On the non-existence of Steiner $(v, k, 2)$ trades with certain volumes

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Abstract

In this note, we prove that there does not exist a Steiner $(v, k, 2)$ trade of volume m, where m is odd, $2k + 3 \leq m \leq 3k - 4$, and $k \geq 7$. This completes the spectrum problem for Steiner $(v, k, 2)$ trades.

1 Introduction

A (v, k, t) *trade* $T = \{T_1, T_2\}$ of *volume* $m = m(T)$ consists of two disjoint collections T_1 and T_2, each containing m k-subsets, called *blocks*, of some set V, such that each t-subset of V is contained in the same number of blocks in T_1 and T_2. The set of elements of V contained in T_1 is denoted by $F(T_1)$. Note that there may exist elements of V which occur in no block of T_1. In this paper since we are not concerned with the value of v we write (k, t) trade instead of (v, k, t) trade.

Definition 1 *A (k, t) trade $T = \{T_1, T_2\}$ is called Steiner (k, t) trade if any t-subset of $F(T_1)$ occurs at most once in T_1.*

Definition 2 *The spectrum $S(k, t)$ of Steiner (k, t) trade is*

$$S(k, t) = \{m \,|\, \text{there exists a Steiner } (k, t) \text{ trade of volume } m\}.$$

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It is well-known that $S(3,2) = \{0,4,6,7,8,\ldots\}$ (see [7]), $S(4,2) = \{0,6,8,9,10,\ldots\}$ (see [2]) and $S(4,3) = \{0,8,12,14,15,16,\ldots\}$ (see [5] and references therein). In [3], Gray and Ramsay show that $S(5,2) = \{0,8,10,12,13,14,\ldots\}$ and $S(6,2) = \{0,10,12,14,15,16,\ldots\}$. They also prove that:

Theorem 3 (1) (See [3]) *If $0 < m < 2k-2$ or $m = 2k-1$, then $m \notin S(k,2)$. If $m = 0$, or $m \geq 3k-3$, or m is even and $2k-2 \leq m \leq 3k-4$, then $m \in S(k,2)$.*
(2) (See [4]) *$2k+1 \in S(k,2)$ precisely when $k \in \{3,4,7\}$.*

So for $k \geq 7$, the inclusion of odd volumes between $2k+3$ and $3k-4$ in $S(k,2)$ has remained undetermined. In this note we prove there does not exist a Steiner $(k,2)$ trade T with $m = m(T)$ odd and $2k+3 \leq m \leq 3k-4$ for $k \geq 7$. This completely settles the spectrum problem for Steiner $(k,2)$ trades.

2 Preliminary Results

First we state some results of [6] and [3].

Definition 4 *For an s-subset S and trade $T = \{T_1, T_2\}$, let $r_S(T_1)$ be the number of blocks in T_1 which contain S. If $S = \{x\}$, we write r_x for $r_{\{x\}}(T_1)$.*

Lemma 5 (See [6]) *If S is an s-subset, $1 \leq s < t$, and T is a (k,t) trade, then*

$$r_S(T) \neq 1,\ m(T) - 1.$$

Lemma 6 (See [3]) *Suppose $T = \{T_1, T_2\}$ is a Steiner $(k,2)$ trade with $r_\alpha = 2$ for some $\alpha \in F(T_1)$ and $m(T) < 4k-10$. If B_1 and B_2 are the two blocks of T_1 containing α, then there exist (distinct) elements $x \in B_1$ and $y \in B_2$ such that at least $k-1$ blocks of T_1 (including B_1) contain x but not y, and at least $k-1$ blocks of T_1 (including B_2) contain y but not x.*

Lemma 7 (See [3]) *Suppose $T = \{T_1, T_2\}$ is a Steiner $(k,2)$ trade, $k > 3$, and there exist distinct elements $x, y \in F(T_1)$ such that $r_x + r_y \geq m(T)$. Then $r_x = r_y = m(T)/2$.*

We also make use of the following lemma in the next sections.

Lemma 8 *Let x, y, z and k be integers with $k \geq 3$ and $\phi(x,y,z) = xz + yz - xy$. If*

(1) *$x + y + z = k-1$; and*

(2) *$0 \leq x \leq y \leq z \leq k-2$,*

then $\phi(x,y,z) \geq k-2$ if $k \neq 4,7$ and $\phi(x,y,z) \geq k-3$ if $k = 4$ or 7. Furthermore, these minimum values are obtained only at

(i) *$(x,y,z) = (0,1,k-2)$ for $k = 3$ or $k \geqslant 8$;*

(ii) $(x, y, z) \in \{(0,1,3), (1,1,2)\}$ *for* $k = 5$*;*

(iii) $(x, y, z) \in \{(0,1,4), (1,2,2)\}$ *for* $k = 6$*;*

(iv) $(x, y, z) = (1,1,1)$ *for* $k = 4$*; and*

(v) $(x, y, z) = (2,2,2)$ *for* $k = 7$*;*

Proof If $x = 0$ then $\phi(0, y, z) = yz$, $0 \le y \le z$ and $y + z = k - 1$. So $\phi(0, y, z) \ge k - 2$. Moreover, $\phi(0, 1, k-2) = k - 2$. Now let $x \ge 1$. Then (2) becomes $1 \le x \le y \le z \le k - 2$ and so $k \ge 4$. From (1) we have $z = k - 1 - x - y$ so

$$\psi(x, y) = \phi(x, y, k - 1 - x - y) = (x + y)(k - 1 - (x + y)) - xy$$

and $x + 2y \le k - 1$. If we assume x, y are real numbers and $0 \le \lambda \le y - x$ then

$$\begin{aligned} \psi(x + \lambda, y - \lambda) &= (x + y)(k - 1 - (x + y)) - (x + \lambda)(y - \lambda) \\ &= (x + y)(k - 1 - (x + y)) - xy + \lambda(x - y) + \lambda^2 \\ &= (x + y)(k - 1 - (x + y)) - xy + \lambda(x - y + \lambda) \\ &\le \psi(x, y). \end{aligned}$$

So the minimum of $\psi(x, y)$ occurs at $x = y$. Letting $y = x$ in $\psi(x, y)$ we find $\psi(x, x) = -5x^2 + 2(k - 1)x$ and by (1) and (2) we have $1 \le x \le (k - 1)/3$. So

$$\psi(x, x) >= \min(\psi(1, 1), \psi((k - 1)/3, (k - 1)/3)).$$

Now if $x = 1$ then

$$\phi(1, 1, k - 3) = 2k - 7 \ge k - 2 \text{ for } k \ge 5$$

and if $x = (k - 1)/3$ then

$$\phi((k - 1)/3, (k - 1)/3, (k - 1)/3) = (x - 1)^2/9 \ge k - 2 \text{ for } k \ge 9.$$

The case $k \in \{4, 5, 6, 7, 8\}$ is left for the reader. □

3 Steiner $(k, 2)$ trades with $k \ge 8$

In this section we prove that for $k \ge 8$ there does not exist a Steiner $(k, 2)$ trade T with $m(T)$ odd and $2k + 3 \le m(T) \le 3k - 4$. We begin with the following crucial lemma.

Lemma 9 *Suppose* $T = \{T_1, T_2\}$ *be a Steiner* $(k, 2)$ *trade with* $k \ge 8$*. If there exists an* $\alpha \in F(T_1)$ *with* $r_\alpha = 3$ *then* $m(T) \ge 3k - 3$*.*

Proof Let B_1, B_2 and B_3 be the three blocks in T_1 which contain the element α and let C_1, C_2 and C_3 be the three blocks in T_2 which contain the element α (see Table 1). Note that $B_1 \cup B_2 \cup B_3 = C_1 \cup C_2 \cup C_3$.

T_1	T_2
$B_1 \ : \ \{\alpha, b_{11}, b_{12}, \ldots, b_{1(k-1)}\}$	$C_1 \ : \ \{\alpha, c_{11}, c_{12}, \ldots, c_{1(k-1)}\}$
$B_2 \ : \ \{\alpha, b_{21}, b_{22}, \ldots, b_{2(k-1)}\}$	$C_2 \ : \ \{\alpha, c_{21}, c_{22}, \ldots, c_{2(k-1)}\}$
$B_3 \ : \ \{\alpha, b_{31}, b_{32}, \ldots, b_{3(k-1)}\}$	$C_3 \ : \ \{\alpha, c_{31}, c_{32}, \ldots, c_{3(k-1)}\}$
$\vdots$	$\vdots$

Table 1

Define $X_{ij} = (C_i \cap B_j) \setminus \{\alpha\}$ and $x_{ij} = |X_{ij}|$ for $1 \le i, j \le 3$. Then it follows that $\sum_{i=1}^{3} x_{ij} = k-1$ and $\sum_{j=1}^{3} x_{ij} = k-1$. Moreover, since T_1 and T_2 are distinct we have $x_{ij} \le k-2$. We also define

$$P_i = \{\{\beta, \gamma\} \mid \beta \in X_{ir},\ \gamma \in X_{is},\ \text{and } 1 \le r < s \le 3\}$$

for $1 \le i \le 3$. Note that each P_i is the edge set of the complete tripartite graph, G_i say, with parts X_{ij}, $1 \le j \le 3$. Now let $A \in T_1 \setminus \{B_1, B_2, B_3\}$ and $P = \{\{x, y\} \mid x, y \in A\}$. Since each element of P_i occurs exactly in one block of T_1 we have $|P \cap P_i| = 0$, 1, or 3. Moreover, if $|P \cap P_i| = 3$ then these three pairs form a triangle. Therefore there must be at least

$$x_{i1}x_{i2} + x_{i1}x_{i3} + x_{i2}x_{i3} - 2(\text{maximum number of triangles in } G_i)$$

blocks in T_1 to cover the pairs in P_i. Assuming $x_{i1} \le x_{i2} \le x_{i3}$, the maximum number of triangles in G_i is $x_{i1}x_{i2}$. So there must be at least

$$x_{i1}x_{i2} + x_{i1}x_{i3} + x_{i2}x_{i3} - 2x_{i1}x_{i2} = x_{i1}x_{i3} + x_{i2}x_{i3} - x_{i1}x_{i2}$$

blocks in T_1 to cover the elements of P_i. On the other hand, no element of P_i and P_j can occur in the same block of T_1 for $i \ne j$. Now applying Lemma 8 we see

$$m(T) \ge 3 + (k-2) + (k-2) + (k-2) = 3k - 3.$$

This completes the proof. □

Lemma 10 *Suppose $T = \{T_1, T_2\}$ is a Steiner $(k, 2)$ trade, $k \ge 8$ and $m(T) \le 3k-4$. Then each block of T_1 contains an element which occurs in exactly two blocks.*

Proof First note that if $r_\alpha = 3$ for some $\alpha \in F(T_1)$ then by Lemma 9 $m(T) \ge 3k-3$ which is a contradiction. Now consider the block $\{a_1, a_2, a_3, \ldots, a_k\} \in T_1$. If $r_{a_i} > 3$ for all $1 \le i \le k$ then since each element $a_1, a_2, a_3, \ldots, a_k$ occurs at least four times in the blocks of T_1 and no pair of these elements can occur in more than one block of T_1 it follows that $m(T) \ge 3k+1$. This is also a contradiction. So each block of T_1 contains an element which occurs exactly in two blocks. □

Theorem 11 *Let m be odd, $k \ge 8$ and $2k+3 \le m \le 3k-4$. Then there does not exist a Steiner $(k, 2)$ trade of volume m.*

Proof Let $T = \{T_1, T_2\}$ be a Steiner $(k, 2)$ trade of volume m. By Lemma 9, $r_\alpha \neq 3$ for all $\alpha \in F(T_1)$. By Lemma 10, there exists an element $\alpha \in F(T_1)$ with $r_\alpha = 2$. Now since for $k > 7$ we have $3k - 4 < 4k - 10$, by Lemma 6 there exist distinct elements x and y in $F(T_1)$ such that at least $k - 1$ blocks of T_1 contain x but not y, and at least $k - 1$ blocks of T_1 contain y but not x, and a block from each of these collections contains α.
(i) If there is a block B in T_1 with $x, y \in B$, then we need at least $(k-1) + (k-1) + (k-2) + 1 = 3k - 3$ blocks in T_1, since by Lemma 5 $r_\beta \geq 2$ for all $\beta \in F(T_1)$, which is a contradiction. So there is no block in T_1 containing both x and y.
(ii) If each block of T_1 contains either x or y then $r_x + r_y \geq m(T)$. Now by Lemma 7 $r_x = r_y = m(T)/2$ which is impossible since m is odd. So there is a block in T_1 which contains neither x nor y.
(iii) Let $B \in T_1$ and $x, y \notin B$. By Lemma 10 there is an element $\gamma \in B$ with $r_\gamma = 2$. Now since $m(T) < 4k - 10$ by Lemma 6 there exist distinct elements z and w in $F(T_1)$ such that at least $k - 1$ blocks of T_1 contain z but not w, and at least $k - 1$ blocks of T_1 contain w but not z, and a block from each of these collections contains γ. If $\{x, y\} \cap \{z, w\} = \emptyset$ then

$$m(T) \geq (k-1) + (k-1) + (k-3) + (k-3) > 3k - 4 \text{ for } k \geq 8.$$

So without loss of generality we can assume $\{x, y\} \cap \{z, w\} = \{x\}$, say $x = w$. By (i) the pair $\{x, z\}$ cannot appear in any block of T_1. If the pair $\{y, z\}$ does not appear in any block of T_1 then

$$m(T) \geq (k-1) + (k-1) + (k-1) = 3k - 3 > 3k - 4.$$

So let $\{y, z\}$ appears in a block of T_1. This forces that $m(T) = 3k - 4$ (so k is odd), $r_x = r_y = r_z = k - 1$ and for any block $A \in T_1$ we have $A \cap \{x, y, z\} \neq \emptyset$. Now for any element $\delta \in F(T_1) \setminus \{x, y, z\}$ we must have $r_\delta = 2$. Therefore

$$k(3k-4) = (k-1) + (k-1) + (k-1) + 2(|F(T_1)| - 3).$$

This is also impossible since left hand side is odd and right hand side is even. This completes the proof. □

4 Non-existence of a Steiner $(7, 2)$ trade of volume 17

In this section we prove that there does not exist a Steiner $(7, 2)$ trade of volume 17. So by [3] and [4] $S(7, 2) = \{0, 12, 14, 15, 18, 19, 20, \ldots\}$.

Lemma 12 *Let $T = \{T_1, T_2\}$ be a Steiner $(7, 2)$ trade with $m(T) = 17$. If there exists an $\alpha \in F(T_1)$ with $r_\alpha = 3$ then $r_x \geq 3$ for all $x \in F(T_1)$.*

Proof Let B_1, B_2 and B_3 be the three blocks in T_1 which contain the element α and let C_1, C_2 and C_3 be the three blocks in T_2 which contain the element α.

Note that $B_1 \cup B_2 \cup B_3 = C_1 \cup C_2 \cup C_3$. Let X_{ij}, x_{ij} and P_i for $1 \le i,j \le 3$ be defined as in Lemma 9. So we have $\sum_{i=1}^{3} x_{ij} = 6$ and $\sum_{j=1}^{3} x_{ij} = 6$. Applying Lemma 8 and the fact $m(T) = 17$ forces $x_{ij} = 2$ for $1 \le i,j \le 3$. This implies that the blocks in T_1 have one of the two structures as shown in Table 2. Note that for both structures $C_1 = \{\alpha, 1, 2, 7, 8, 13, 14\}$, $C_2 = \{\alpha, 3, 4, 9, 10, 15, 16\}$, and $C_3 = \{\alpha, 5, 6, 11, 12, 17, 18\}$. Moreover $r_x \ge 3$ for $x \in \{1, 2, 3, \ldots, 18\}$.

	T_1 (Structure 1)		T_1 (Structure 2)
B_1 :	$\{\alpha,1,2,3,4,5,6\}$	B_1 :	$\{\alpha,1,2,3,4,5,6\}$
B_2 :	$\{\alpha,7,8,9,10,11,12\}$	B_2 :	$\{\alpha,7,8,9,10,11,12\}$
B_3 :	$\{\alpha,13,14,15,16,17,18\}$	B_3 :	$\{\alpha,13,14,15,16,17,18\}$
B_4 :	$\{1,7,13,*,*,*,*\}$	B_4 :	$\{1,7,*,*,*,*,*\}$
B_5 :	$\{1,8,14,*,*,*,*\}$	B_5 :	$\{1,13,*,*,*,*,*\}$
B_6 :	$\{2,7,14,*,*,*,*\}$	B_6 :	$\{7,13,*,*,*,*,*\}$
B_7 :	$\{2,8,13,*,*,*,*\}$	B_7 :	$\{1,8,14,*,*,*,*\}$
B_8 :	$\{3,9,15,*,*,*,*\}$	B_8 :	$\{2,7,14,*,*,*,*\}$
B_9 :	$\{3,10,16,*,*,*,*\}$	B_9 :	$\{2,8,13,*,*,*,*\}$
B_{10} :	$\{4,9,16,*,*,*,*\}$	B_{10} :	$\{3,9,15,*,*,*,*\}$
B_{11} :	$\{4,10,15,*,*,*,*\}$	B_{11} :	$\{3,10,16,*,*,*,*\}$
B_{12} :	$\{5,11,17,*,*,*,*\}$	B_{12} :	$\{4,9,16,*,*,*,*\}$
B_{13} :	$\{5,12,18,*,*,*,*\}$	B_{13} :	$\{4,10,15,*,*,*,*\}$
B_{14} :	$\{6,11,18,*,*,*,*\}$	B_{14} :	$\{5,11,17,*,*,*,*\}$
B_{15} :	$\{6,12,17,*,*,*,*\}$	B_{15} :	$\{5,12,18,*,*,*,*\}$
B_{16} :	$\{*,*,*,*,*,*,*\}$	B_{16} :	$\{6,11,18,*,*,*,*\}$
B_{17} :	$\{*,*,*,*,*,*,*\}$	B_{17} :	$\{6,12,17,*,*,*,*\}$

Table 2

Case 1 Let the blocks in T_1 have Structure 1.

(i) If a block contains an element which occurs exactly in three blocks it cannot contain an element which occurs in more than five blocks.

(ii) There are at least five elements in B_1 which occur in exactly three blocks.

Case 2 Let the blocks in T_1 have Structure 2.

(i) If a block contains an element which occurs exactly in three blocks it cannot contain an element which occurs in more than four blocks.

(ii) There are at least five elements in B_1 which occur in exactly three blocks.

Now let $\alpha, \beta, \gamma \in F(T_1)$ with $r_\alpha = r_\beta = r_\gamma = 3$. Then we need at least six blocks in T_1 for these three elements. Now let $r_\delta = 2$ for some $\delta \in F(T_1)$. Since $m(T) = 17 < 4.7 - 10 = 18$ by Lemma 6 there exist (distinct) elements x and y in $F(T_1)$ such that at least 6 blocks of T_1 contain x but not y, and at least 6 blocks of T_1 contain y but not x. So by (i) $m(T) \geq 6+6+6$ which is a contradiction. Therefore, if $r_\alpha = 3$ for some $\alpha \in F(T_1)$ then $r_x \geq 3$ for all $x \in F(T_1)$. This completes the proof. □

In a similar manner to Theorem 11 we prove the following lemma.

Lemma 13 *Let $T = \{T_1, T_2\}$ be a Steiner $(7,2)$ trade with $m(T) = 17$. Then $r_x \geq 3$ for all $x \in F(T_1)$.*

Proof Let $r_\alpha = 2$ for some $\alpha \in F(T)$. Since $m(T) = 17 < 18 = 4.7 - 10$ then by Lemma 6 there exist (distinct) elements x and y in $F(T_1)$ such that at least 6 blocks of T_1 contain x but not y, and at least 6 blocks of T_1 contain y but not x.
(i) If there is a block B in T_1 with $x, y \in B$ then since by Lemma 5 $r_\alpha \geq 2$ for all $\alpha \in F(T_1)$ we need at least $6+6+5+1 = 18$ blocks in T_1 which is impossible. So there is no block in T_1 containing both x and y.
(ii) If each block of T_1 contains either x or y then $r_x + r_y \geq m(T)$. Now by Lemma 7 $r_x = r_y = 17/2$ which is impossible. So there is a block in T_1 which contains neither x nor y.
(iii) Let $B \in T_1$ and $x, y \notin B$. If each element of B occurs in more then three blocks then $m(T) \geq 3.7 + 1 = 22$ which is impossible. Moreover, by Lemma 12 and the fact that $F(T_1)$ has an element which occurs in exactly two blocks, B contains no element which occurs in exactly three blocks. Therefore there is an element $\beta \in B$ with $r_\beta = 2$. So by Lemma 6 there exist (distinct) elements z and w in $F(T_1)$ such that at least 6 blocks of T_1 contain z but not w, and at least 6 blocks of T_1 contain w but not z. If $\{x, y\} \cap \{z, w\} = \emptyset$ then $m(T) \geq 6+6+4+4 = 20$. So without loss of generality we can assume $\{x, y\} \cap \{z, w\} = \{x\}$, say $x = w$. By (i) the pair $\{x, z\}$ cannot appear in any block of T_1. If the pair $\{y, z\}$ does not appear in any block of T_1 then

$$m(T) \geq 6+6+6 = 18 > 17.$$

So let $\{y, z\}$ appears in a block of T_1. Since $m(T) = 17$ we have $r_x = r_y = r_z = 6$ and for any block $A \in T_1$ we have $A \cap \{x, y, z\} \neq \emptyset$. Now for any element $\gamma \in F(T_1) \setminus \{x, y, z\}$ we must have $r_\gamma = 2$. Therefore

$$7.17 = 119 = 6+6+6+2(|F(T_1)| - 3).$$

This is also impossible since left hand side is odd and right hand side is even. This completes the proof. □

The proof of the following lemma is left for the reader.

Lemma 14 *Let $T = \{T_1, T_2\}$ be a Steiner $(7,2)$ trade with $m(T) = 17$.*
(i) Any block of T_1 contains at most two elements which occur in more than three blocks.
(ii) If a block $B \in T_1$ contains two elements which occur in more than three blocks then $A \cap B \neq \emptyset$ for any block $A \in T_1$.

Theorem 15 *There does not exist a Steiner $(7,2)$ trade with volume 17.*

Proof Let $T = \{T_1, T_2\}$ be a Steiner $(7,2)$ trade with $m(T) = 17$. Then by Lemmas 12 and 13 the blocks of T_1 have one the two structures as shown in Table 2.

Case 1 Let T_1 have Structure 1 as shown in Table 2. By Lemma 14 part (i), $|B_{16} \cap \{1, 2, 3, \ldots, 18\}| = i$, where $0 \leq i \leq 2$. It is straightforward to check that if $i = 0, 1$ then $m(T) > 17$. If $i = 2$ then by Lemma 14 part (ii), $B_{16} \cap B_j \neq \emptyset$ for $1 \leq j \leq 17$. This is impossible since $B_1 \cap B_2 \cap B_3 = \{\alpha\}$.

Case 2 Let T_1 have Structure 2 as shown in Table 2. By Lemma 14 part (i) $B_4 \cap \{1, 2, 3, \ldots, 18\} = \{1, 7\}$. So $B_4 \cap B_j \neq \emptyset$ for $1 \leq j \leq 17$. This is impossible since $B_3 \cap B_4 = \emptyset$. □

5 Conclusion

Here we summarize the results on the spectrum of Steiner $(k, 2)$ trades.

Theorem 16 *There exists a Steiner $(k, 2)$ trade of volume m if and only if*

(1) $m = 0$;

(2) $m \geq 3k - 3$;

(3) *m is even and* $2k - 2 \leq m \leq 3k - 4$; *or*

(4) $m = 2k + 1$ *when* $k \in \{3, 4, 7\}$.

The result of this paper contributes to the understanding of Steiner 2-designs for (at least) the following reasons.

(1) It lays the foundation for solving the intersection problem for these designs, which heretofore has only been solved for block sizes 3 and 4. (See [1] for a survey on intersection problem.)

(2) It aids in the investigation of defining sets (see [8]) for these designs. A defining set must meet every trade; so every bit of information about possible trades gives another small step towards understanding the seemingly unfathomable secrets of defining sets.

(3) In practical applications of design theory (e.g. design of experiments), many appropriate designs can be found. As the experiment progresses, an additional constraint on the design might surface. Do we have to scrap it all and start over with a new design, or can we just wiggle the existing design a bit, so as to satisfy the new constraint, and only have to repeat some of the trials? Exactly what is needed here is a small trade. Our results here indicate just what kind of small trades are possible.

Open problem. What is the spectrum of Steiner $(k, 3)$ trades?

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