(3, λ)-GROUP DIVISIBLE COVERING DESIGNS

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Abstract Let u, g, k and λ be positive integers with $u \geq k$. A (k, λ)-group divisible covering design ((k, λ)-GDCD) with type g^u is a λ-cover of pairs by k-tuples of a gu-set X with u holes of size g, which are disjoint and spanning. The covering number, C(k, λ; g^u), is the minimum number of blocks in a (k, λ)-GDCD of type g^u. In this paper, the determination of the function C(3, λ; g^u) begun by [6] is completed.

1. Introduction

Let u, g, k and λ be positive integers with $u \geq k$..

Roughly speaking, a (k, λ)-*group divisible covering design* ((k, λ)-GDCD) with type g^u is a λ-cover of pairs by k-tuples of a gu-set X with u holes of size g, which are disjoint and spanning. More formally, a (k, λ)-GDCD of type g^u is defined to be a triple (X, G, B) which satisfies the following properties:

(1) G is a partition of a set X (of *points*) into subsets called *groups or holes*,

(2) B is a set of k-subsets of X (called *blocks*) such that a group and a block contain at most one common point,

(3) every pair of points from distinct groups occurs in at least λ blocks.

The *group-type* (or *type*) of the GDCD is the multiset T = { |G|: G $\in$ G }, and it will be denoted by an "exponential" notation: a type $1^i 2^r 3^k$... denotes i occurrences of 1, r occurrences of 2, etc.

For any pair $e = \{x, y\}$ of points in X, let m(e) be the number of blocks in *B* that cintain e. The excess of the GDCD is the multigraph spanned by all pairs e of points from distinct groups with multiplicity $m(e) - \lambda$.

The concept of a covering design with holes has played an important role in the discussion of various covering problems. As a general covering problem, the main problem here is to determine the values of the covering number $C(k, \lambda; g^u)$, that is, the minimum number of blocks in a (k, λ)-GDCD of type g^u. Let

$$L(k, \lambda; g^u) = \lceil gu / k \lceil \lambda g(u - 1) / (k - 1) \rceil \rceil$$

where $\lceil x \rceil$ denotes the least integer not less than x. It is evident that

$$C(k, \lambda; g^u) \geq L(k, \lambda; g^u) \qquad (1.1)$$

The lower bound (1.1) for $C(k, \lambda; g^u)$ is not always best possible. In particular, we have the following result, which is a modification of [2, Lemma 7.2].

Theorem 1.1 Suppose that $\lambda(u\text{-}1)g \equiv 0 \pmod{k\text{-}1}$ and $\lambda u(u\text{-}1)g^2 \equiv 1 \pmod{k}$. Then $C(k, \lambda; g^u) \geq L(k, \lambda; g^u) + 1$.

Theorem 1.1 and the bound (1.1) together imply that

$$C(k, \lambda; g^u) \geq B(k, \lambda; g^u) \qquad (1.2)$$

where $B(k, \lambda; g^u)$ is defined by $B(k, \lambda; g^u) = L(k, \lambda; g^u) + 1$ if $\lambda(u\text{-}1)g \equiv 0 \pmod{k -1}$ and $\lambda u(u -1)g^2 \equiv 1 \pmod{k}$, and $B(k, \lambda; g^u) = L(k, \lambda; g^u)$ otherwise.

In view of (1.2), a (k, λ)-GDCD of type g^u with $B(k, \lambda; g^u)$ blocks is said to be minimal. Upper bounds on $C(k, \lambda; g^u)$ are generally given by construction of a minimal k-GDCD of type g^u.

The first author [6] has proved that $C(3, 1; g^u) = B(3, 1; g^u)$ for all positive integers g and $u \geq 3$ with the possible exception of the pairs $(g, u) \in \{(7, 8), (11, 14)\}$. In this paper, we will remove these two exceptional pairs and show that $C(3, \lambda; g^u) = B(3, \lambda; g^u)$ for all positive integers g, $\lambda \geq 2$ and $u \geq 3$. Thus the determination of the function $C(3, \lambda; g^u)$ is completed.

We use as our standard design theory reference Beth, Jungnickei and Lenz [1]. Following Hanani [2] we denote by $B(K, \lambda; v)$ a pairwise balanced design (PBD) of order v with block sizes from K and index λ. By (K, λ)-GDD we mean a group divisible design (GDD) with block sizes from K and index λ. As usual, we use 'exponential' notation to describe the type of a GDD. We simply write k for K whenever $K = \{k\}$. Using this notation, a PBD $B(k, \lambda; v)$ is a balanced incomplete block design (BIBD) with parameters v, k and λ. The notation $B(K \cup \{w^*\}, 1; v)$ stands for a PBD of order v and index unity having blocks of sizes from K, except for one block of size w when $w \notin K$. If $w \in K$, then $B(K \cup \{w^*\}, 1; v)$ is a PBD of order v and index unity having blocks of sizes from K containing at least one block of size w. A similar terminology applies to GDDs.

If we remove one or more subdesigns from a GDD, we obtain a holey GDD (HGDD). In the sequel, we write (k, λ)-HGDD for a structure $(X, \{Y_i\}_{1 \le i \le t}, \mathcal{G}, \mathcal{B})$ where X is a gu-set (of points), $\mathcal{G} = \{G_1, G_2, \ldots, G_u\}$ is a partition of X into u groups of g points each, $\{Y_1, Y_2, \ldots, Y_t\}$ is a partition of X into t holes, each hole Y_i $(1 \le i \le t)$ is a set of uh_i points such that $|Y_i \cap G_j| = h_i$ for $1 \le j \le u$, and $\mathcal{B}$ is a collection of k-subsets of X (called blocks) such that no block contains two distinct points of any group or any hole, but any other pairset of points of X is contained in exactly λ blocks of $\mathcal{B}$. The pair (u, T) is referred to as the type of the design where T is the multiset $\{h_i: 1 \le i \le t\}$and will be denoted by an "exponential" notation. In the case of one hole, say Y, the HGDD $(X, \{Y\}, \mathcal{G}, \mathcal{B})$ is called an incomplete group divisible design (IGDD). We denote it (k, λ)-IGDD and write $(g, h)^u$ for its type where $|G \cap Y| = h$ for any $G \in \mathcal{G}$. Note that if $Y = \varnothing$, then the IGDD is a GDD.

For all practical purpose, we record the following existence results.

Theorem 1.2 [2] The necessary and sufficient condition for the existence of a $(3, \lambda)$-GDD of type g^u are

(1) $u \ge 3$;

(2) $\lambda(u-1)g \equiv 0 \pmod 2$; and

(3) $\lambda u(u-1)g^2 \equiv 0 \pmod 6$.

Theorem 1.3 [4] The necessary and sufficient conditions for the existence of a $(3, \lambda)$-IGDD of type $(g, h)^u$ are

(1) $g \geq 2h$;

(2) $\lambda g(u-1) \equiv 0 \pmod 2$;

(3) $\lambda(g - h)(u - 1) \equiv 0 \pmod 2$; and

(4) $\lambda u(u - 1)(g^2 - h^2) \equiv 0 \pmod 6$.

Theorem 1.4 [3] Let u and t be positive integers not less than 3. The necessary and sufficient conditions for the existence of a 3-HGDD of type (u, h^t) are

(1) $\lambda(u -1)(t -1)h \equiv 0 \pmod 2$; and

(2) $\lambda uht(u -1)(t -1)h \equiv 0 \pmod 6$.

Theorem 1.5 [5] There exists a B({3, 5*}, 1; v) for any positive integer $v \equiv 5 \pmod 6$.

It is worth mentioning that the notion of a GDCD is a natural generalization of standard packing designs and group divisible designs. A (u, k, λ) covering design is (k, λ)-GDCD with type 1^u. When a (k, λ)-GDD exists, it is actually a minimal (k, λ)-GDCD.

2. The determination for $C(3, 1; 7^8)$ and $C(3, 1; 11^{14})$

In this section, we deal with the two outstanding cases mentioned in Section 1. This completes the determination of the function $C(3, 1; g^u)$.

Lemma 2.1 There exists a minimal (3, 1)-GDCD of type 7^8.

Proof In this case, $B(3, 1; 7^4) = L(3, 1; 7^4) = 467$. Let the point set be $X = Z_{56}$ and the group set be {{j, j+8, j+16, j+24, j+32, j+40, j+48}: j = 0, 1, ..., 7}. Then the required blocks are

{0, 1, 6}	(mod 56)	{0, 3, 7}	(mod 56)
{0, 2, 23}	(mod 56)	{0, 11, 26}	(mod 56)
{0, 12, 39}	(mod 56)	{0, 13, 38}	(mod 56)
{0, 14, 36}	(mod 56)		
{0, 27, 55}			

{j, j+9, j+46} (j = 9, 10, ..., 55)
{j, j+28, j+46} (j = 0, 1, ..., 8)
{j, j+9, j+37} (j = 0, 1, ..., 8)
{j+9, j+18, j+46} (j = 0, 1, ..., 8)

The excess of this GDCD consists of the following 29 pairs:

{j, j+37} {j+9, j+18} {j+28, j+46} {0, 55} {0, 27}

where j = 0, 1, ..., 8. □

Lemma 2.2 There exists a minimal (3, 1)-GDCD of type 11^{14}.

Proof In this case, B(3, 1; 11^{14}) = L(3, 1; 11^{14}) = 154×24. Let the point set be X = Z_{154} and the group set be {{j, j+14, j+28, j+42, ..., j+140}: j = 0, 1, ..., 13}. Let H_0 be the subgroup of order 77 in Z_{154}. Then the blocks of required design are given below.

{0, 11, 24} (mod 154)	{0, 12, 44} (mod 154)	{0, 15, 34} (mod 154)
{0, 18, 67} (mod 154)	{0, 20, 65} (mod 154)	{0, 21, 64} (mod 154)
{0, 22, 63} (mod 154)	{0, 23, 62} (mod 154)	{0, 25, 60} (mod 154)
{0, 26, 59} (mod 154)	{0, 27, 58} (mod 154)	{0, 29, 69} (mod 154)
{0, 36, 74} (mod 154)	{0, 51, 124} (mod 154)	

{0, 75, 76} (translated by H_0)	{0, 2, 50} (translated by H_0)
{1, 54, 58} (translated by H_0)	{0, 6, 61} (translated by H_0)
{0, 10, 47} (translated by H_0)	{0, 8, 54} (translated by H_0)
{0, 16, 68} (translated by H_0)	{0, 5, 77} (translated by H_0)
{0, 66, 137} (translated by H_0)	{1, 4, 11} (translated by H_0)
{1, 67, 76} (translated by H_0)	{0, 1, 72} (translated by H_0)
{1, 17, 78} (translated by H_0)	{1, 3, 8} (translated by H_0)
{1, 38, 55} (translated by H_0)	{0, 3, 9} (translated by H_0)
{0, 57, 107} (translated by H_0)	{1, 5, 53} (translated by H_0)
{1, 56, 109} (translated by H_0)	{1, 9, 77} (translated by H_0)

The excess of this GDCD consists of the following 77 pairs:

{j, j+77} (j = 0, 1, ..., 76). □

As an immediate consequence of (1.2) and the above lemmas, we obtain the following.

Corollary 2.3 If (g, u) ∈ {(7, 8), (11, 14)}, then C(3, 1; g^u) = B(3, 1; g^u).

Combining the results in [6] and Corollary 2.3 gives the following theorem.

Theorem 2.4 Let g and $u \geq 3$ be positive integers. Then $C(3, 1; g^u) = B(3, 1; g^u)$ where $B(3, 1; g^u) = \lceil gu / 3 \lceil g(u-1)/2 \rceil \rceil$.

3. Covering numbers for $2 \leq \lambda \leq 5$

In this section, we determine completely the covering number $C(3, \lambda; g^u)$ for $2 \leq \lambda \leq 5$. We shall prove that the lower bound (1.2) on the function $C(3, \lambda; g^u)$ is achieved for all positive integer g, $u \geq 3$ and $2 \leq \lambda \leq 5$. More specifically, we show the following.

Theorem 3.1 Let g, λ and u be positive integers satisfying $u \geq 3$ and $2 \leq \lambda \leq 5$. Then $C(3, \lambda; g^u) = B(3, \lambda; g^u)$, in which $B(3, \lambda; g^u) = \lceil gu / 3 \lceil \lambda g(u-1)/2 \rceil \rceil + 1$ whenever

(1) $\lambda = 2$, $g \equiv 1$ or $2 \pmod 3$ and $u \equiv 2 \pmod 3$;

(2) $\lambda = 5$, $g \equiv 2$ or $4 \pmod 6$ and $u \equiv 2 \pmod 3$;

(3) $\lambda = 5$, $g \equiv 1$ or $5 \pmod 6$ and $u \equiv 5 \pmod 6$,

and $B(3, \lambda; g^u) = \lceil gu / 3 \lceil \lambda g(u-1)/2 \rceil \rceil$ otherwise.

As already mentioned earlier, in order to prove Theorem 3.1 we need only to construct a minimal GDCD for each statede values of g, u and λ. Note that the result for $g = 1$ in Theorem 3.1 has been proved by Hanani [2]. So, we may also assume that $g \geq 2$ below.

We now present our constructions for the required (3, λ)-GDCDs, which split into four lemmas depending on the values of λ.

Lemma 3.2 For all integers $g \geq 2$ and $u \geq 3$, $C(3, 2; g^u) = B(3, 2; g^u)$.

Proof For the case where $g \equiv 1, 2 \pmod 3$ and $u \equiv 0, 1 \pmod 3$ or $g \equiv 0 \pmod 3$ and $u \geq 3$, the results follows from Theorem 1.2 where the GDCD is exact.

For the remaining case where $g \equiv 1, 2 \pmod 3$ and $u \equiv 2 \pmod 3$, first note that $B(3, 2; g^u) = \lceil gu / 3 \lceil 2g(u-1)/2 \rceil \rceil + 1$. The construction then is as follows.

Start with a B({3, 5*}, 1; 2u+1) which exists by Theorem 1.5. Delete one point not belonging to the block of size 5 to create a ({3, 5*},1)-GDD of type 2^u. Replace the distinguished block by a minimal (3, 2)-GDCD of type 1^5 and take two copies of all blocks of size 3 from the GDD. This gives a minimal (3, 2)-GDCD of type 2^u whose excess consists of four pairs. Now we take a (3, 2)-IGDD of type $(g, 2)^u$ from Theorem 1.3 and fill in its hole by the above minimal (3, 2)-GDCD of type 2^u to obtain the required minimal (3, 2)-GDCD of type g^u. □

Lemma 3.3 For all integers $g \geq 2$ and $u \geq 3$, $C(3, 3; g^u) = B(3, 3; g^u)$.

Proof Theorem 1.2 takes care of the case where $g \equiv 0 \pmod 2$ and $u \geq 3$ or $g \equiv 1 \pmod 2$ and $u \equiv 1 \pmod 2$.

For the case where $g \equiv 1 \pmod 2$ and $u \equiv 0 \pmod 6$, note that a (3, 3)-HGDD of type $(u, 1^g)$ exists by Theorem 1.4. Replacing each of holes in a (3, 3)-HGDD of type $(u, 1^g)$ by a copy of a minimal (3, 3)-GDCD of type 1^u produces the result.

For the case where $g \equiv 1 \pmod 2$ and $u \equiv 4 \pmod 6$, a minimal (3, 3)-GDCD of type g^u is obtained by taking a minimal (3, 1)-GDCD and a (3, 2)-GDD with type g^u.

It remains to treat the case where $g \equiv 1 \pmod 2$ and $u \equiv 2 \pmod 6$. We distinguish the constructions into three cases according the values of $g \pmod 6$.

Case 1 $g \equiv 1 \pmod 6$

In this case, the excess of a minimal (3, 3)-GDCD of type g^u consists of $(gu / 2) + 2$ pairs and the construction is as follows.

(1) Take a minimal (3, 1)-GDCD of type g^u from Theorem 2.4. According to the construction of the design, we can know that its excess contains $(gu / 2) + 1$ pairs. We may also assume that two disjoint pairs {b, c} and {d, e} are contained in the excess.

(2) Take a minimal (3, 2)-GDCD of type g^u constructed in Lemma 3.2, which contains a sub-GDCD of type 1^5. Assume that the sub-GDCD is based on {a, b, c, d, e}. Replace the sub-GDCD by the following 7 blocks:

(a, b, e} {a, c, d} {a, c, e} {a, b, d} {b, d, e} {b, c, e} {c, d, e}

It is readily checked that the above two steps yield a minimal (3, 3)-GDCD of type g^u.

Case 2 $g \equiv 3 \pmod 6$

In this case, a minimal (3, 3)-GDCD of type g^u is obtained by taking a minimal (3, 1)-GDCD and a (3, 2)-GDD with type g^u.

Case 3 $g \equiv 5 \pmod 6$

In this case, the procedure is the same as the above Case 1. □

Lemma 3.4 For all integers $g \geq 2$ and $u \geq 3$, $C(3, 4; g^u) = B(3, 4; g^u)$.

Proof The case where $g \equiv 1, 2 \pmod 3$ and $u \equiv 0, 1 \pmod 3$ or $g \equiv 0 \pmod 3$ and $u \geq 3$, are covered by Theorem 1.2 where the GDCD is exact.

For the case where $g \equiv 1, 2 \pmod 3$ and $u \equiv 2 \pmod 3$, the construction is similar to that of Lemma 3.2, with a minor modification. A minimal (3, 4)-GDCD of type 2^u is formed by taking four copies of all blocks of size 3 from a ({3, 5*},1)-GDD of type 2^u and then replacing the distinguished block by a minimal (3, 4)-GDCD of type 1^5. Then we take a (3, 4)-IGDD of type $(g, 2)^u$ from Theorem 1.3 and fill in its hole by the above minimal (3, 4)-GDCD of type 2^u to obtain the required minimal (3, 4)-GDCD of type g^u. This completes the proof. □

Lemma 3.5 For all integers $g \geq 2$ and $u \geq 3$, $C(3, 5; g^u) = B(3, 5; g^u)$.

Proof If one of the following congruences is satisfied:

(1) $g \equiv 1, 5 \pmod 6$ and $u \equiv 1, 3 \pmod 6$;

(2) $g \equiv 2, 4 \pmod 6$ and $u \equiv 0, 1 \pmod 3$;

(3) $g \equiv 3 \pmod 6$ and $u \equiv 1 \pmod 2$;

(4) $g \equiv 0 \pmod 6$ and $u \geq 3$,

the results follows from Theorem 1.2 where the GDCD is exact.

For the case where $g \equiv 1, 5 \pmod 6$ and $u \equiv 0$ or $4 \pmod 6$, the required minimal (3, 5)-GDCD of type g^u is given by taking a minimal (3, 1)-GDCD and a (3, 4)-GDD with the same type g^u.

For the case where $g \equiv 1 \pmod 6$ and $u \equiv 2 \pmod 6$, it was shown in Theorem 1.4 that a (3, 5)-HGDD of type $(u, 1^g)$ exists. Filling in each hole of such HGDD by a minimal (3, 5)-GDCD of type 1^u produces the desired GDCD.

For the case where $g \equiv 5 \pmod 6$ and $u \equiv 2 \pmod 6$, the required minimal (3, 5)-GDCD of type g^u is given by taking a minimal (3, 1)-GDCD and a minimal (3, 4)-GDCD with the same type g^u.

For the case where $g \equiv 1, 5 \pmod 6$ and $u \equiv 5 \pmod 6$, or $g \equiv 2, 4 \pmod 6$ and $u \equiv 2 \pmod 3$, $B(3, 2; g^u) = \lceil gu / 3 \lceil 2g(u - 1) / 2 \rceil \rceil + 1$. The required minimal (3, 5)-GDCD of type g^u is given by taking a minimal (3, 2)-GDCD and a (3, 3)-GDD with the same type g^u.

Finally, for the case where $g \equiv 3 \pmod 6$ and $u \equiv 0 \pmod 2$, the required minimal (3, 5)-GDCD of type g^u is obtained by taking a minimal (3, 1)-GDCD and a (3, 4)-GDD with the same type g^u. □

4. Conclusion

As a consequence of Theorems 2.4 and 3.1, we have

Theorem 4.1 Let g, λ and u be positive integers satisfying $u \geq 3$. Then $C(3, \lambda; g^u) = B(3, \lambda; g^u)$, in which $B(3, \lambda; g^u) = \lceil gu / 3 \lceil \lambda g(u - 1) / 2 \rceil \rceil + 1$ when one of the following congruences is satisfied:

(1) $\lambda \equiv 2 \pmod 6$, $g \equiv 1$ or $2 \pmod 3$ and $u \equiv 2 \pmod 3$;

(2) $\lambda \equiv 5 \pmod 6$, $g \equiv 2$ or $4 \pmod 6$ and $u \equiv 2 \pmod 3$;

(3) $\lambda \equiv 5 \pmod 6$, $g \equiv 1$ or $5 \pmod 6$ and $u \equiv 5 \pmod 6$,

and $B(3, \lambda; g^u) = \lceil gu / 3 \lceil \lambda g(u - 1) / 2 \rceil \rceil$ otherwise.

Proof The result for $\lambda \leq 5$ was established in Theorems 2.4 and 3.1. For $\lambda \geq 6$, let $\lambda = 6m + \lambda'$. In this case, a minimal (3, λ)-GDCD of type g^u is obtained by taking a minimal (3, λ')-GDCD of type g^u and m times a (3, 6)-GDD of type g^u (for the existence of this see Theorem 1.2). The conclusion then follows from (1.2). □

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(Received 15/3/96)