# Equitable Partial Cycle Systems

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**Abstract**

In this paper, we give necessary and sufficient conditions for the existence of equitable partial 4-cycle and 5-cycle systems. Furthermore, we construct equitable partial 4-cycle and 6-cycle systems of $K_{n,n}$.

# 1 Introduction

A *(partial) m-cycle system* of order $n$ is an ordered pair $(V, C)$, where $V$ is a set of $n$ vertices and $C$ is a collection of $m$-cycles defined on $V$ such that every pair of vertices in $V$ is adjacent in exactly (at most) one $m$-cycle of $C$. In graph theoretical terms, a (partial) $m$-cycle system is a decomposition of (a subset of) the edges of $K_n$ into $m$-cycles. We define a *(partial) bipartite m-cycle system* of order $n$ to be a partition of (a subset of) the edges of $K_{a,n-a}$ into $m$-cycles.

Let $c(i)$ denote the number of $m$-cycles which contain a vertex $i \in V$. A partial $m$-cycle system is said to be *equitable* if $|c(i) - c(j)| \leq 1$, for all $i, j \in V$.

The *leave* of an $m$-cycle system $(V, C)$ of order $n$ is the graph on $n$ vertices which contains the edges of $K_n$ that are not found in any $m$-cycle of $C$. A *maximum packing* of $K_n$ with $m$-cycles is a (partial) $m$-cycle system whose leave contains the fewest number of edges possible. Let $M(m, n)$ denote the number of $m$-cycles in a maximum packing of $K_n$. It has been shown by Schönheim and Bialostocki [7] that

$$M(4,n) = \begin{cases} \lfloor \frac{n}{4} \lfloor \frac{n-1}{2} \rfloor \rfloor & \text{if } n \not\equiv 5 \text{ or } 7 \pmod 8 \\ \lfloor \frac{n}{4} \lfloor \frac{n-1}{2} \rfloor \rfloor - 1 & \text{otherwise.} \end{cases}$$

Furthermore, it has been shown by Rosa and Znám [6] that

$$M(5,n) = \begin{cases} (n^2 - n - \delta_n)/10 & \text{if } n \text{ is odd,} \\ (n^2 - 2n - \delta_n)/10 & \text{if } n \text{ is even,} \end{cases}$$

where

$$\delta_n = \begin{cases} 0 & \text{if } n \equiv 0,1,2,5 \pmod{10}, \\ 4 & \text{if } n \equiv 6 \pmod{10}, \\ 6 & \text{if } n \equiv 3 \pmod{10}, \\ 8 & \text{if } n \equiv 4,8 \pmod{10}, \\ 12 & \text{if } n \equiv 7,9 \pmod{10}. \end{cases}$$

Andersen, Hilton, and Mendelsohn considered equitable partial Steiner triple systems of order $n$ (EPSTS($n$)s) [1]. They found that if there exists a PSTS($n$) with $t$ triples, then there exists an EPSTS($n$) with $t$ triples. In particular, EPSTS($n$)s with $t$ triples exist when $1 \leq t \leq \mu(n)$, where $\mu(n)$ denotes the number of triples in a maximum PSTS($n$). This result was crucial in obtaining the smallest known embedding for partial Steiner triple systems. Subsequently, Rodger and Stubbs [5] generalized the result in [1] for partial triple systems of all indices. They found that if there exists a partial triple system of order $n$ and index $\lambda$ (or PTS($n, \lambda$)) with $t$ triples, then there exists an equitable PTS($n, \lambda$) with $t$ triples. Recently, Raines and Rodger [4] found necessary and sufficient conditions for the existence of EPSTS($n$)s whose leave contains a particular matching. This result was crucial in obtaining small embeddings of partial extended triple systems and partial totally symmetric quasigroups [2, 3]. In this paper we extend the result in [1] for 4-cycle and 5-cycle systems and include results for equitable partial bipartite 4-cycle and 6-cycle systems of $K_{n,n}$.

# 2 The Constructions

We start by generalizing the result in [1] for partial 4-cycle systems.

**Theorem 2.1** *Let $n$ and $t$ be positive integers where $1 \leq t \leq M(4,n)$. If there exists a partial 4-cycle system of order $n$ which contains $t$ cycles, then there exists an equitable partial 4-cycle system of order $n$ which contains $t$ cycles.*

**Proof:** Let $(V, C)$ be a partial 4-cycle system of order $n$ which contains $t$ cycles, and let $c(i)$ denote the number of 4-cycles which contain a particular vertex $i \in V$. If $(V, C)$ is equitable, then there is nothing to prove, so we assume that $c(1) \leq c(2) \leq \cdots \leq c(n)$ and $c(1) \leq c(n) - 2$. Let $C'$ denote the set of 4-cycles of $C$ which do not contain both vertices 1 and $n$. Form a graph $G$ on the vertex set $\{2, \ldots, n-1\}$ in which: for every cycle of the form $1, y_1, y_2, y_3, 1 \in C'$, let $G$ contain the 2-path $y_1, y_2, y_3$, and color the edges of this 2-path with color 1; and for every cycle of the form $n, z_1, z_2, z_3, n \in C'$, let $G$ contain the 2-path $z_1, z_2, z_3$, and color the edges of this 2-path with the color $n$. Since no cycle in $C'$ contains both vertices 1 and $n$, there are no edges in $G$ corresponding to 4-cycles that contain both vertices 1 and $n$. Now the number of 2-paths colored $n$ exceeds the number colored 1 by at least 2. Therefore, there must be at least two trails $T_1 = \alpha_1, \alpha_2, \ldots, \alpha_{4k+3}$ and $T_2 = \beta_1, \beta_2, \ldots, \beta_{4m+3}$ of 2-paths, alternately colored 1 and $n$, which start and end with a 2-path whose edges are colored $n$ (since $T_1$ and $T_2$ are maximal, it follows that the edges $1\alpha_1$ and $1\alpha_{4k+3}$ occur in no cycle in $C'$). Notice that the first and last vertices in each 2-path in $T_1$ and the first and last vertices in each 2-path in $T_2$ are adjacent to an edge colored $n$. Now each vertex $\alpha_i \in T_1$ and each vertex $\beta_j \in T_2$ is the beginning or end of at most one 2-path of each color; otherwise, the edge $1\alpha_i$ or the edge $n\alpha_i$ would be included in more than one 4-cycle. Similarly, we can show that the edges $1\beta_j$ and $n\beta_j$ would be included in more than one 4-cycle (though it is possible that a vertex can be the middle vertex of more than one 2-path).

The edges $1\alpha_1$ and $1\alpha_{4k+3}$ occur in no cycle in $C'$ since $\alpha_1$ and $\alpha_{4k+3}$ are vertices which are the beginning or end of no 2-paths colored 1. However, the possibility remains that the edge $1\alpha_1$ is contained in some cycle of $C \setminus C'$. If such a cycle exists, it must be of the form $1, \alpha_1, \gamma, n, 1$, where $\gamma$ is either a middle vertex of some 2-path in $T_1$ or not in $T_1$ at all. Vertex $\gamma$ cannot be the beginning vertex of some 2-path in $T_1$ since this would mean that the edge $n\gamma$ would appear in more than one 4-cycle (recall that each vertex that is the beginning of some 2-path in $T_1$ is adjacent to an edge colored $n$). In addition, there may be any number of cycles in $C \setminus C'$ of the form $1, x, n, y, 1$, where each vertex $x$ and $y$ will either be a middle vertex of some 2-path in $T_1$ or not in $T_1$ at all. Certainly, neither $x$ nor $y$ can be the beginning vertices of any 2-path in $T_1$.

Assume, without loss of generality, that $C \setminus C'$ contains some cycle $1, \alpha_1, \gamma, n, 1$. We first observe that the edges $1\beta_1$ and $1\beta_{4m+3}$ are contained in no 4-cycle of $C'$, so now we must show that the edge $1\beta_1$ is contained in no 4-cycle of $C \setminus C'$ (a similar argument shows that the edge $1\beta_{4m+3}$ is contained in no 4-cycle of $C \setminus C'$). If the edge $1\beta_1$ is contained in a 4-cycle of $C \setminus C'$, then this cycle must be of the form $1, \beta_1, z, n, 1$ (since the edge $n\beta_1$ is already contained in some 4-cycle of $C'$), but this implies that $\beta_1 = \alpha_1$ and $z = \gamma$ since the edge $1n$ is contained in at most one 4-cycle, namely $1, \alpha_1, \gamma, n, 1$. However, this implies that $T_1 = T_2$ since any vertex is the beginning or end of at most one 2-path of a particular color, but we have assumed that $T_1 \neq T_2$. Therefore, there is no edge of the form $1\beta_1$ or $1\beta_{4m+3}$ in any cycle of $C$. Now we define a new partial 4-cycle system $(V, C^*)$ in which $C^* = C \setminus (\{n, \beta_1, \beta_2, \beta_3, n\} \cup \{n, \beta_{4i+1}, \beta_{4i+2}, \beta_{4i+3}, n\} \cup \{1, \beta_{4i-1}, \beta_{4i}, \beta_{4i+1}, 1\}) \cup (\{1, \beta_{4i+1}, \beta_{4i+2}, \beta_{4i+3}, 1\} \cup \{n, \beta_{4i-1}, \beta_{4i}, \beta_{4i+1}, n\} \cup \{1, \beta_1, \beta_2, \beta_3, 1\})$, for $1 \leq i \leq m$. Let $c^*(i)$ denote the number of cycles in $C^*$ which contain a vertex $i \in V$. Certainly, $c^*(1) = c(1) + 1$, $c^*(n) = c(n) - 1$, and $c^*(j) = c(j)$, for $2 \leq j \leq n-1$. Repetition of this process among all pairs of vertices produces the desired equitable partial 4-cycle system. □

We can also use Theorem 2.1 to prove the following theorems.

**Corollary 2.2** *Let $n$ and $t$ be positive integers. An equitable partial 4-cycle system* *of order $n$ containing $t$ 4-cycles exists if and only if* $1 \le t \le M(4,n)$.

**Proof:** Certainly, since $M(4,n)$ is the number of 4-cycles in a maximum packing of $K_n$ with 4-cycles, there cannot exist an equitable partial 4-cycle system with more than $M(4,n)$ 4-cycles, so $1 \le t \le M(4,n)$. Furthermore, there exists a partial 4-cycle system of order $n$ which contains $t$ cycles for all values of $t$ between 1 and $M(4,n)$ since we can always start with a partial 4-cycle system with $M(4,n)$ 4-cycles and arbitrarily throw out $M(4,n)-t$ of these 4-cycles to form a partial 4-cycle system of order $n$ which contains $t$ cycles. So by Theorem 2.1 there exists an equitable partial 4-cycle system of order $n$ which contains $t$ 4-cycles if $1 \le t \le M(4,n)$. □

Theorem 2.1 also allows us to obtain a nice construction of equitable partial bipartite 4-cycle systems of $K_{n,n}$.

**Corollary 2.3** *Let $n$ and $t$ be positive integers. If there exists a partial bipartite 4-cycle system of $K_{n,n}$ which contains $t$ cycles, then there exists an equitable partial bipartite 4-cycle system of $K_{n,n}$ which contains $t$ cycles.*

**Proof:** Let $(V,C)$ be a partial 4-cycle system of order $2n$ with $t$ cycles on $K_{n,n}$, and again let $c(i)$ denote the number of 4-cycles which contain a particular vertex $i \in V$. We assume that $(V,C)$ is not equitable. Suppose that the sets $X = \{1,\dots,n\}$ and $Y = \{n+1,\dots,2n\}$ form the partition of the vertices of $K_{n,n}$. Apply the technique in the proof of Theorem 2.1 separately to vertices in $X$ until $|c(i)-c(j)| \le 1$, for all $i,j \in X$. Then apply the same technique to vertices in $Y$ until $|c(i)-c(j)| \le 1$, for all $i,j \in Y$. Clearly, this gives an equitable partial bipartite 4-cycle system of $K_{n,n}$ which contains $t$ cycles. □

Now that we have constructed equitable partial 4-cycle systems and equitable partial bipartite 4-cycle systems, we proceed by considering equitable partial 5-cycle systems and equitable partial bipartite 6-cycle systems. We have the following theorem regarding equitable partial 5-cycle systems.

**Theorem 2.4** *Let $n$ and $t$ be positive integers. If there exists a partial 5-cycle system of order $n$ which contains $t$ cycles, then there exists an equitable partial 5-cycle system of order $n$ which contains $t$ cycles.*

**Proof:** Let $(V,C)$ be a partial 5-cycle system of order $n$ which contains $t$ cycles, and let $c(i)$ denote the number of 5-cycles of $C$ which contain vertex $i \in V$. We suppose, without loss of generality, that $c(1) \le c(2) \le \cdots \le c(n)$. If $c(n)-c(1) \le 1$, then there is nothing to prove, so we assume that $c(n) \ge c(1)+2$.

The first goal is to form a partial 5-cycle system $(V,C')$ which contains $t$ cycles and in which $|c'(n)-c'(1)| \le 1$ (where $c'(i)$ denotes the number of 5-cycles in $C'$ which contain vertex $i \in V$). We do this by first forming a multigraph $G$ with vertex set $V(G) = V \setminus \{1,n\}$. For each 5-cycle of the form $1, x_1, x_2, x_3, x_4, 1$, where $x_1 \ne n$ and $x_4 \ne n$, we form the edge $x_1x_4$ in $G$ and color it with the color 1. For each 5-cycle of the form $n, y_1, y_2, y_3, y_4, n$, where $y_1 \ne 1$ and $y_4 \ne 1$, we form the edge $y_1y_4$ in $G$ and color it with the color $n$. So in $G$, a particular pair $i$ and $j$ of vertices is joined by at most one edge colored 1 and by at most one edge colored $n$ because the

2-paths $i, 1, j$ and $i, n, j$ occur in at most one 5-cycle of $C$. Furthermore, since the edge $1i$ (resp. $ni$) occurs in at most one cycle of $C$, it follows that there is at most one edge colored 1 (resp. colored $n$) incident with each vertex $i$ in $G$. Therefore, each vertex in $G$ has degree at most 2, and the edges of $G$ are properly 2-edge-colored. Now $C$ may contain a cycle which contains the edge $1n$, say $1, n, z_1, z_2, z_3, 1$. If such a cycle exists, then there can be no edge in $G$ colored 1 that is incident with $z_3$, and there can be no edge in $G$ colored $n$ that is incident with $z_1$. Otherwise, $C$ would contain at least one of the edges $nz_1$ and $1z_3$ in more than one cycle..

Consider the components of $G$. Each vertex of $G$ has degree at most 2, and $G$ is properly 2-edge-colored, so each component must either be a doubled edge, an even cycle, or a path. Since $c(n) \geq c(1) + 2$, the number of edges colored $n$ exceeds the number of edges colored 1 by at least 2. Therefore, $G$ must contain at least two paths of odd length which start and end with edges colored $n$. Suppose $C$ contains the cycle $1, n, z_1, z_2, z_3, 1$. Then one of the paths of odd length may contain $z_3$ as an endpoint, but at least one path does not. Notice also that any path which starts and ends with an edge colored $n$ and which does not contain $z_3$ as an endpoint does not contain $z_1$, for each vertex along the path is incident with an edge colored $n$. Select such a path $P = \alpha_1, \alpha_2, \ldots, \alpha_{2k}$. Switch colors along the edges of $P$ so that it now starts and ends with an edge colored 1. We use this new coloring of $G$ to define a new partial 5-cycle system which contains $t$ cycles.

We trace back the edges of the path $P$ to the cycles of $C$ from which they came. We form a new partial 5-cycle system $(V, C')$ from $(V, C)$ by modifying the cycles from which we produce the path $P$. On each of these cycles we perform a cycle switch by replacing 1 with $n$ and $n$ with 1 (see Figure 1). Notice that if $n$ and 1 both appear in some cycle, say $1, a, n, b, c, 1$, then $a$ is adjacent to both $b$ and $c$ in $G$, so if $a$ is in the recolored path, then so are $b$ and $c$. Hence our coloring does not duplicate vertices in any cycle. We replace the cycle $1, a, n, b, c, 1$ with the cycle $n, a, 1, b, c, n$ (see Figure 2). After having performed the necessary cycle switches, the number of cycles containing vertex 1 will increase by one, and the number of cycles containing $n$ will decrease by one. Each other vertex is contained in exactly the same number of cycles.

For $2 \leq i \leq n-1$, $c'(i) = c(i)$, $c'(1) = c(i) + 1$ and $c'(n) = c(n) - 1$. Repetition of this process yields a partial 5-cycle system $(V, C^*)$ with $t$ cycles in which $|c^*(1) - c^*(n)| \leq 1$ and in which $c^*(i) = c(i)$ for $2 \leq i \leq n-1$ (where $c^*(i)$ denotes the number of 5-cycles in $C^*$ which contain vertex $v \in V$). Furthermore, repetition of the process on each other pair of vertices in $V$ produces an equitable partial 5-cycle system which contains $t$ 5-cycles. □

**Corollary 2.5** *Let $n$ and $t$ be positive integers. There exists an equitable partial 5-cycle system of order $n$ which contains $t$ cycles if and only if $1 \leq t \leq M(5, n)$.*

**Proof:** Since $M(5, n)$ is the number of 5-cycles in a maximum packing of $K_n$ with 5-cycles, there cannot exist an equitable partial 5-cycle system with more than $M(5, n)$ 5-cycles, so $1 \leq t \leq M(5, n)$. Furthermore, there exists a partial 5-cycle system of order $n$ which contains $t$ cycles for all values of $t$ between 1 and $M(5, n)$

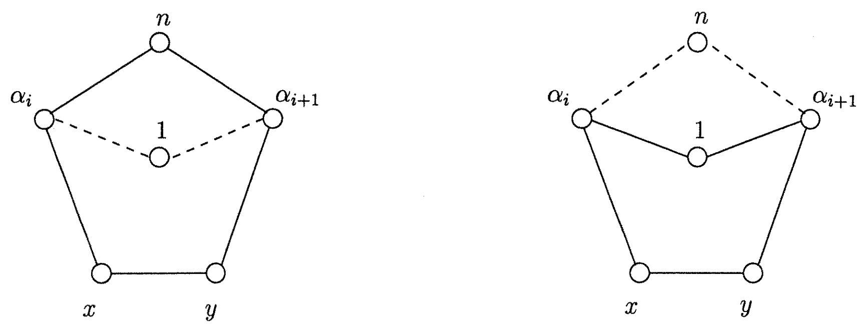


Figure 1: A cycle switch when the cycle contains exactly one of the vertices 1 and $n$

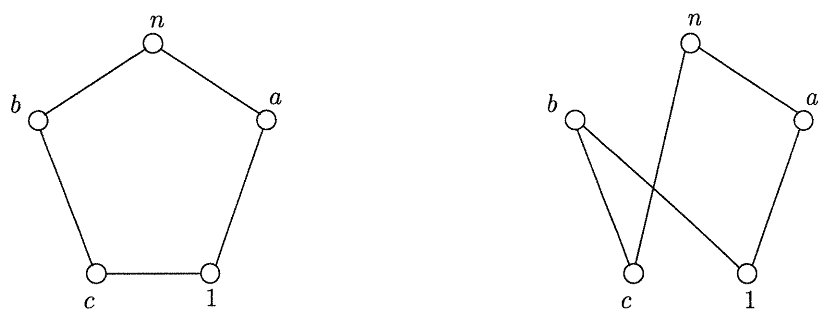


Figure 2: A cycle switch when the cycle contains both vertices 1 and $n$

since we can always start with a partial 5-cycle system with $M(5,n)$ 5-cycles and arbitrarily discard $M(5,n)-t$ of these 5-cycles to form a partial 5-cycle system of order $n$ which contains $t$ cycles. So by Theorem 2.4 there exists an equitable partial 5-cycle system of order $n$ which contains $t$ 5-cycles if $1 \leq t \leq M(5,n)$. □

Finally, we present a result concerning equitable partial bipartite 6-cycle systems.

**Theorem 2.6** *Let $n$ and $t$ be positive integers. If there exists a partial bipartite 6-cycle system of $K_{n,n}$ which contains $t$ cycles, then there exists an equitable partial bipartite 6-cycle system of $K_{n,n}$ which contains $t$ cycles.*

**Proof:** Let $(V,C)$ be a partial bipartite 6-cycle system of $K_{n,n}$ which contains $t$ 6-cycles. We assume that $(X=\{1,2,\ldots,n\}, Y=\{n+1,n+2,\ldots,2n\})$ forms the bipartition of the vertex set of $K_{n,n}$. Let $c(i)$ denote the number of 6-cycles in $C$ which contain vertex $i \in V$. We can assume without loss of generality that $c(1) \leq c(2) \leq \cdots \leq c(n)$, that $c(n+1) \leq c(n+2) \leq \cdots \leq c(2n)$, and that

$c(1) \leq c(n) + 2$. The goal is to form a partial bipartite 6-cycle system of $K_{n,n}$ such that $|c(i) - c(j)| \leq 1$ for each $i, j \in V$.

We begin by forming a partial bipartite 6-cycle system in which $|c(i) - c(j)| \leq 1$ for each $i, j \in X$. Consider the vertices 1 and $n$. Now $c(n) - c(1) \geq 2$. Form a multigraph $G$ with vertex set $V(G) = Y$. Place an edge $ab$ in $G$ if and only if $a, 1, b$ or $a, n, b$ is a 2-path contained in a cycle of $C$. If $a, 1, b$ is contained in a 6-cycle in $C$, then color the edge $ab$ in $G$ with the color 1, and if $a, n, b$ is contained in a 6-cycle, then color the edge $ab$ in $G$ with the color $n$. Observe that neither $a$ nor $b$ can be 1 or $n$, since both 1 and $n$ occur in $X$. So, in particular, the distance between vertices 1 and $n$ will never be 3 in any cycle. So this graph $G$ will have the same properties as the graph obtained when forming equitable partial 5-cycle systems. That is, edges in $G$ which are formed from cycles containing both vertices 1 and $n$ will be adjacent to each other on some path. Furthermore, the number of paths of odd length in $G$ which begin and end with edges colored 1 will be at least 2 less than the number of paths of odd length which begin and end with edges colored $n$. As before, we choose one of these paths which begin and end with an edge colored $n$ and switch the colors. Subsequently, we perform the appropriate cycle switches. Repetition of this process on all pairs of vertices in $X$ and then on all pairs of vertices in $Y$ gives the desired partial bipartite 6-cycle system.

## 3 Open Questions

It would be interesting to know if equitable cycle systems exist whenever non-equitable systems exist. In particular, is there a method other than construction to show the existence of equitable cycle systems with cycle length 6 or more? An affirmative answer for this question could also settle the existence of equitable bipartite systems, but it is possible that their existence could be shown independently.

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