The 3-connected Graphs with a Longest Path Containing Precisely Two Contractible Edges

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ABSTRACT. Previously the authors characterized the 3-connected graphs with a Hamilton path containing only two contractible edges. In this paper we extend this result by showing that if a 3-connected graph has a diameter containing only two contractible edges, then that diameter is a Hamilton path.

INTRODUCTION AND TERMINOLOGY

All graphs in this paper are finite, undirected and simple.

Let G be a 3-connected graph. An edge $e = xy$ in G is said to be contractible if the graph obtained from G by contracting e is also 3-connected. Otherwise, e is said to be noncontractible. For $G \not\cong K_4$ and $e = xy \in E(G)$, one easily sees that e is noncontractible if and only if there exists $s \in V(G)$ such that $S = \{x, y, s\}$ is a 3-cutset of G; in that case we say that e and S are associates of each other. We use $E_c(G)$ to denote the set of all contractible edges of G and $E_n(G)$ for the set of all noncontractible edges. For H a subgraph of G we set $E_c(H) = E_c(G) \cap E(H)$ and $E_n(H) = E_n(G) \cap E(H)$. We also let $G[H]$ denote the subgraph induced by $V(H)$. If no confusion can arise, we will often use H for any of $V(H)$, $E(H)$ or the subgraph H. For $x \in V(G)$, $N(x)$ will denote the set of neighbours of x in G.

A consequence of a result in Dean, Hemminger and Toft [DHT87] is that every diameter of a 3-connected graph G contains at least two contractible edges of G. In [ACH93] the authors characterized the 3-connected graphs with a Hamilton path containing only two contractible edges; we denote this class by $\mathcal{H}_2$. Now let $\mathcal{D}_2$ denote the class of 3-connected graphs G that have a diameter containing only two

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contractible edges of G. In this paper we show that such a diameter is in fact a Hamilton path. That is, our goal is to prove the following.

Theorem. $\mathcal{D}_2 = \mathcal{H}_2$.

We refer the reader to [ACH93] for other background information. Since we will refer to several results from that paper it seems desirable to keep the numbering of them unchanged. Therefore, we will use letters or names to refer to some of the remaining results. As was done there we hereafter let G denote a graph in $\mathcal{D}_2$ and let $P = (x_1, x_2, \cdots, x_n)$ denote a fixed diameter in G that contains only two contractible edges of G. Of course, $N(x_1), N(x_n) \subseteq V(P)$ since P is a diameter. And, by way of contradiction, we will assume throughout that P is not a Hamilton path. Now it is known [AHO93] that if G has a longest cycle that contains at most three contractible edges of G, then G is hamiltonian. Thus, we also assume hereafter that x_1 is not adjacent to x_n in G. And since we know from computer checks that the theorem is true for $n < 10$, we will avoid messy small cases by assuming that $n \geq 10$. We will refer to x_1 as the *left end of* P and using this order, we let e_L and e_R denote the two contractible edges in P where e_L is to the left of e_R. We use the notation $[x_i, x_j]$ for $1 \leq i \leq j \leq n$ to denote the path $(x_i, x_{i+1}, \cdots, x_j)$.

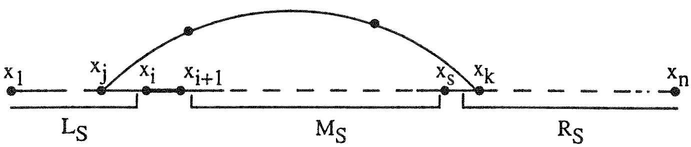
We will refer to a 3-cut $S = \{x_i, x_{i+1}, v\}$ associated with $f = x_i x_{i+1} \in E_n(P)$ simply as a *cut*; it is called a *bad cut* if $v \notin V(P)$ and a *good cut* if $v \in V(P)$. An edge $f \in E_n(P)$ is called a *bad edge* if it has no good cut associated with it. Of course, a consequence of our theorem here is that there are no bad cuts or bad edges! Never mind; a cut S separates P into either two or three segments (any one of which can be empty– but no more than one according to the following lemma) which we denote by L_S, M_S and R_S where $L_S(R_S)$ is to the left (right) of S while M_S is between the edge f and the vertex v when $v \in V(P)$ and is not adjacent to f in P.

A result, so basic to all that we do that we will seldom refer to it again as such, is Lemma 1 of [DHT87].

Lemma DHT. If S is a cut associated with $f = x_i x_{i+1} \in E_n(P)$, then every component of $G - S$ intersects P.

For $N \in \{L_S, M_S, R_S\}$ and nonempty, the component of $G - S$ that contains N might contain one (but not two by Lemma DHT) of the other members of $\{L_S, M_S, R_S\}$; but if it contains neither, then we say that S isolates N. Moreover, if M_S is isolated by S, we call S a *natural cut*. If M_S is not isolated by S, then we call S an *unnatural cut*; it is *unnatural to the right (left)* if $R_S(L_S)$ is isolated by S (see Figure 1). Thus bad cuts and cuts consisting of three consecutive vertices of

P are unnatural both to the left and to the right.



The cut $S = \{x_i, x_{i+1}, x_s\}$ is natural in the above graph while it is left-unnatural in the graph below (the jumper from x_i to x_j indicates the components in each case).

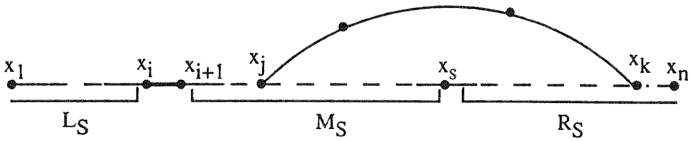


Figure 1

Using the above terminology, Lemma 2 of [DHO89] is as follows.

Lemma 1 [DHO89]. If S is a natural cut, then M_S contains an endvertex of at least one of e_L or e_R.

This lemma is used to locate contractible edges in P so often that we often use it implicitly. In that connection, we note that all cuts associated with x_1x_2 and $x_{n-1}x_n$ are natural ones.

As suggested by the notation L_S, M_S and R_S, one would expect, in the case of a good cut, that $G - S$ commonly has three components. We now show that is not the case here, whether S is a natural cut or not.

Lemma 2. If S is a cut, then $G - S$ has only two components.

Proof. The claim is immediate by Lemma DHT if S is a bad cut. So let $S = \{x_i, x_{i+1}, x_s\}$ with $x_s \in V(P)$. Obviously we may assume that $2 \leq i \leq n-2$, that $i + 3 \leq s \leq n - 1$ and, by way of contradiction, that L_S, M_S and R_S are each isolated by S. Therefore, by Lemma 1, at least one of L_S or R_S fails to contain an endvertex of a contractible edge of G. Since the two cases are similar we only consider one. So assume that e_L is to the right of x_{i+1}. Thus we have a cut $T = \{x_1, x_2, x_t\}$ with $x_t \in V(P)$. We consider the possible values of t. First note that $t \notin [3, i+1]$; for if $t \in [3, i+1]$, M_T contains an endvertex of e_L or e_R by Lemma 1. Thus, since x_i must be adjacent to vertices in the component containing R_S, we must have $t \geq s+1$. But this contradicts that L_S and R_S are isolated by S since x_1 must be adjacent to vertices in R_T. ∎

Natural cuts are generally better behaved than unnatural cuts and many of the results for cuts from [AH93] carry over to them, almost verbatim, as in [ACH93]. The minor differences from [ACH93] are in the proofs; if $x \in S$ and C is a component of $G - S$, then there is an edge from x to C. In the previous case, when P was a Hamilton path, this was an edge to a vertex in P. Now it just gives us a *P-jumper* (or more simply, a *jumper* since they will always refer to P) from x to a vertex $y \in C \cap P$; that is, an xy-path that is openly disjoint from P. We will use the notation $P_{i,j}$ for a jumper between distinct vertices $x_i, x_j \in P$. Likewise, if S is an unnatural cut to the left, for example, then there is a jumper from M_S to R_S but none to L_S. On the other hand, if $S = \{x_i,\ x_{i+1},\ v\}$ is a bad cut, then there are jumpers from L_S to R_S, but they must all go through v. Since we will need to refer to the part of these paths from v to P we will call them *semi-jumpers.*

One of the useful results about cuts concerns "crossed natural cuts", except now natural cuts can cross in four different ways; to the inside, to the outside, to the right side, and to the left side. So let $S = \{x_i, x_{i+1}, x_s\}$ and $T = \{x_j, x_{j+1}, x_t\}$ be natural cuts with $i+1 \leq j$. Then S and T are *crossed to the inside* if $i+1 \leq t < s \leq j$; they are *crossed to the outside* if $t \leq i$ and $j+1 \leq s$; they are *crossed to the right* if $j+1 \leq s < t$; and they are *crossed to the left* if $s < t \leq i$.

Lemma 3 [Crossed Natural Cuts]. Let $S = \{x_i, x_{i+1}, x_s\}$ and $T = \{x_j, x_{j+1}, x_t\}$ be crossed natural cuts with $i+1 \leq j$. Then
(1) $S' = \{x_i, x_{i+1}, x_t\}$ is a natural cut if $M_{S'} \neq \emptyset$,
(2) $T' = \{x_j, x_{j+1}, x_s\}$ is a natural cut if $M_{T'} \neq \emptyset$, and
(3) if $j \neq i+1$, then x_s and x_t are consecutive in P (in the case of being crossed to the outside, this means that $t = 1$ and $s = n$).

Proof. Suppose that S and T are crossed to the inside and let $x_k \in M_{S'}$; that is, $i+1 < k < t$. Now there can be no jumpers from x_k to any vertices in $[x_1, x_{i-1}] \cup [x_{s+1}, x_n]$ since S is a natural cut; but neither can there be any jumpers from x_k to vertices in $[x_{t+1}, x_s] \subseteq M_T$ since T is a natural cut. Thus $M_{S'}$ is isolated by S' as claimed in (1). (2) is a symmetric version of (1) and by the same type of argument we see in (3) that $\{x_s, x_t\}$ is a 2-cut if $t+1 < s$. The proof of (1) and (2) for the other type cuts is equally straightforward. For them however, $j = i+1$ is possible, and in that case $\{x_s, x_t\}$ no longer needs be a 2-cut; for example if S and T are crossed to the right with $j = i+1$ and $s+1 < t$, we can still have a jumper from x_{i+1} to x_{s+1}. ∎

A similar, and equally useful, result for good cuts is the following.

Lemma 4. If $S = \{x_i, x_{i+1}, x_s\}$ and $T = \{x_j, x_{j+1}, x_t\}$ are good cuts with $i+1 \leq j$, $j+1 \leq s$ and $j+1 < t$, then we don't have both that R_S is isolated by S and that R_T is isolated by T.

Proof. By way of contradiction, suppose that they are. Then we have a jumper from x_i to $[x_{s+1}, x_n]$ and another from x_{j+1} to $[x_{t+1}, x_n]$. But the first jumper forces $t \geq s+1$, while the second forces $s \geq t+1$. ∎

Of course there is a symmetric version of this lemma involving L_S and L_T.

Our first "new" lemma is a similar "crossing" result for bad cuts.

Lemma ACH. If $S = \{x_i, x_{i+1}, x_s\}$ is a good cut and $f = x_j x_{j+1}$ is a bad edge with $i+1 \leq j < s$, then R_S is not isolated by S.

Proof. Suppose that R_S is isolated by S and let $B = \{x_j, x_{j+1}, v\}$ be a bad cut associated with f. So we have a jumper Q from x_i to $x_q \in [x_{s+1}, x_n]$. If $i+1 < j$, then we must also have a jumper from x_{i+1} to $[x_{s+1}, x_n]$. But these two jumpers must go through v since B is a bad cut, which contradicts that P was a diameter (and $(x_1 x_2 \cdots x_i \cdots v \cdots x_{i+1} \cdots x_n)$ is longer than P). So we must have $j = i+1 > 2$; the latter since $N(x_1) \subset V(P)$. But then, all paths in $G[L_B \cup \{v\}]$ from x_1 to v must pass through x_i; or such a path united with the portion of Q from v to x_q contradicts that R_S is isolated by S. Thus $\{x_i, x_{i+1}, x_{i+2}\}$ is a cut and so f is not a bad edge. ∎

Now suppose that $x_1 x_2 \in E_n(P)$ and that $S = \{x_1, x_2, x_s\}$ is an associated good cut (clearly, $x_1 x_2$ cannot be a bad edge). Then e_L is to the left of x_s by Lemma 1. Moreover, we can pick x_s so that e_R is to the right of x_s. This is obvious if $e_R = x_{n-1} x_n$ so assume that $x_{n-1} x_n \in E_n(P)$ and let $T = \{x_{n-1}, x_n, x_t\}$ be an associated cut. Thus, by Lemma 3, we can take $s \leq t$ and so, by Lemma 1, x_s is between e_L and e_R. Furthermore, since P is a diameter, all neighbours of x_1 lie on P, so there is an edge $x_1 x_p$ with $x_p \in R_S$, that is, with $p > s$. In the following lemma we will divide into cases depending on whether there is such an x_p to the left of e_R or not. In either case, we find that there is only one such x_p.

Lemma 5. Let $S = \{x_1, x_2, x_s\}$ be a cut. Then there is a unique $p > s$ with $x_1 x_p \in E(G)$. Moreover, if $e_R \in [x_p, x_n]$, then $N(x_1) = \{x_2, x_3, x_p\}$, $e_L = x_3 x_4$

and $N(x_3) = \{x_1, x_2, x_4\}$ (that is, $s = 4$); otherwise $N(x_1) = \{x_2, x_3, x_{n-1}\}$.

Proof. Suppose first that $e_R \in [x_p, x_n]$ for some $p > s$ with $x_1 x_p \in E(G)$. So we have $4 \leq s < p < n$. Thus there are no bad edges in $[x_1, x_{p-1}]$. Note that there is a jumper Q from $\{x_1, x_2\}$ to $x_q \in R_S$ with $q > s$ and $q \neq p$; otherwise $\{x_s, x_p\}$. is a 2-cut separating x_1 from x_n.

Let $e_L = x_h x_{h+1}$. Since e_R is to the right of x_p, there are cuts A and B associated with $x_{h-1}x_h$ and $x_{h+1}x_{h+2}$,respectively. And, as noted above, A is a good cut, say $A = \{x_{h-1}, x_h, x_a\}$ (note that $A = S$ is possible). And we claim that we can choose B to be a good cut as well. For suppose that $B = \{x_{h+1}, x_{h+2}, v\}$ is a cut with $v \in V(G) - V(P)$. Thus, as noted above, $p = s + 1 = h + 2$ and Q must be (x_2, v, x_q) since it goes from L_B to R_B and since P is a diameter. Now v is not in the component containing M_S because of the edge vx_q and so there are no edges from v to M_S; consequently, $\{x_{h+1}, x_{h+2}, x_2\}$ is a good cut associated with $x_{h+1}x_{h+2}$. So as claimed, we can assume that $B = \{x_{h+1}, x_{h+2}, x_b\}$ is a good cut associated with $x_{h+1}x_{h+2}$.

The remainder of the proof of this lemma now proceeds just as that of Lemma 5 in [ACH93]. ∎

Theorem 6. The pair e_L, e_R is one of the following:

(1) $x_1x_2,\ x_{n-1}x_n$, or

(2) $x_1x_2,\ x_{n-3}x_{n-2}$ or $x_3x_4,\ x_{n-1}x_n$, or

(3) $x_3x_4,\ x_{n-3}x_{n-2}$.

Proof. Suppose that we don't have (1). Thus, by symmetry, we assume that $x_1x_2 \in E_n(P)$ and let $S = \{x_1, x_2, x_s\}$ be an associated cut. Consequently, there is an edge x_1x_p with $s < p$ and, by Lemma 1, with e_L to the left of x_s. If $e_R = x_{n-1}x_n$, then $p < n$ since $x_1x_n \notin E(G)$ and so (2) holds by Lemma 5.

So suppose that $x_{n-1}x_n \in E_n(P)$ as well and let $T = \{x_{n-1}, x_n, x_t\}$ be an associated cut. Thus, as with S, we have an edge x_nx_q with $q < t$ and with e_R to the right of x_t. So by Lemma 5, $dg(x_1) = dg(x_n) = 3$. Using symmetry and Lemma 5, we can assume that we have one of the following two situations: (a) x_p and x_q are both between e_L and e_R or (b) e_L and e_R are both in $[x_1, x_p]$. We are done in case (a), since then e_L and e_R are as in (3) by Lemma 5.

And case (b) does not occur. This is because of the edges x_1x_{n-1} and $x_{n-2}x_n$ (the latter is given by Lemma 5) and the assumption that P is a diameter but not a Hamilton path. For suppose that vx_k is an edge with $v \in V(G) - V(P)$ and $x_k \in P$. So $k \neq 1, n$ and, if $1 < k < n-1$, then the path $(v, x_k, x_{k+1}, \cdots, x_{n-2}, x_n, x_{n-1}, x_1, x_2, \cdots, x_{k-1})$ contradicts that P is a diameter. Likewise, the path $(v, x_{n-1}, x_n, x_{n-2}, x_{n-3}, \cdots, x_1)$ shows that $k \neq n-1$. ∎

We can now extend Lemma 5.

Corollary 7. $dg(x_1) = dg(x_n) = 3$ and $x_1x_3, x_{n-2}x_n \in E(G)$.

Proof. By Lemma 5 and symmetry we only need to consider the case with $e_L = x_1x_2$. Thus, by Theorem 6, e_R is either $x_{n-3}x_{n-2}$ or $x_{n-1}x_n$.

By way of contradiction, assume that there exist i, j with $x_i, x_j \in N(x_1)$ and with $4 \leq i < j \leq n-1$. Again by Theorem 6 we have that e_R is to the right of x_i (since $i = n-2$ implies that $x_{n-1}x_n$ is contractible). And since there can be no bad edges in $[x_1, x_i]$, we let $S = \{x_2, x_3, x_s\}$ and $T = \{x_3, x_4, x_t\}$ be good cuts associated with x_2x_3 and x_3x_4, respectively. So $s \neq 1$ and hence, by Lemma 1, $s > i$ since e_R is to the right of x_i. Thus $s \geq j$ with R_S isolated by S unless $s = j$ and S is natural. But, by Theorem 6, the latter forces $e_R = x_{n-3}x_{n-2}$ and $s = n-2$. But then, since $n \geq 10$, $S' = \{x_2, x_3, x_{n-3}\}$ is a natural cut, in contradiction to Lemma 1. Hence S isolates R_S and so we have a jumper from x_2 to x_w with $x_w \in R_S$. Because of that jumper $t \neq 1$ and so, as with s, we have $t \geq j$. But by Lemma 3, R_T cannot be isolated by T so we must have $i = 4$, $t = j$ and, by Lemma 1, $e_R = x_{n-3}x_{n-2} \in [x_i, x_j]$. But now, by the symmetric version of Lemma 5, we have $j = n-1$. Thus $s = n-1$ since $s \geq j$ and R_S is isolated by S. This is a contradiction since $x_{n-2}x_n \in E(G)$ by Lemma 5. ∎

Corollary 8. Let $x_i, x_u, x_j, x_v \in V(P)$ with disjoint jumpers $P_{i,v}$ from x_i to x_v and $P_{u,j}$ from x_u to x_j, respectively.

(1) If $i < u < v < j$, if x_ux_{u+1} is a good edge and if $v > u+2$, then $[x_u, x_v]$ contains a contractible edge of G.

(2) If $i < u < j < v$, if x_ux_{u+1} is a good edge and if $j > u+2$, then $[x_u, x_j]$ contains a contractible edge of G.

Proof. We only prove (1) since (2) follows in a like manner. So assume that $v \geq u+3$ and that $[x_u, x_v]$ contains no contractible edges. By assumption we have a good cut, call it Q, associated with x_ux_{u+1}. And there are no bad cuts associated with $x_{u+1}x_{u+2}$ since an associated vertex would have to be on each of the two disjoint jumpers in the hypothesis. Thus we have cuts $Q = \{x_u, x_{u+1}, x_q\}$ and $S = \{x_{u+1}, x_{u+2}, x_s\}$. We must have $s \in [1, i] \cup [j, n]$ and since the two cases are similar, we only consider the one with $s \in [1, i]$. Thus L_S is isolated by S because of the jumper $P_{u,j}$. Since L_S is isolated by S, we have jumpers from both x_{u+1} and x_{u+2} to L_S; say $P_{u+1,w}$ and $P_{u+2,z}$, respectively. Note that we can take $w \neq z$ unless $L_S = \{x_1\}$. Thus, if $x_{n-1}x_n \in E_n(P)$ with associated cut $B = \{x_{n-1}, x_n, x_b\}$, then we must have $b \geq u+2$ by Lemma 4 for B and S. But then $v = n-1$ is not possible by Lemma 1, so we have $b \geq v$ and e_R is to the right of x_v. $e_R \in \{x_{n-3}x_{n-2}, x_{n-1}x_n\}$. Again by Theorem 6, Of course we get the same conclusion if $e_R = x_{n-1}x_n$.

Now consider the possible values of q. By Lemma 4 for Q and S, we must have $q \geq i$ because of the jumper $P_{i,v}$. If $i \leq q \leq u-1$, then Q is a natural cut because of the jumper $P_{u+2,z}$. Hence $[x_i, x_u]$ contains the contractible edge e_L, so $e_L = x_3x_4$ and $s \leq i \leq 3$. But by Lemma 5, $i \neq 3$ and $s \neq 2$ because of the edge x_1x_3. And $s \neq 1$ since L_S is a component of $G - S$. So we don't have $i \leq q \leq u-1$ either. Therefore, because of the jumpers $P_{u+2,z}$ and $P_{i,v}$, we must have $q \geq v$ with R_Q a component of $G - Q$ and with jumpers from x_u and x_{u+1} to R_Q. Thus, by Lemma ACH, we can take the cut associated with $x_{u+2}x_{u+3}$ to be a good one, say $T = \{x_{u+2}, x_{u+3}, x_t\}$.

But now, because of these jumpers and Lemma 4 for S and T, we cannot have $t < u$. And we can't have $u \leq t \leq v$ by Lemma 1 since in that case T would be a natural cut because of the jumper $P_{i,v}$. Moreover, we can't have $t > j$ by Lemma 4 for Q and T. Thus $v < t \leq j$. But such a t doesn't give a cut! That completes the proof of (1). ∎

PROOF OF THE THEOREM

Suppose that G is a 3-connected graph containing a longest path with precisely two contractible edges and consider a qualifying diameter $P = (x_1, x_2, \cdots, x_n)$ with $n \geq 10$. We shall show that P is a hamiltonian path in G.

Assume that P is not a hamiltonian path and let x_i be the first vertex from the left that has a neighbour not in P; that is, $N(x_h) \subset V(P)$ if $h < i$, while we have a vertex $v \notin P$ with $x_iv \in E(G)$. And since G is 3-connected, we have three openly disjoint semi-jumpers from v to P which, by the choice of i, we can take to be P_i, P_j and P_k to x_i, x_j and x_k, respectively, with the edge x_iv as P_i and with $2 < i+1 < j < k-1$ (the inequalities since P is a diameter). Thus, $k \geq i+4$ and, by the choice of i, all jumpers with one endvertex in $[x_1, x_{i-1}]$ must be edges. We also pick such v and k so that $k - i$ is as large as possible, and after that choice we choose j as small as possible.

Now by Lemma 5 ($dg(x_3) = 3$ if $e_L = x_3x_4$) and Theorem 6, e_L is to the left of x_i and, by symmetry, e_R is to the right of x_k; that is, all edges in $[x_i, x_k]$ are noncontractible. Moreover, we claim that, because of the choice of i, all edges in $[x_1, x_{i+2}]$ are good edges. For let $B = \{x_h, x_{h+1}, w\}$ be any bad cut. Then we must have a semi-jumper from w to L_B. Thus we immediately have that all edges in $[x_1, x_{i+1}]$ are good edges. And if $h = i+1$, then all semi-jumpers from w to L_B must go to x_i; thus $\{x_i, x_{i+1}, x_{i+2}\}$ is a good cut.

So let $S = \{x_{i+1}, x_{i+2}, x_s\}$ be a good cut. Thus, by Lemma 1, we have that S

is an unnatural cut with $s \in [x_1, x_i] \cup [x_k, x_n]$.

We first assume that $s \geq k$. If S isolates R_S, then there is a jumper from x_{i+1} to R_S, which contradicts Corollary 8 since $k \geq i+4$. Hence, S is unnatural to the left, and so $s = k$ and $j = i+2$. But there must be a jumper P' from $x_{i+1} \in S$ to $x_w \in M_S \cup R_S$ and, by Corollary 8, the only possibility is $x_w = x_{i+3}$. But then $(x_1, \cdots, x_i, v, P_j \longrightarrow x_{i+2}, x_{i+1}, P' \longrightarrow x_{i+3}, \cdots, x_n)$ is a longer path than P.

If $s \leq i$, then S isolates L_S. So we have edges $x_{i+1}x_w$ and $x_{i+2}x_z$ with $x_w, x_z \in L_S$. Moreover, $x_w \neq x_1$ or we have a longer path than P. So $s > 1$ and we can take $w \neq z$; otherwise, $L_S - \{x_w = x_z\}$ is contained in a component of $G - \{x_w, x_s\}$. If $w, z \leq i-2$, then, by Corollary 8, $[x_{max\{w,z\}}, x_{i+1}]$ contains e_L. Hence $e_L = x_3x_4$, $x_w = x_2$ and $x_z = x_1$. But now, since $x_1x_3 \in E(G)$ in that case, $(v, x_i, x_{i-1}, \cdots, x_3, x_1, x_2, x_{i+1}, x_{i+2}, \cdots, x_n)$ is a longer path than P. But $w, z \leq i-2$ if $s < i$.

So we can assume that $max\{w, z\} = i-1$ and that $s = i$, that is, that $S = \{x_i, x_{i+1}, x_{i+2}\}$ is a cut. Because of this cut we can now show that we have a structure on $[x_i, x_k]$ that resembles what was called a "span" in [ACH93]. Now from Corollary 8, the cut S, and our choice of i, there are no jumpers from $[x_1, x_{i-1}]$ to $[x_{i+3}, x_n]$. Likewise , by our choice of k, there are no jumpers from $[x_{k+1}, x_n]$ to $[x_1, x_{k-3}]$ except possibly an edge x_ix_u with $u > k$. But the latter forces $k = j+2$ which in turn forces a squaring jumper over x_j (or $\{x_i, x_j\}$ is a 2-cut) resulting in a longer path using that jumper and $P_j \cup P_k$. So there is no such edge. Next we note that, because we chose j as small as possible, there are no semi-jumpers from $P_j - \{x_j\}$ to $[x_{i+1}, x_{j-1}]$. Thus, by Corollary 8, there can be no semi-jumpers from a vertex in $P_k - \{v, x_k\}$ to $[x_{i+1}, x_{j-3}]$; nor one to $\{x_{j-2}, x_{j-1}\}$ or we have a longer path. Thus all jumpers that we now consider must be openly disjoint from $P_i \cup P_j \cup P_k$. So what is to prevent $[x_{j+1}, x_n]$ from being contained in a component of $G - \{x_i, x_j\}$? By the preceding it can only be a squaring jumper from x_{j-1} to x_{j+1} or a jumper from x_k to $\{x_{j-2}, x_{j-1}\}$. Thus, for $j \geq i+4$, the only jumpers on $[x_{i+1}, x_{j-1}]$ are the squaring jumpers $P_{q,q+2}$, $i+1 \leq j-3$; and each must be there or $\{x_i, x_{q+1}\}$ is a 2-cut.

Because of these squaring jumpers, we have, for $j \geq i+3$, a path Q from x_{i+2} to x_{i+1} with $V(Q) = V([x_{i+1}, x_{j-1}])$: if $j = i+3$, it is (x_{i+2}, x_{i+1}); if $j = i+4$, it is $(x_{i+2}, P_{i+2,i+4} \longrightarrow x_{i+4}, x_{i+3}, P_{i+3,i+1} \longrightarrow x_{i+1})$; if $j = i+5$, it is $(x_{i+2}, P_{i+2,i+4} \longrightarrow x_{i+4}, x_{i+5}, P_{i+5,i+3} \longrightarrow x_{i+3}, P_{i+3,i+1} \longrightarrow x_{i+1})$, and so on, depending on whether $j-i$ is odd or even. We also let P_R denote the path $(x_{i+1}, x_i, v, P_j \longrightarrow x_j, x_{j+1}, \cdots, x_n)$.

The coup de grace will come shortly by combining these paths with another path produced by using the "leap frog technique" which is based on the following lemma. We remind the reader that all jumpers into vertices to the left of x_i are in fact edges by our choice of i. However, we continue to refer to them as jumpers, since it saves us from specifying each time that they are not edges of P.

Lemma [Leap Frog]. If $P_{r,t}$ is a jumper with x_t between e_L and x_r, with $t < i-1$ and with no jumper from x_r further to the left than x_t, then there is a jumper $P_{t+1,u}$ with $u < t$.

Proof. Let $Z = \{x_t, x_{t+1}, x_z\}$ be a cut associated with $x_t x_{t+1}$. Since Z is a minimal cut the only problem situation is clearly when the only choice for Z is as an unnatural cut to the right with $z > t+1$, and hence with a jumper Q_1 from x_t to R_Z. And since $\{x_t, x_{t+1}, x_{t+2}\}$ is not a cut, there is a jumper Q_2 from $x_a \in L_Z$ to $x_b \in M_Z \cup \{x_z\}$ with $b > t+2$. But applying Corollary 8 to Q_1 and Q_2 forces $b = r$, which contradicts our choice of t. ∎

So suppose that we have a jumper Y' from $y \in \{x_{i+1}, x_{i+2}\}$ to x_t with $t < i-1$. Now using the Leap Frog Lemma, we will produce a path P_L on $V([x_1, x_{i-1}]) \cup \{y\}$ from y to x_{i-1}. This will be achieved by producing two disjoint paths Y and W to x_1 from y and x_{i-1}, respectively. Since this is done by an iterative procedure, we will let Y and W denote the paths at each stage. So initially we take $Y = Y'$ and $W = \{x_{i-1}\}$. Now, assuming that e_L is to the left of x_t, "leaping over" x_t to x_u with $u < t$ by a jumper that the Leap Frog Lemma assures us exists. If e_L is to the left of x_u, then we extend Y in a like manner, that is, by "running down" P to x_{u+1} and "leaping over" x_u. We continue this procedure until our current jumper goes to the left of e_L; say for example, that it extends Y by going from x_{b+1} to x_a. If $e_L = x_1 x_2$, then $a = 1$ and we complete W by adding $[x_1, x_b]$ to it. If $e_L = x_3 x_4$, then by Lemma 5, $x_a \in \{x_1, x_2\}$; if $a = 1$ complete W as before and, if $a = 2$, complete W by adding $(x_b, x_{b-1}, \cdots, x_3, x_1)$ to it while completing Y by adding the edge $x_1 x_2$ to it.

We are now ready to put things together.

First suppose that $j = i+2$. So we have a jumper from x_{i+1} to x_t with $t < i$. If $t = i-1$, then $(x_1, x_2, \cdots, x_{i-1}, x_{i+1})$ followed by P_R is a longer path than P. If $t < i-1$, then P_L connected to P_R by the edge $x_{i-1} x_i$ is a longer path than P.

So we try $j \geq i+3$. This time we use a jumper from x_{i+2} to x_t with $t < i$. If $t = i-1$, then $(x_1, x_2, \cdots, x_{i-1}, x_{i+2})$ followed by Q, which in turn is followed by P_R, gives a longer path than P. If $t < i-1$, then we use P_L followed by Q, which in turn, is followed by P_R to get a longer path than P.

Thus, if we let $A = x_1$, then, by the leap frog technique, we have completed the proof of the Theorem by producing a path W from x_{i-1} to A followed by a path Y to x_{i+2}. This completes the proof. ∎

REMARK. The Leap Frog Lemma obviously applies in the more general setting of $\mathcal{H}_2$ and so, starting at one end of P and applying the above technique, we see that all members of $\mathcal{H}_2$ are in fact hamiltonian. In this context, we have examples to show that $\mathcal{H}_k \neq \mathcal{D}_k$ for $k \geq 6$, but we don't know what happens for $k = 3, 4$ and 5.

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