of a signed dominating function of G. The upper signed domination number of a graph G, denoted $\Gamma_s(G)$, equals the maximum weight of a minimal signed dominating function of G. In this paper we present a variety of algorithmic results on the complexity of signed and upper signed domination in graphs.

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1 Introduction

In this paper we shall use the terminology of [3]. Specifically, if T is a rooted tree with root r and v is a vertex of T, then the *level number* of v, which we denote by $l(v)$, is the length of the unique r-v path in T. If a vertex v of T is adjacent to u and $l(u) > l(v)$, then u is called a *child* of v, and v is the *parent* of u. A vertex w is a *descendant* of v (and v is an *ancestor* of w) if the level numbers of the vertices on the v-w path are monotonically increasing. We will refer to an end-vertex of T as a *leaf*.

Let $G = (V, E)$ be a graph and let v be a vertex in V. The *closed neighborhood* $N[v]$ of v is defined as the set of vertices within distance 1 from v, i.e., the set of vertices $\{u \mid d(u,v) \leq 1\}$. The *open neighborhood* $N(v)$ of v is $N[v] - \{v\}$. For a set S of vertices, we define the open neighborhood $N(S) = \cup N(v)$ over all v in S and the closed neighborhood $N[S] = N(S) \cup S$. A set S of vertices is a *dominating set* if $N[S] = V$. The *domination number* of a graph G, denoted $\gamma(G)$, is the minimum cardinality of a dominating set in G. Similarly, the *upper domination number* $\Gamma(G)$ is the maximum cardinality of a minimal dominating set in G.

Let $g : V \to \{0,1\}$ be a function which assigns to each vertex of a graph an element of the set $\{0, 1\}$. To simplify notation we will write $g(S)$ for $\sum g(v)$ over all v in the set S of vertices, and we define the *weight* of g to be $g(V)$. We say g is a *dominating function* if for every $v \in V$, $g(N[v]) \geq 1$. The domination number and upper domination number of a graph G can be defined as $\gamma(G) = \min\ \{ \text{g(V)} \mid \text{g is a dominating function on G}\}$ and $\Gamma(G) = \max\ \{ \text{g(V)} \mid \text{g is a minimal dominating function on G}\}$.

A signed dominating function has been defined similarly in [6]. A function $g : V \to \{-1, 1\}$ is a *signed dominating function* if $g(N[v]) \geq 1$ for every $v \in V$. A signed dominating function is minimal if and only if for every vertex $v \in V$ with $g(v) = 1$, there exists a vertex $u \in N[v]$ with $g(N[u]) \in \{1, 2\}$. The *signed domination number* for a graph G is $\gamma_s(G) = \min\ \{g(V) \mid g$ is signed dominating function on $G\}$ and the *upper signed domination number* for a graph G is $\Gamma_s(G) = \max\ \{g(V) \mid g$ is a minimal signed dominating function on $G\}$. In [6] various properties of the signed domination number are presented.

There is a variety of possible applications for this variation of domination. By assigning the values -1 or $+1$ to the vertices of a graph we can model such things as networks of positive and negative electrical charges, networks of positive and negative spins of electrons, and networks of people or organizations in which global decisions must be made (e.g. yes-no, agree-disagree, like-dislike, etc.). In such a context, for example, the signed domination number represents the minimum number of people whose positive votes can assure that all local groups of voters (represented by closed neighorhoods) have more positive than negative voters, even though the entire network may have far more people who vote negative than positive. Hence this variation of domination studies situations in which, inspite of the presence of negative vertices, the closed neighorhoods of all vertices are required to maintain a positive sum.

In this paper we present a variety of algorithmic results. We show that the decision problem corresponding to the problem of computing γ_s is NP-complete, even when restricted to bipartite graphs or chordal graphs. For a fixed k, we show that the problem of determining if a graph has a signed dominating function of weight at most k can also be NP-complete. We then show that the decision problem corresponding to the problem of computing Γ_s is NP-complete, even when restricted to bipartite graphs. A linear time algorithm for finding a minimum signed dominating function in an arbitrary tree is presented.

2 Complexity Issues for Signed Domination

The following decision problem for the domination number of a graph is known to be NP-complete, even when restricted to bipartite graphs (see Dewdney [5]) or chordal graphs (see Booth [1] and Booth and Johnson [2]).

Problem: **DOMINATION** (DM)
INSTANCE: A graph $G = (V, E)$ and a positive integer $k \leq |V|$.
QUESTION: Is $\gamma(G) \leq k$?

We will demonstrate a polynomial time reduction of this problem to our signed domination problem.

Problem: **SIGNED DOMINATION** (SD)
INSTANCE: A graph $H = (V, E)$ and a positive integer $j \leq |V|$.
QUESTION: Is $\gamma_s(H) \leq j$?

Theorem 1 *Problem SD is NP-complete, even when restricted to bipartite or chordal graphs.*

Proof. It is obvious that SD is a member of NP since we can, in polynomial time, guess at a function $f : V \to \{-1, 1\}$ and verify that f has weight at most j and is a signed dominating function.

We next show how a polynomial time algorithm for SD could be used to solve DM in polynomial time. Given a graph $G = (V, E)$ and a positive integer k construct the graph H by adding to each vertex v of G a set of $deg_G(v) + 1$ paths of length two. Let $m = |E|$ and $n = |V|$. We have $|V(H)| = n + 2\sum_{v \in V(G)}(deg_G(v) + 1) = 3n + 4m$

and $|E(H)| = m + 2\sum_{v \in V(G)}(deg_G(v) + 1) = 2n + 5m$, and H can be constructed in polynomial time. Note that if G is a bipartite or chordal graph, then so too is H.

Clearly, for any signed dominating function g of a graph F, if v is an end-vertex and w is its neighbor (so $N_F(v) = \{w\}$), then $g(v) = g(w) = 1$. In particular, for any signed dominating function g of H, if $g(v) = -1$, then $v \in V(G) \subseteq V(H)$. Further, we note that if $g : V(H) \to \{-1, 1\}$, $v \in V(G)$ and $g(w) = -1$ for every $w \in N_G[v]$, then, because $|N_G[v]| = |N_H(v) - N_G[v]| = deg_G(v) + 1$, we would have $g(N_H[v]) \leq 0$. That is, if $g : V(H) \to \{-1, 1\}$ is a signed dominating function for H, then $g(v) = 1$ for $v \in V(H) - V(G)$, and $\{v \in V(G) \mid g(v) = 1\}$ is a dominating set for G. It follows that if we let $j = |V(H)| - 2(n - k) = 4m + n + 2k$, then $\gamma(G) \leq k$ if and only if $\gamma_s(H) \leq j$. The completes the proof of Theorem 1. □

Problem DM is polynomial for fixed k. To see this, let $G = (V, E)$ be a graph with $|V| = p$. If $k \geq p$, then V is a dominating set of G of cardinality at most k. On the other hand, if $k < p$, then consider all the r-subsets of V, where $r = 1, \ldots, k$. There are $\sum_{r=1}^{k} \binom{p}{r}$ of these subsets, which is bounded above by the polynomial $\sum_{r=1}^{k} p^r$. It takes a polynomial amount of time to verify that a set is or is not a dominating set. These remarks show that it takes a polynomial amount of time to verify whether G has a dominating set of cardinality at most k when k is fixed. Hence for fixed k, problem $DM \in P$.

In contrast, we now show that for a fixed k, the problem SD can be NP-complete. To see this, we will demonstrate a polynomial time reduction of the signed domination problem to the following zero signed domination problem.

Problem: **ZERO SIGNED DOMINATION** (ZSD)
 INSTANCE: A graph $G = (V, E)$.
 QUESTION: Does G have a signed dominating function of weight at most 0?

Theorem 2 *Problem ZSD is NP-complete, even when restricted to bipartite or chordal graphs.*

Proof. It is obvious that ZSD is a member of NP since we can, in polynomial time, guess at a function $f : V(G) \to \{-1, 1\}$ and verify that f has weight at most 0 and is a signed dominating function.

We next show how a polynomial time algorithm for ZSD could be used to solve SD in polynomial time. Let L be the graph of Figure 1. Then L has a signed dominating function of weight -1 as illustrated. In fact, $\gamma_s(L) = -1$. Note that L is chordal.

Given a graph $H = (V, E)$ and a positive integer j, let $G = H \cup \bigcup_{i=1}^{j} L_i$, where $L_i \cong L$ for $i = 1, \ldots, j$. It is clear that G can be constructed in polynomial time. Note that if H is chordal, then so too is G.

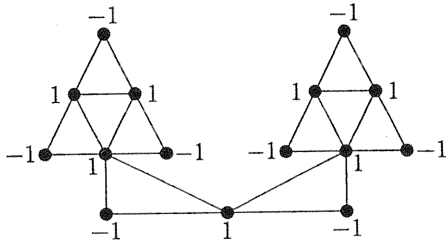

Figure 1: The graph L.

We now show that $\gamma_s(H) \le j$ if and only if $\gamma_s(G) \le 0$. Suppose first that $\gamma_s(H) \le j$ and that f is a signed dominating function of H of weight $\gamma_s(H)$. Let f_i be any signed dominating function of weight -1 for L_i for $i = 1, \ldots, j$. Define $g : V(G) \to \{-1, 1\}$ by $g(x) = f_i(x)$ if $x \in V(L_i)$, $(i = 1, \ldots, j)$, while $g(x) = f(x)$ for $x \in V(H)$. Then g is a signed dominating function of G of weight $\gamma_s(H) + j(-1) \le j - j = 0$, whence $\gamma_s(G) \le 0$. Conversely, suppose that $\gamma_s(G) \le 0$ and that g is a signed dominating function of weight $\gamma_s(G)$. Let f be the restriction of g on $V(H)$ and let f_i be the restriction of g on $V(L_i)$ for $i = 1, \ldots, j$. Then $\gamma_s(H) + j(-1) = \gamma_s(H) + \sum_{i=1}^{j} \gamma_s(L_i) \le f(V(H)) + f_i(V(L_i)) = g(V(G)) = \gamma_s(G) \le 0$, so that $\gamma_s(H) \le j$.

Let F be the graph of Figure 2. Then F has a signed dominating function of weight -1 as illustrated. In fact, $\gamma_s(F) = -1$. Note that F is bipartite. Given a graph $H = (V, E)$ and a positive integer j, let $G = H \cup \cup_{i=1}^{j} F_i$, where $F_i \cong F$ for $i = 1, \ldots, j$. It is clear that G can be constructed in polynomial time. Note that if H is bipartite, then so too is G. Proceeding now as in the preceeding paragraph, we may show that $\gamma_s(H) \le j$ if and only if $\gamma_s(G) \le 0$. This completes the proof of the theorem. □

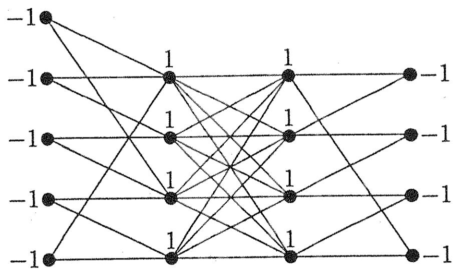

Figure 2: The graph F.

Next we consider the following decision problem corresponding to the problem of computing $\Gamma_s(G)$. If a graph G is bipartite or chordal, then it is known that $\beta(G) = \Gamma(G)$, where $\beta(G)$ is the maximum cardinality of an independent set of G (see [4] and [9]). Since the maximum independent set problem can be solved in polynomial time for these two families of graphs, so too can the problem of finding $\Gamma(G)$ for G either bipartite or chordal. We show that the decision problem

Problem: **UPPER SIGNED DOMINATION** (USD)
INSTANCE: A graph $G = (V, E)$ and a positive integer $k \leq |V|$.
QUESTION: Is there a minimal signed dominating function of weight at least k for G?

is NP-complete, even when restricted to bipartite or chordal graphs, by describing a polynomial transformation from the following known NP-complete decision problem [7]:

Problem: **ONE-IN-THREE 3SAT** ($OneIn3SAT$)
INSTANCE: A set U of variables, and a collection C of clauses over U such that each clause $c \in C$ has $|c| = 3$ and no clause contains a negated variable.
QUESTION: Is there a truth assignment for U such that each clause in C has exactly one true literal?

Theorem 3 *Problem USD is NP-complete, even when restricted to bipartite graphs.*

Proof. It is obvious that USD is a member of NP since we can, in polynomial time, guess at a function $f : V \rightarrow \{-1, 1\}$ and verify that f has weight at least k and is a minimal signed dominating function. To show that USD is an NP-complete problem when restricted to bipartite graphs, we will establish a polynomial transformation from the NP-complete problem $OneIn3SAT$. Let I be an instance of $OneIn3SAT$ consisting of the (finite) set $C = \{c_1, \ldots, c_m\}$ of three literal clauses in the n variables $u_1, u_2, \ldots, u_n$. We transform I to the instance (G_I, k) of USD in which $k = 3n + 4m$ and G_I is the bipartite graph constructed as follows.

Let H be the path u, v_1, v_2, v_3, v_4 and let $H_1, H_2, \ldots, H_n$ be n disjoint copies of H. Corresponding to each variable u_i we associate the graph H_i. Let $u_i, v_{i,1}, v_{i,2}, v_{i,3}, v_{i,4}$ be the names of the vertices of H_i that are named u, v_1, v_2, v_3 and v_4, respectively, in H. Corresponding to each 3-element clause c_j we associate a path F_j on four vertices with one end-vertex labeled c_j. The construction of our instance of USD is completed by joining the vertex c_j to the three special vertices that name the three literals in clause c_j. Let G_I denote the resulting bipartite graph.

It is easy to see that the construction can be accomplished in polynomial time. All that remains to be shown is that I has a one-in-three satisfying truth assignment if and only if $\Gamma_s(G_I) \geq k$, where $k = 3n + 4m$.

First suppose that I has a one-in-three satisfying truth assignment. We construct a minimal signed dominating function f of G_I of weight k, which will show that $\Gamma_s(G_I) \geq k$. For each $i = 1, 2, \ldots, n$, do the following. If $u_i = T$, then let $f(u_i) = f(v_{i,2}) = f(v_{i,3}) = f(v_{i,4}) = 1$ and let $f(v_{i,1}) = -1$. On the other hand, if $u_i = F$, then let $f(u_i) = -1$ and let $f(v_{i,1}) = f(v_{i,2}) = f(v_{i,3}) = f(v_{i,4}) = 1$. For each $j = 1, 2, \ldots, m$, let $f(v) = 1$ for each vertex v of F_j. In each case $f(N[v_{i,1}]) = 1$ and $f(N[v_{i,4}]) = 2$. Since I has a one-in-three satisfying truth assigment, it follows that $f(N[c_j]) = 1$ for all $j = 1, 2, \ldots, m$. Hence, for every vertex v of G_I with $f(v) = 1$, there exists a vertex $u \in N[v]$ with $f(N[u]) \in \{1, 2\}$. Since f has weight k and $f(N[v]) \geq 1$ for all $v \in V(G_I)$, the function f is a minimal signed domination function of weight k, implying that $\Gamma_s(G_I) \geq k$.

Conversely, assume that $\Gamma_s(G_I) \geq k$. Let g be a minimal signed dominating function of weight at least k. Note that at least one vertex of H_i, $i = 1, 2, \ldots, n$ must be assigned a -1 under g, since otherwise $g(N[w]) \notin \{1, 2\}$ for all $w \in N[v_{i,2}]$, contradicting the minimality of g. Hence $g(V(H_i)) \leq 3$ for all $i = 1, 2, \ldots, n$. Since $g(V(F_j)) \leq 4$ for all $j = 1, 2, \ldots, m$, we have that $g(V(G_I)) \leq 3n + 4m$ with equality if and only if $g(V(H_i)) = 3$ for all i and $g(V(F_j)) = 4$ for all j. Equality holds, since we have assumed that $g(V(G_I)) \geq k$. Let $j \in \{1, 2, \ldots, m\}$. The two central vertices of F_j each have closed neighborhood sum 3 under g. The minimality of g implies that there exists a vertex in the closed neighborhood of the vertex adjacent to c_j in F_j with closed neighborhood sum 1 or 2 under g. It follows that $g(N[c_j]) = 1$ or 2. This implies that the sum of the function values under g of the three special vertices that name the three literals in clause c_j, and that are joined to the vertex c_j, is either equal to -1 or 0. The first possibility implies that exactly two of these three special vertices joined to c_j are assigned the value -1 under g and one is assigned the value 1 under g, while the second possibility cannot occur. We now obtain a truth assignment $t : \{u_1, u_2, \ldots, u_n\} \to \{T, F\}$ as follows. We merely set $t(u_i) = T$ if $g(u_i) = 1$ and $t(u_i) = F$ if $g(u_i) = -1$. By our construction of the graph G_I, it follows that each clause c_j of I contains exactly one variable u_i with $g(u_i) = 1$. Hence I has a one-in-three satisfying truth assignment. Therefore, I has a one-in-three satisfying truth assignment if and only if $\Gamma_s(G_I) \geq k$, completing the proof. □

3 A Linear Algorithm for Trees

Next we present a linear algorithm for finding a minimum signed dominating function in a nontrivial tree T. The algorithm roots the tree T and associates various variables with the vertices of T as it proceeds. For any vertex v, the variable $MinSum$ denotes the miminum possible sum of values that may be assigned to v and its children. So $MinSum = 1$ or 2 for the root v, depending on whether v has even or odd degree, respectively; otherwise, $MinSum = 0$ or 1, depending on whether v has even or odd degree, respectively. The variable $ChildSum$ denotes the sum of the values assigned to the children of v, while the variable $Sum(v)$ denotes the sum of the values assigned to v and the children of v.

Algorithm SD. *Given a nontrivial tree T on n vertices, root the tree T and relabel the vertices of T from 1 to n so that $label(w) > label(y)$ if the level of vertex w is less than the level of vertex y. Note the root of T will be labeled n.*

```
for i := 1 to n do
begin
    1.  deg i ← degree of the vertex i in T;

    2.  if i = n
        then if deg i is odd
             then MinSum ← 2
             else MinSum ← 1;

    3.  if i < n
        then if deg i is odd
             then MinSum ← 1
             else MinSum ← 0;

    4.  if vertex i is a leaf and i < n
        then ChildSum ← 0
        else ChildSum ← sum of the values of the children of
                        vertex i;

    5.  if ChildSum < MinSum
        then begin
                 5.1.  while ChildSum < MinSum - 1 do
                             increase the value of the children
                             of vertex i;

                 5.2.  f(i) ← 1;
             end
    6.       else if vertex i has a child w with Sum(w) = 0 or 1
             6.1. then f(i) ← 1
             6.2. else f(i) ← -1;

    7.  Sum(i) ← ChildSum + f(i);
end;
```

We now verify the validity of Algorithm SD.

Theorem 4 *Algorithm SD produces a minimum signed dominating function in a nontrivial tree.*

Proof. Let $T = (V, E)$ be a nontrivial tree of order n, and let f be the function produced by Algorithm SD. Then $f : V \rightarrow \{-1, 1\}$. For convenience, the variables $ChildSum$ and $MinSum$, which were used by Algorithm SD when it considered the vertex v, will be denoted by $ChildSum(v)$ and $MinSum(v)$, respectively.

Lemma 1 *When Algorithm SD assigns a value to the root r' of a subtree (or tree) T', the following three conditions will hold:*

1. *For any vertex $v \in T' - \{r'\}$, $f(N[v]) \geq 1$.*
2. *$Sum(r') \geq MinSum(r')$.*
3. *The initial value assigned to r' is the minimum value it can receive given the values of its descendants under f.*

Proof. We proceed by induction on the order in which the vertices were labeled. The first vertex assigned a value will be a leaf. Vacuously, the first condition holds. In the case of a leaf i, $ChildSum(i) = 0$ and $MinSum(i) = 1$, so that statements in Step 5 will be executed. The leaf i will be assigned the value 1 in Step 5.2, so that $Sum(i) = MinSum(i) = 1$ and the second and third conditions hold.

Next we assume that Algorithm SD assigns values to the first k vertices so that Conditions 1, 2 and 3 hold. We show that these conditions hold after the $(k+1)$st vertex is assigned a value.

We begin with Condition 1. Before the $(k+1)$st vertex is assigned a value, we can assume by the inductive hypothesis that all its descendants, other than its children, have closed neighborhood sums of at least one. These descendants will continue to have closed neighborhood sums of at least one after the $(k+1)$st vertex is asssigned a value, because even if some children of the $(k+1)$st vertex are reassigned values in Step 5.1 of the algorithm, their closed neighborhood sums will not decrease. Also, by the inductive hypothesis, any child w of vertex $k+1$ will have $Sum(w) \geq MinSum(w) \geq 0$. If $ChildSum(k+1) < MinSum(k+1)$, then the $(k+1)$st vertex is assigned the value 1 in Step 5.2 and $f(N[w]) = Sum(w) + 1 \geq 1$. Suppose, then, that the case of Step 6 of the algorithm holds. If $Sum(w) = 0$ or 1, then the case in Step 6.1 of the algorithm holds and the $(k+1)$st vertex will be assigned the value 1. So $f(N[w]) \geq 1$. If $Sum(w) > 1$, then $f(N[w]) \geq 1$, regardless of what value is assigned to vertex $k+1$. Thus, all descendants of the $(k+1)$st vertex will have closed neighborhood sums of at least one. The $(k+1)$st vertex, therefore, satisfy Condition 1.

We show next that after the $(k+1)$st vertex is assigned a value, $Sum(k+1) \geq MinSum(k+1)$. This is enforced in Steps 5 and 6 of the algorithm. In Step 5, if $Childsum(k+1) < MinSum(k+1)$, then the $(k+1)$st vertex is given the value 1 and the values assigned to its children are increased as much as necessary to bring $Sum(k+1)$ up to $MinSum(k+1)$. If $ChildSum(k+1) \geq MinSum(k+1)$, then, since $ChildSum(k+1)$ and $MinSum(k+1)$ differ in parity, $ChildSum(k+1) \geq MinSum(k+1)+1$. Hence $Sum(k+1) \geq MinSum(k+1)$, regardless of what value is assigned to the $(k+1)$st vertex. Thus the vertex r' satisfies Condition 2.

It remains to consider Condition 3. Let $v = r'$. Suppose the initial value assigned to v is 1. If v was assigned the value 1 in Step 5.2, then the values assigned to its children were increased until $ChildSum(v) = MinSum(v) - 1$. Thus $ChildSum(v) = -1$ or 0 if $deg\ v$ is even or odd, respectively, and v is not the root of T; otherwise, $ChildSum(v) = 0$ or 1 if $deg\ v$ is even or odd, respectively, and v is the root of T. It follows that for f to be a signed dominating function of T, the value for $f(v)$ must be 1. If v was assigned the value 1 in Step 6.1, then v has a child w with $Sum(w) = 0$ or 1. Thus $1 \leq f(N[w]) = Sum(w) + f(v) \leq f(v) + 1$, so $f(v) \geq 0$. Once again, the value for $f(v)$ must be 1. This completes the proof of the lemma. □

Since $MinSum(n) \geq 1$, an immediate consequence of Lemma 1 is the following:

Corollary 1 *The function f produced by Algorithm SD is a signed dominating function for T.*

To show that the signed dominating function f obtained by Algorithm SD is minimum, let g be any minimum signed dominating function for the rooted tree T. If $f \neq g$, then we will show that g can be transformed into a new minimum signed dominating function g' that will differ from f in fewer values than g did. This process will continue until $f = g$. Suppose, then, that $f \neq g$. Let v be the lowest labeled vertex for which $f(v) \neq g(v)$. Then *all* descendants of v are assigned the same value under g as under f. An immediate corollary of Lemma 1 now follows.

Corollary 2 *If $g(v) < f(v)$, then the initial value assigned to the vertex v was increased in Step 5 of Algorithm SD.*

Lemma 2 *If $g(v) < f(v)$, then the function g' defined by $g'(u) = f(u)$ if $u \in N[parent(v)]$ and $g'(u) = g(u)$ if $u \notin N[parent(v)]$ is a minimum signed dominating function of T that differs from f in fewer values than does g.*

Proof. By Corollary 2, the initial value assigned to the vertex v was increased in Step 5.1 of Algorithm SD and this occurs when the parent of v was being assigned a value. Let w be the parent of v. Thus g' is defined by $g'(u) = f(u)$ if $u \in N[w]$ and $g'(u) = g(u)$ for all remaining vertices u in V. Then $f(w) = 1$ and $Sum(w) = MinSum(w)$.

If w is the root of T, then $f(N[w]) = Sum(w) = MinSum(w)$. If $deg\ w$ is even, then $g(N[w])$ is odd, so $g(N[w]) \geq 1 = MinSum(w) = f(N[w])$. If $deg\ w$ is odd, then $g(N[w])$ is even, so $g(N[w]) \geq 2 = MinSum(w) = f(N[w])$. Hence $f(N[w]) \leq g(N[w])$. Furthermore, all vertices in $V - N[w]$ have the same values under g as under f. Thus $g' = f$ and $f(V) \geq g(V) = g(V - N[w]) + g(N[w]) = f(V - N[w]) + g(N[w]) \geq f(V - N[w]) + f(N[w]) = f(V)$. Consequently, we must have equality throughout. In particular, $g(V) = f(V)$; so $g' = f$ is a minimum signed dominating function of T.

If w is not the root, then $1 \leq f(N[w]) = Sum(w) + f(parent(w)) = MinSum(w) + f(parent(w)) \leq 1 + f(parent(w))$, so $f(parent(w)) \geq 0$. Thus $f(parent(w)) = 1$ and $f(N[w]) = MinSum(w) + 1$. Since all the descendants of w, other than its children, have the same values under g as under f, $g'(N[u]) = f(N[u])$ if $u = w$ or if u is a descendant of w. Furthermore, since $f(w) = 1$ and $f(parent(w)) = 1$, $g'(N[u]) \geq g(N[u])$ for all vertices u different from w or a descendant of w. Thus, since f and g are signed dominating functions on T, so too is g'. If $deg\ w$ is even, then $g(N[w])$ is odd, so $g(N[w]) \geq 1 = MinSum(w) + 1 = f(N[w])$. If $deg\ w$ is odd, then $g(N[w])$ is even, so $g(N[w]) \geq 2 = MinSum(w) + 1 = f(N[w])$. Hence $f(N[w]) \leq g(N[w])$. Consequently, $g'(V) = g(V - N[w]) + f(N[w]) \leq g(V - N[w]) + g(N[w]) = g(V)$. Hence, g' is a minimum signed dominating function of T that differs from f in fewer values than does g. □

It remains for us to consider the case where $f(v) < g(v)$. Here the vertex v is not the root of T, for otherwise $f(V) < g(V) = \gamma_s(T)$, which is impossible. Since the labeling of the vertices was arbitrary at each level, if any vertex x at the same level as v has $g(x) < f(x)$, we can use Lemma 2 to find a signed dominating function g' that agrees with f in more values than under g. So we may assume in what follows that every vertex x at the same level as v has $f(x) \leq g(x)$.

Since $f(v) < g(v)$, it follows that $f(v) = -1$ and $g(v) = 1$. By the minimality of g, there exists a vertex $x \in N[v]$ such that $g(N[x]) \in \{1, 2\}$. Let w be the parent of v and let u be the parent of w. If $f(w) \leq g(w)$ and $f(u) \leq g(u)$, then $f(N[x]) = f(N[x] - \{v\}) + f(v) \leq g(N[x] - \{v\}) + g(v) - 2 = g(N[x]) - 2 \leq 0$, which is a contradiction. Hence $f(w) > g(w)$ or $f(u) > g(u)$.

If $f(w) > g(w)$, i.e., $f(w) = 1$ and $g(w) = -1$, define a function $g' : V \to \{-1, 1\}$ by $g'(y) = g(y)$ if $y \in V - \{v, w\}$, $g'(v) = -1$ and $g'(w) = 1$. Note that $f(v) = g'(v) = -1$ and $f(w) = g'(w) = 1$. The only vertices whose neighborhood sums are decremented under g' are the children of v. However, these closed neighborhood sums under g' are at least as large as under f. Thus, since g are f are signed dominating functions, so too is g'. Furthermore, $g'(V) = g(V)$, so that g' is a minimum signed dominating function which differs from f in fewer values than does g.

Assume, therefore, that $f(w) \leq g(w)$. It follows that $f(u) > g(u)$, i.e., $f(u) = 1$ and $g(u) = -1$. Define a function $g' : V \to \{-1, 1\}$ by $g'(y) = g(y)$ if $y \in V - \{v, u\}$, $g'(v) = -1$ and $g'(u) = 1$. As before, g' is a minimum signed dominating function which differs from f in fewer values than does g. This completes the proof of Theorem 3. □

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