On Vertex-disjoint Complete Subgraphs of a Graph

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Abstract

We conjecture that if G is a graph of order sk, where $s \geq 3$ and $k \geq 1$ are integers, and $d(x)+d(y) \geq 2(s-1)k$ for every pair of non-adjacent vertices x and y of G, then G contains k vertex-disjoint complete subgraphs of order s. This is true when $s = 3$, [6]. Here we prove this conjecture for $k \leq 6$.

1 Introduction

We put forward a conjecture which would generalize a deep theorem proved by Hajnal and Szemerédi [4]. They proved that if G is a graph of order sk, where $s \geq 3$ and $k \geq 1$ are integers, and the minimum degree of G is at least $(s-1)k$, then G contains k vertex-disjoint complete subgraphs of order s. The case $s = 3$ was first obtained by Corrádi and Hajnal [3]. We propose the following conjecture.

Conjecture A *Let s and k be integers with $s \geq 3$ and $k \geq 1$. Let G be a graph of order sk. If $d(x) + d(y) \geq 2(s-1)k$ for every pair of non-adjacent vertices x and y of G, then G contains k vertex-disjoint complete subgraphs of order s.*

Considering complements of graphs, this conjecture takes the following form.

Conjecture B *Let s and k be integers with $s \geq 3$ and $k \geq 1$. Let G be a graph of order sk. If $d(x) + d(y) \leq 2k - 2$ for every pair of adjacent vertices x and y of G, then G contains k mutually disjoint independent sets of cardinality s.*

In [6], we proved a stronger result than Conjecture A for the case $s = 3$, that is,

Theorem 1 *Let k be an integer with $k \geq 1$. Let G be a graph of order $3k$. If $d(x) + d(y) \geq 4k - 1$ for every pair of non-adjacent vertices x and y of G, then G contains k vertex-disjoint triangles.*

It is well known [2, 5] that if a graph G of order $n \geq 3$ has a pair of non-adjacent vertices x and y with $d(x) + d(y) \geq n$, then G is Hamiltonian if and only if $G + xy$ is

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Hamiltonian. If $n = 2k$ and $d(x)+d(y) \geq 2k-1$ instead, then it is easy to see that G contains k vertex-disjoint copies of K_2 if and only if $G + xy$ does. However, we have the following example for vertex-disjoint copies of K_3 in a graph. Let G be a graph of order $3k$ consisting of a path $P = xzy$ and a complete graph of order $3(k-1)$ such that they are vertex-disjoint, x and y are not adjacent, $d(x) = d(y) = 3k-2$ and $d(z) = 2$. It is clear that G does not contain k vertex-disjoint copies of K_3 but $G + xy$ does. It is also clear that y is the only vertex not adjacent to x and vice versa. As for vertex-disjoint copies of K_s $(s \geq 4)$ in a graph, this example can be easily generalized.

To further support the conjecture, we prove it for $k \leq 6$. We state the result as follows:

Theorem 2 *Let s and k be integers with $s \geq 3$ and $1 \leq k \leq 6$. Let G be a graph of order sk. If $d(x) + d(y) \leq 2k-2$ for every pair of adjacent vertices x and y of G, then G contains k mutually disjoint independent sets of cardinality s.*

We shall deduce some general propositions based on Conjecture B being false. Then we use these propositions to prove Theorem 2. We will use the following terminology and notation. Let $G = (V, E)$ be a graph. Let $x \in V$ and $Y \subseteq V$. We use $N(x, Y)$ to denote the set of neighbors of x that are in Y and let $d(x, Y) = |N(x, Y)|$. Thus $d(x, V) = d(x)$, i.e., the degree of x in G. For a subset $X \subseteq V$, $N(X, Y) = \cup_{x \in X} N(x, Y)$. A partition $(Y_1, Y_2, \ldots, Y_m)$ of Y is called an *s-uniform partition* (s-UP in short) of Y if $s = |Y_i|$ and Y_i is an independent set of G for all $i \in \{1, 2, \ldots, m\}$, and it is called an *$s$-chain* of Y if Y_i is an independent set of G for all $i \in \{1, 2, \ldots, m\}$ such that $s-1 = |Y_1|$, $s+1 = |Y_m|$ and $s = |Y_i|$ for all $i \in \{2, 3, \ldots, m-1\}$. We define $d(xy) = d(x)+d(y)$ for each edge $xy \in E$. Let $\Delta_2(Y)$ be the maximum of $d(xy)$ for all $xy \in E$ with $\{x, y\} \subseteq Y$. Set $\Delta_2(G) = \Delta_2(V)$. For two disjoint subsets A and B of V, $E(A, B)$ is the set of edges of G between of A and B and let $e(A, B) = |E(A, B)|$. We consider only finite simple graphs. Unexplained terminology and notation are adopted from [1].

2 Preliminaries

First, we note that Conjecture A is true when $s \in \{1, 2\}$. This is trivial if $s = 1$. If $s = 2$, G contains k independent edges as G is Hamiltonian by Ore's theorem [5].

We suppose that Conjecture B fails. Let $G = (V, E)$ be a graph of order sk with $s \geq 3$, $k \geq 1$ and $\Delta_2(G) \leq 2k-2$ such that G is a counter-example to Conjecture B with $|E|$ as small as possible. We use the idea in [1, pp.351–357] to prove the following propositions.

Proposition 2.1. *V has an s-chain.*

Proof. Let $xy \in E$. By the minimality of G, V has a partition $(U_1, U_2, \ldots, U_k)$ such that $|U_i| = s$ and U_i is an independent set of $G - xy$ for all $i \in \{1, 2, \ldots, k\}$. Hence $\{x, y\} \subseteq U_i$ for some $i \in \{1, 2, \ldots, k\}$, say $\{x, y\} \subseteq U_1$. As $d(xy) \leq 2(k-1)$, we may assume that $d(x) \leq k-1$. Thus, $d(x, V - U_1) \leq k-2$. This implies that $d(x, U_i) = 0$

for some $i \in \{2,3,\ldots,k\}$, say $d(x,U_k)=0$. Then $(U_1-\{x\},U_2,\ldots,U_{k-1},U_k\cup\{x\})$ is an *s-chain*. □

Let $(V_1,V_2,\ldots,V_k)$ be an *s-chain* of V. We say that V_m is *accessible* if there are distinct indices $i_1,i_2,\ldots,i_n \in \{1,2,\ldots,k\}$ and a vertex $x_{i_j} \in V_{i_j}$ for each $j \in \{2,3,\ldots,n\}$ such that $i_1=1, i_n=m$ and $d(x_{i_j},V_{i_{j-1}})=0$ for all $j \in \{2,3,\ldots,n\}$. In this case, we also say that x_{i_n} is *accessible* or an *accessible* vertex of V_{i_n}. Furthermore, we say that the set $\{V_{i_1},V_{i_2},\ldots,V_{i_n}\}$ is a justification of the accessibility of V_{i_n} or x_{i_n}, respectively. Clearly, each V_{i_j} in this justification in *accessible*, too. In particular, V_1 is *accessible*. By this definition, V_k is not *accessible*. For if $i_n=k$ in the above, then we obtain an s-UP $(V_1',V_2',\ldots,V_k')$ of V where $V_1'=V_1\cup\{x_{i_2}\}$, $V_{i_j}'=V_{i_j}\cup\{x_{i_{j+1}}\}-\{x_{i_j}\}$ for all $j\in\{2,3,\ldots,n-1\}$, $V_k'=V_k-\{x_{i_n}\}$ and $V_i'=V_i$ for each $i\in\{1,2,\ldots,k\}-\{i_1,i_2,\ldots,i_n\}$. For two distinct *accessible* sets V_i and V_j, we write $V_i \prec V_j$ if every justification of the accessibility of V_j contains V_i. An *accessible* set V_t is said to be terminal if $V_t \not\prec V_i$ for every *accessible* set V_i. Clearly, there is a terminal set. From these definitions, it is easy to see that if V_t is a terminal set such that $V_t \neq V_1$ and $x \in V_t$, then x is *accessible* if and only if $d(x,V_i)=0$ for some *accessible* set $V_i \neq V_t$.

Let A be the union of all *accessible* sets in $(V_1,\ldots,V_k)$ and set $B=V-A$. Assume that A includes p *accessible* sets V_i as subsets and so B includes $q=k-p$ *inaccessible* sets V_j as subsets.

Proposition 2.2. $\Delta_2(B)\le 2(q-1)$. *Furthermore, for any non-empty set $X\subseteq B$ with $|X|\equiv 0 \pmod q$, X has an s'-UP where $s'=|X|/q$. In particular, $B-\{x\}$ has an s-UP for all $x\in B$.*

Proof. As every vertex of B is *inaccessible*, we see that $d(x,V_i)\ge 1$ for each vertex $x\in B$ and each *accessible* set V_i. Therefore $d(x,B)\le d(x)-p$ for all $x\in B$. This implies that $\Delta_2(B)\le\Delta_2(G)-2p\le 2(k-1)-2p=2(q-1)$. As $\Delta_2(X)\le 2q-2$, the second statement of the proposition follows by the minimality of G. □

Proposition 2.3. *Let V_t be a terminal set. If $V_t\neq V_1$, then for each accessible vertex $x\in V_t$, $(A-V_t)\cup\{x\}$ has an s-UP.*

Proof. As x is *accessible*, there exists an *accessible* set V_m such that $m\neq t$ and $d(x,V_m)=0$. As V_t is terminal, there is a justification $\{V_{i_1},V_{i_2},\ldots,V_{i_n}\}$ of the accessibility of V_m (with $i_1=1$ and $i_n=m$) such that V_t does not belong to it. Let $x_{i_j}\in V_{i_j}$ be such that $d(x_{i_j},V_{i_{j-1}})=0$ for each $j\in\{2,3,\ldots,n\}$. Clearly, $(V_1\cup\{x_{i_2}\},V_{i_2}\cup\{x_{i_3}\}-\{x_{i_2}\},\ldots,V_{i_n}\cup\{x_{i_{n+1}}\}-\{x_{i_n}\})$ is an s-UP of $\cup_{j=1}^n V_{i_j}\cup\{x\}$ where $x_{i_{n+1}}=x$. This, together with *accessible* sets not in the justification except V_t, forms an s-UP of $(A-V_t)\cup\{x\}$. □

Proposition 2.4. *Let V_t be a terminal set. Let $y\in B$ and $x\in V_t$. Suppose $V_t\neq V_1$, $d(y,V_t)=1$ and $xy\in E$. Then x is inaccessible.*

Proof. If x is *accessible*, then by Proposition 2.3, $(A-V_t)\cup\{x\}$ has an s-UP. By Proposition 2.2, $B-\{y\}$ has an s-UP. Clearly, $V_t\cup\{y\}-\{x\}$ is an independent set. Therefore V has an s-UP, a contradiction. □

3 Proof of Theorem 2

We still use notation and terminology of Section 2. Let $G = (V, E)$ be a counter-example to Conjecture B as defined in Section 2. We choose an *s-chain* of V, say $(V_1, V_2, \ldots, V_k)$ such that

$$|A| \text{ is maximum.} \tag{1}$$

Subject to (1), we further choose $(V_1, V_2, \ldots, V_k)$ such that

$$e(A, B) \text{ is minimum.} \tag{2}$$

Let V_t be an arbitrary terminal set, and define

$$B_t = \{x \in B | d(x, V_t) = 1\} \text{ and } R_t = N(B_t, V_t); \tag{3}$$

$$b_t = |B_t| \text{ and } r_t = |R_t|. \tag{4}$$

We may assume that $A = V_1 \cup V_2 \cup \cdots \cup V_p$ and $B = V_{p+1} \cup \cdots \cup V_{p+q}$ where $k = p + q$. We shall prove $p \geq 6$ and therefore Theorem 2 follows. It is easy to see that Conjecture B is true when $k = 1$ or $k = 2$. Thus we have $k \geq 3$.

Proposition 3.1. *For each $x \in V_t$, $d(x, B) \leq 2k - p - 3$.*

Proof. Let y be arbitrary in $N(x, B)$. As y is *inaccessible*, $d(y, V_i) \geq 1$ for all i, $1 \leq i \leq p$. Therefore $d(x, B) \leq d(xy) - d(y) \leq 2k - p - 2$. If $d(x, B) = 2k - p - 2$, we must have that $d(y, B) = 0$, $d(y, V_i) = 1$ and $d(x, V_i) = 0$ for all $i, 1 \leq i \leq p$. If $p \geq 2$, then $V_t \neq V_1$ and therefore x is *accessible.* This is in contradiction to Proposition 2.4 as $d(y, V_t) = 1$. If $p = 1$, let $(U_2, U_3, \ldots, U_k)$ be an $(s-1)$-UP of $B - \{x_1, x_2, \ldots, x_k\}$ where $N(x, B) = \{x_1, x_2, \ldots, x_{2k-3}\}$, whose existence is quaranteed by Proposition 2.2. Clearly, some U_i does not contain any of $x_{k+1}, \ldots, x_{2k-3}$, and we may assume it is U_2. Note that y is arbitrary in $\{x_1, x_2, \ldots, x_{2k-3}\}$. It follows that $(V_1 \cup \{x_1, x_2\} - \{x\}, U_2 \cup \{x\}, U_3 \cup \{x_3\}, \ldots, U_k \cup \{x_k\})$ is an s-UP of V, a contradiction. □

Proposition 3.2. *For each $x \in R_t$, $d(x, B) \leq k - p$.*

Proof. Suppose that there exists $x_0 \in R_t$ such that $d(x_0, B) \geq k - p + 1$. Let $x_1 \in B_t$ be such that $x_0 x_1 \in E$. By Proposition 2.4, x_0 is *inaccessible* and therefore $d(x_0, V_i) \geq 1$ for all $i \neq t$, $1 \leq i \leq p$. Thus $d(x_0) \geq k$. By Proposition 2.2, $B - \{x_1\}$ has an s-UP $(U_{p+1}, \ldots, U_k)$. As $\Delta_2(G) \leq 2k - 2$, we have $d(x_0, B - \{x_1\}) \leq 2k - 2 - d(x_1) - d(x_0, A) - 1 \leq 2k - 2 - 2p = 2(q-1)$. This implies that some U_i, say $U_i = U_{p+1}$, contains at most one neighbor of x_0. If $U_{p+1} \cap N(x_0, B) = \emptyset$, we add x_0 to U_{p+1}. If $U_{p+1} \cap N(x_0, B) = \{y\}$, then $d(y, B) \leq 2k - 2 - d(x_0) - d(y, A) \leq 2k - 2 - k - p = q - 2$. This implies $d(y, U_j) = 0$ for some $j, p + 2 \leq j \leq k$. We then move y to U_j and add x_0 to U_{p+1}. In either case, we obtain an *s-chain* $(V_t - \{x_0\}, V'_{p+1}, V'_{p+2}, \ldots, V'_k)$ of $B \cup V_t - \{x_1\}$. Let $V'_t = V_t \cup \{x_1\} - \{x_0\}$ and $V'_i = V_i$ for all $i \neq t, 1 \leq i \leq p$. Then $(V'_1, \ldots, V'_k)$ is an *s-chain* of V. Clearly, each *accessible* vertex with respect to $(V_1, \ldots, V_k)$ is still *accessible* with respect to $(V'_1, \ldots, V'_k)$. Therefore $V'_1, \ldots, V'_p$ are *accessible* sets in $(V'_1, \ldots, V'_k)$. Let $A' = \cup_{i=1}^{p} V'_i$ and $B' = V - A'$. Then we have

$$e(A', B') \leq e(A, B) - d(x_0, B) - d(x_1, A) + d(x_0, A) + d(x_1, B) + 2$$

$$\begin{aligned}&\le e(A,B)+2k-2-2d(x_0,B)-2d(x_1,A)+2\\&\le e(A,B)+2k-2-2(k-p+1)-2p+2\\&= e(A,B)-2.\end{aligned}$$

This is in contradiction with (2) while (1) is maintained. □

By Proposition 3.2,

$$|R_t| \ge |B_t|/(k-p), \ \textit{i.e.}, \ (k-p)r_t \ge b_t. \tag{5}$$

Let $d_t = \max\{d(x,B)|x \in V_t\}$. By Proposition 3.2, we obtain

$$(k-p)|R_t| + d_t(s-|R_t|) \ge e(V_t,B) \ge 2((k-p)s+1) - |B_t|. \tag{6}$$

Combining (5) and (6), we obtain

$$(2(k-p)-d_t)r_t \ge (2(k-p)-d_t)s+2. \tag{7}$$

By Proposition 2.4, $R_t \neq V_t$ if $V_t \neq V_1$ and so $r_t \le s-1$. We deduce from (7) that $d_t > 2(k-p)$. As $d_t \le 2k-p-3$ by Proposition 3.1, we obtain

$$p \ge 4 \textit{ and } k \ge 5. \tag{8}$$

Let $U_t = V_t - R_t$ and $s_t = s - r_t = |U_t|$. Let $W_t = B - N(R_t,B)$. By Proposition 3.2, $|N(R_t,B)| \le (k-p)r_t$. Hence $|W_t| \ge (k-p)s_t + 1$. Clearly,

$$e(U_t,W_t) = \sum_{x\in W_t} d(x,U_t) \ge 2|W_t| \ge 2(k-p)s_t+2. \tag{9}$$

To show $p \ge 6$. We distinguish two cases: $p=4$ or $p=5$.

Case 1. $p=4$.

In this case, $d(u,W_t) \le 2k-7$ for all $u \in U_t$ by Proposition 3.1. Let $U'_t = \{u \in U_t | d(u,W_t) = 2k-7\}$ and $W'_t = N(U'_t, W_t)$. By (9) with $p=4$, $U'_t \neq \emptyset$. We claim

For every $uw \in E(U'_t, W'_t)$ with $u \in U'_t$ and $w \in W'_t$, $d(u,A) = 0$, $d(w,V_i) = 1$ for all $i \in \{1,2,3,4\} - \{t\}$ and $d(w,U_t) = 2$. (10)

Proof of (10). As $w \notin N(R_t,B)$ and w is *inaccessible*, we have $d(w,U_t) \ge 2$ and $d(w,V_i) \ge 1$ for all $i \in \{1,2,3,4\} - \{t\}$. As $d(uw) \le 2k-2$, (10) follows.

Without loss of generality, assume V_4 is terminal. We claim

For every $w \in W'_4$, there exists a unique $x_w \in U_4 - U'_4$ such that $wx_w \in E$ and $d(x_w, W_4) \le 2k-10$. (11)

Proof of (11). Let $u \in U'_4$ be such that $uw \in E$. By (10), $d(w,U_4) = 2$ and $d(w,A) = 5$. Let $x_w \in U_4 - \{w\}$ with $wx_w \in E$. We need to show that $d(x_w,W_4) \le 2k-10$. Suppose that $d(x_w,W_4) \ge 2k-9$. As $d(wx_w) \le 2k-2$, $d(x_w,V_i) = 0$ for some $i \in \{1,2,3\}$, i.e., x_w is *accessible*. By Proposition 2.3, $(A-V_4)\cup\{x_w\}$ has an s-UP (V'_1, V_2, V'_3). As $d(u,A) = 0$ by (10), $V_2 \not\to V_4$ and $V_3 \not\to V_4$. It follows that, if $V_2 \prec V_3$ then V_3 is terminal, and if $V_2 \not\prec V_3$ then V_2 is terminal. Without loss of generality, say V_3 is terminal. By (10), there exists $v \in U_3'$ such that $d(v, A) = 0$. We have $d(w, V_3) = 1$ by (10). By Proposition 2.4, $wv \notin E$ as v is *accessible*. Without loss of generality, say $v \in V_3'$. Then $(V_1', V_2', V_3' \cup \{u\} - \{v\}, V_4' \cup \{v, w\} - \{u, x_w\})$ together with an s-UP of $B - \{w\}$ forms an s-UP of V, a contradiction. So (11) holds.

By (10) and (11), $N(u, W_4) \cap N(v, W_4) = \emptyset$ for any $\{u, v\} \subseteq U_4'$ with $u \neq v$. Hence $|W_4'| = (2k-7)|U_4'|$. It follows that $|N(W_4', U_4 - U_4')| \geq |W_4'|/(2k-10) > |U_4'|$. Let $X \subseteq N(W_4', U_4 - U_4')$ with $|X| = |U_4'|$. Then $e(X, W_4) + e(U_4', W_4) \leq (2k-10)|X| + (2k-7)|U_4'| < (2k-8)|X \cup U_4'|$. It follows $e(U_4, W_4) < (2k-8)s_4$, contradicting (9) with $t = p = 4$.

The idea of Case 1 is used in Case 2. However, Case 2 is more complicated.

Case 2. $p = 5$.

In this case, $d(u, W_t) \leq 2k-8$ for all $u \in U_t$ by Proposition 3.1. Let

$$U_t^1 = \{u \in U_t | d(u, W_t) = 2k-8\};$$
$$U_t^2 = \{u \in U_t | d(u, W_t) = 2k-9\};$$
$$U_t^3 = U_t - (U_t^1 \cup U_t^2) \text{ and } W_t' = N(U_t^1 \cup U_t^2, W_t).$$

By (9) with $p = 5$, we see that $U_t^1 \cup U_t^2 \neq \emptyset$. Similar to the proof of (10), we can readily show

For every $uw \in E(U_t^1 \cup U_t^2, W_t')$ with $u \in U_t^1 \cup U_t^2$ and $w \in W_t'$, $d(u, A) \leq 1$, $1 \leq d(w, V_i) \leq 2$ for all $i \in \{1, 2, 3, 4, 5\} - \{t\}$ and $2 \leq d(w, U_t) \leq 3$. (12)

We divide case 2 into the following two subcases.

Case 2.1. There exist two distinct terminal sets V_i and V_j such that $d(x, V_j) \geq 1$ for all $x \in V_i$.

Without loss of generality, say $i = 5$ and $j = 4$. As $\Delta_2(G) \leq 2k-2$ and by (12) with $t = 5$, we have

For every $uw \in E(U_5^1 \cup U_5^2, W_5')$ with $u \in U_5^1 \cup U_5^2$ and $w \in W_5'$, $d(u, V_1 \cup V_2 \cup V_3) = 0$, $d(u, V_4) = 1$, $d(u, W_5) = 2k-9$, $d(w, V_i) = 1$ for all $i \in \{1, 2, 3, 4\}$ and $d(w, U_5) = 2$. (13)

By (13), $U_5^1 = \emptyset$. We claim that one of V_2 and V_3 is terminal. To see this, let $u_0 \in U_5^2$ and $u_0' \in U_4^1 \cup U_4^2$. By (12), $d(u_0', A) \leq 1$. As $d(u_0, V_1) = 0$, $V_2 \not\prec V_5$ and $V_3 \not\prec V_5$. As either $d(u_0', V_1) = 0$ or $d(u_0', V_5) = 0$, we see that $V_2 \not\prec V_4$ and $V_3 \not\prec V_4$. It follows that, if $V_2 \prec V_3$ then V_3 is terminal, and if $V_2 \not\prec V_3$ then V_2 is terminal. This shows the claim.

Without loss of generality, say V_3 is terminal. We shall show that $d(x, V_5) \geq 1$ for all $x \in V_3$. To see this , we suppose that $d(v_0, V_5) = 0$ for some $v_0 \in V_3$, and therefore v_0 is *accessible*. Then we claim that, for each $w \in W_5'$, there exists a unique $x_w \in U_5^3$ such that $wx_w \in E$ and $d(x_w, W_5) \leq 2k-12$. By (13), $d(w, U_5) = 2$. Let $N(w, U_5) =$

$\{u, x_w\}$ with $u \in U_t^2$. We need to show that $d(x_w, W_5) \leq 2k - 12$. Suppose instead that $d(x_w, W_5) \geq 2k-11$. As $d(wx_w) \leq 2k-2$ and $d(w, A) = 6$ by (13), $d(x_w, V_i) = 0$ for some $i \in \{1, 2, 3\}$. Hence x_w is *accessible*. Note that as $d(w, V_3) = 1$ by (13), $v_0 w \notin E$ by Proposition 2.4. Let $N(u, V_4) = \{u'\}$. If $(A - V_5) \cup \{x_w\}$ has an s-UP (V_1', V_2', V_3', V_4') such that $\{v_0, u'\} \not\subseteq V_i'$ for every $i \in \{1, 2, 3, 4\}$, say without loss of generality $v_0 \in V_4'$, then $(V_1', V_2', V_3', V_4' \cup \{u\} - \{v_0\}, V_5 \cup \{w, v_0\} - \{u, x_w\})$ together with an s-UP of $B - \{w\}$ forms an s-UP of V, a contradiction. Therefore, all we need is to show that there is such an s-UP of $(A - V_5) \cup \{x_w\}$. This is obvious if there exists a justification of the accessibility of x_w which does not contain V_3 or V_4. In particular, this is true if $d(a, V_1) = 0$ for some $a \in V_3$ and $d(b, V_1) = 0$ for some $b \in V_4$. Therefore we may assume that either $d(z, V_1) \geq 1$ for all $z \in V_3$, or $d(z, V_1) \geq 1$ for all $z \in V_4$. Without loss of generality, say the former holds. Similar to obtaining (13), we see that $U_3^1 = \emptyset$ and $d(z, A) = 1 = d(z, V_1)$ for all $z \in U_3^2$. By (9) with $p = 5$ and $t = 3$, U_3^2 has at least two distinct vertices, say v_1 and v_1'. Without loss of generality, say $v_1' = v_0$. Clearly, any justification containing no V_5 of the accessiblity of V_3 is a justification of the accessibility of both v_0 and v_1. Then we see that a desired s-UP of $(A - V_5) \cup \{x_w\}$ is yielded from any given justification of the accessibility of x_w. Therefore our claim is true. This claim, together with (13), implies that $N(u, W_5) \cap N(v, W_5) = \emptyset$ for any $\{u, v\} \subseteq U_5^2$ with $u \neq v$. Therefore $|W_5'| = (2k-9)|U_5^2|$ and $|N(W_5', U_5^3)| \geq |W_5'|/(2k-12) > |U_5^2|$. As in Case 1, it follows that $e(U_5, W_5) \leq (2k-10)s_5$, contradicting (9) with $t = p = 5$. This shows that $d(x, V_5) \geq 1$ for all $x \in V_3$.

With V_3, V_4 and V_5 playing the roles of V_5, V_3 and V_4, respectively in the above argument, we obtain $d(x, V_3) \geq 1$ for all $x \in V_4$. Similar to obtaining (13), we see that for each $i \in \{3, 4, 5\}$, there exists a vertex $u_i \in U_i^2$ such that $d(u_i, V_1 \cup V_2) = 0$. Therefore V_2 is terminal. By (12), there is a vertex $u_2 \in U_2^1 \cup U_2^2$ such that $d(u_1, A) \leq 1$. Hence $d(u_2, V_i) = 0$ for some $i \in \{3, 4, 5\}$, say without loss of generality $d(u_2, V_5) = 0$. With V_2 playing the role of V_3 in the above argument, we again obatin $e(U_5, W_5) \leq (2k-10)s_5$, a contradiction.

Case 2.2. For any two distinct terminal sets V_i and V_j, there exist $x \in V_i$ and $y \in V_j$ such that $d(x, V_j) = 0$ and $d(y, V_i) = 0$.

In this subcase, we claim first that V_i is terminal for all $i \in \{2, 3, 4, 5\}$. As there is a terminal set, say without loss of generality V_5 is terminal. Let $u_5 \in U_5^1 \cup U_5^2$. Then $d(u_5, A) \leq 1$ by (12). If $d(u_5, V_1) = 0$, then $V_i \not\prec V_5$ for all $i \in \{2, 3, 4\}$, and consequently, V_i is terminal for some $i \in \{2, 3, 4\}$. If $d(u_5, V_1) = 1$, then $d(u_5, A - V_1) = 0$ and there exists exactly one of V_2, V_3 and V_4, say V_2, such that $V_2 \prec V_5$. Then $V_3 \not\prec V_2$ and $V_4 \not\prec V_2$. Therefore one of V_3 and V_4 is terminal. In either case, say without loss of generality V_4 is terminal. Let $u_4 \in U_4^1 \cup U_4^2$. Then $d(u_4, A) \leq 1$ by (12). If $V_2 \prec V_3$, then for each $i \in \{4, 5\}$, $V_3 \not\prec V_i$ as either $d(u_i, V_1) = 0$ or $d(u_i, V_2) = 0$, and consequently, V_3 is terminal. If $V_2 \not\prec V_3$ and V_2 is not terminal, then $V_2 \prec V_4$ or $V_2 \prec V_5$. Say without loss of generality $V_2 \prec V_5$. Then $d(u_5, V_1) = 1 = d(u_5, A)$, $V_3 \not\prec V_5$ and there exists $a \in V_2$ such that $d(a, V_1) = 0$. Thus $V_3 \not\prec V_2$. As either $d(u_4, V_1) = 0$ or $d(u_4, V_5) = 0$, we see that $V_3 \not\prec V_4$, and consequently V_3 is terminal. Finally, we need to show that V_2 is terminal. If V_2 is not terminal, then $V_2 \prec V_i$

for some $i \in \{3,4,5\}$, say without loss of generality $V_2 \prec V_5$. Then $d(u,V_1) \geq 1$ for all $u \in V_5$ and $d(a,V_1) = 0$ for some $a \in V_2$. Similar to obtaining (13), with $d(u,V_1) = 1$ replacing $d(u,V_4) = 1$, we see that all the other equalities in (13) hold. Then for each $w \in W_5'$, it is easy to see that if $N(w,U_5) = \{u, x_w\}$ with $u \in U_5^2$ and $d(x_w, V_i) = 0$ for some $i \in \{2,3,4\}$, then x_w is *accessible* and $(A - V_5) \cup \{x_w\}$ has an s-UP (V_1', V_2', V_3', V_4') with $x_w \notin V_1' \supseteq V_1$. Moreover, either $v_0 \notin V_1'$ or $v_1 \notin V_1'$ where $d(v_0, V_5) = 0$ and $d(v_1, V_5) = 0$ with $v_0 \in V_3$ and $v_1 \in V_4$. Say w.l.o.g, $d(v_0,V_5) = 0$ and $v_0 \in V_4'$. Then $(V_1', V_2', V_3', V_4' \cup \{u\} - \{v_0\}, V_5 \cup \{v_0, w\} - \{u, x_w\})$ together with an s-UP of $B - \{w\}$ forms an s-UP of V, a contradiction. Hence $d(x_w, V_i) \geq 1$ for all $i \in \{1,2,3,4\}$, and therefore $d(x_w, W_5) \leq 2k - 12$ as $d(wx_w) \leq 2k-2$ and $d(w,A) = 6$. As in Case 2.1, this yields $e(U_5, W_5) \leq (2k-10)s_5$, a contradiction. Hence we conclude that V_i is terminal for all $i \in \{2,3,4,5\}$.

For each $i \in \{2,3,4\}$, let $a_i \in V_i$ be such that $d(a_i, V_5) = 0$. So each $a_i (2 \leq i \leq 4)$ is *accessible*. We claim

For every $w \in W_5'$, there exists $x_w \in U_5^3$ such that $wx_w \in E$ and $d(x_w, W_5) \leq 2k - 12$. (14)

Proof of (14). Let $u \in U_5^1 \cup U_5^2$ be such that $uw \in W$. By (12), we may set $N(w, U_5) = \{u, x_w, x_w'\}$ with $x_w = x_w'$ if $d(w,U_5) = 2$. Suppose, for a contradiction, that $d(x_w, W_5) \geq 2k-11$ and $d(x_w', W_5) \geq 2k - 11$. Assume first that $d(w, U_5) = 3$. As $d(uw) \leq 2k-2$ and by (12), we see that $d(u,A) = 0$ and $d(w,V_i) = 1$ for all $i \in \{1,2,3,4\}$. As $d(wx_w) \leq 2k-2$ and $d(wx_w') \leq 2k-2$, it follows that $d(x_w, A) \leq 2$ and $d(x_w', A) \leq 2$. Hence there exists $\{i,j\} \subseteq \{1,2,3,4\}$ with $i \neq j$ such that $d(x_w, V_i) = 0$ and $d(x_w', V_j) = 0$. Let $r \in \{1,2,3,4\} - \{i,j\}$ be such that $r = 1$ if $1 \notin \{i,j\}$. Without loss of generality, say $r = 1, i = 2$ and $j = 3$. By Proposition 2.4, $a_2 w \notin E$ and $a_3 w \notin E$. Then $(V_1 \cup \{u\}, V_2 \cup \{x_w\} - \{a_2\}, V_3 \cup \{x_w'\} - \{a_3\}, V_4, V_5 \cup \{w, a_2, a_3\} - \{u, x_w, x_w'\})$ together with an s-UP of $B - \{w\}$ forms an s-UP of V, a contradiction. Hence $x_w = x_w'$. Then we see that $d(x_w, A) \leq 3$ as $d(wx_w) \leq 2k-2$. Hence $d(x_w, V_i) = 0$ for some $i \in \{1,2,3,4\}$, i.e., x_w is *accessible.* Clearly, $(A - V_5) \cup \{x_w\}$ has an s-UP (V_1', V_2', V_3', V_4') such that $\{a_2, a_3, a_4\} \not\subseteq V_i'$ for all $i \in \{1,2,3,4\}$. As $d(uw) \leq 2k-2$ and by (12), we see that if $d(u,A) = 1$, then $d(w, V_i) = 1$ and therefore $wa_i \notin E$ by Proposition 2.4 for all $i \in \{2,3,4\}$, and if $d(u,A) = 0$, then $d(w,V_i) \geq 2$ for at most one $i \in \{1,2,3,4\}$, and therefore by Proposition 2.4, $a_i w \in E$ for at most one $i \in \{2,3,4\}$. Hence we can always choose an a_j and a V_i' such that $a_j \in V_i'$, $a_j w \notin E$ and $d(u, V_i') = 0$. Without loss of generality, say $i = j = 4$. Then $(V_1', V_2', V_3', V_4' \cup \{u\} - \{a_4\}, V_5 \cup \{w, a_4\} - \{u, x_w\})$ together with an s-UP of $B - \{w\}$ forms an s-UP of V, a contradiction. This proves (14).

Let $\{u_i, u_i'\} (1 \leq i \leq r)$ be a list of all distinct pairs of vertices of $U_5^1 \cup U_5^2$ such that $N(u_i, W_5) \cap N(u_i', W_5) \neq \emptyset$ for all $i \in \{1,2,\ldots,r\}$. As $\Delta_2(G) \leq 2k-2$ and by (12) and (14), we see that for each $i \in \{1,2,\ldots,r\}$, $\{u_i, u_i'\} \subseteq U_5^2$, and $x_w \in U_5^3$ and $d(w, U_5) = 3$ for all $w \in N(u_i, W_5) \cap N(u_i', W_5)$. For each $i \in \{1,2,\ldots,r\}$, we choose a fixed $w_i \in N(u_i, W_5) \cap N(u_i', W_5)$. Then $w_i (1 \leq i \leq r)$ are distinct. Let $v_i (1 \leq i \leq n)$ be a list of the vertices in $U_5^1 \cup U_5^2 - \{u_i, u_i' | 1 \leq i \leq r\}$. Let Q be the bipartite graph induced by the edges in $\{w_iu_i, w_iu_i'|1 \le i \le r\}$. Then $d_Q(w_i) = 2$ for all $i \in \{1, 2, \dots, r\}$. This implies that each block of Q is either a cycle or an edge. Let $A = V(Q) \cap U_5$ and $D = V(Q) \cap W_5$. Let $Q_i (1 \le i \le m)$ be a list of components of Q. For each $i \in \{1, 2, \dots, m\}$, let $A_i = V(Q_i) \cap A$ and $D_i = V(Q_i) \cap D$, and then we see that $|A_i| \le |D_i| + 1$. Furthermore, we have

$$N(A_i, W_5) \cap N(A_j, W_5) = \emptyset, 1 \le i < j \le m; \quad (15)$$
$$N(A_i, W_5) \cap N(v_j, W_5) = \emptyset, 1 \le i \le m \text{ and } 1 \le j \le n; \quad (16)$$
$$N(v_i, W_5) \cap N(v_j, W_5) = \emptyset, 1 \le i < j \le n. \quad (17)$$

By (15)-(17), $|W_5'| \ge (2k-9)(n+m)$. Let $X = \{x_w | w \in W_5'\}$. Then $|X| \ge (2k-9)(n+m)/(2k-12) > n+m$. Let $Y \subseteq X$ with $|Y| = n+m$. If $Z = U_5 - (A \cup Y \cup \{v_i | 1 \le i \le n\})$, then

$$\begin{aligned} e(U_5, W_5) &= \sum_{x\in A} d(x, W_5) + \sum_{x\in Y} d(x, W_5) + \sum_{i=1}^{n} d(v_i, W_5) + \sum_{x \in Z} d(x, W_5) \\ &\le (2k-9)|A| + (n+m)(2k-12) + (2k-8)n + (2k-10)|Z| \\ &\le (2k-10)|A| + (|D|+m) + (2k-12)m + 2(2k-10)n + (2k-10)|Z| \\ &\le (2k-10)s_5 + |D|, \end{aligned}$$

and on the other hand, we have

$$\begin{aligned} e(U_5, W_5) &= \sum_{x \in D} d(x, U_5) + \sum_{y \in W_5 - D} d(y, U_5) \\ &\ge |D| + 2|W_5| \ge |D| + 2(k-5)s_5 + 2. \end{aligned}$$

This is a contradiction. This completes the proof of the theorem.

Remarks. It seems possible to prove the conjecture for more small values of k by refining the above idea. However, it seems very difficult to prove the conjecture in general. It would be interesting to prove it for $s = 4$.

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