Path decompositions and perfect path double covers

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Abstract

We consider edge-decompositions of regular graphs into isomorphic paths. An m-PPD (perfect path decomposition) is a decomposition of a graph into paths of length m such that every vertex is an end of exactly two paths. An m-PPDC (perfect path double cover) is a covering of the edges by paths of length m such that every edge is covered exactly two times and every vertex is an end of exactly two paths of the covering.

We show that if $m \leq 2g - 3$ then:

(1) every $2m$-regular graph G of girth g has an m-PPD,

(2) for even m, every m-regular bipartite graph G of girth g has a decompositon into paths of length m; moreover such a graph has an m-PPDC.

1 Introduction

In this paper we discuss edge-decompositions of regular graphs. Let G and H be graphs. We say that G has a *decomposition* into H if the edge set of G can be partitioned into subsets inducing subgraphs isomorphic to H. Let us denote by g the girth of G and by P_m the m-edge path.

The following conjecture was posed by Graham and Häggkvist (c.f. [3]).

Conjecture 1.1 *Let T be an m-edge tree. Every $2m$-regular graph G can be decomposed into T.*

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*While visiting L.R.I

Note that this is a far-reaching generalization of the famous Ringel's conjecture which is just the special case of the above statement for the complete graph $G = K_{2m+1}$.

Häggkvist [3] gave also a bipartite version of the conjecture (posed independently by Jacobson *et al.* [5]).

Conjecture 1.2 *Let T be an m-edge tree. Every m-regular bipartite graph G can be decomposed into T.*

Jacobson, Truszczyński and Tuza [5] verified this conjecture for every m-regular bipartite graph G of girth at least $m+1$ and for G of arbitrary girth and $T = P_4$.

A general method of attacking problems of this type (called a "packed porcupine method") was presented by Häggkvist [3]. In particular this method allows one to prove both conjectures for graphs G of girth at least m.

In this paper we deal with decompositions of regular graphs into paths. We develop a method of decomposing a $2m$-regular (respectively an m-regular bipartite) graph G into paths P_m for $m \leq 2g-3$. Our method enables us to prove Conjecture 1.1 if $T = P_m$ and $m \leq 2g-3$ and Conjecture 1.2 if $T = P_m$, $m \leq 2g-3$ and m is even.

The results of this paper are related to problems of so-called "perfect covers of graphs" (see Bondy [2]). We say that a graph G has a *perfect path double cover* (PPDC) if the edges of G can be covered with paths such that every edge of G is covered exactly two times and every vertex of G is an end of exactly two paths. Li [6] proved that every graph has a PPDC.

Bondy [2] defined an m-PPDC as a perfect path double cover where every path has m edges and he gave the following conjecture.

Conjecture 1.3 *Every m-regular graph has an m-PPDC.*

The conjecture is trivial for $m = 1, 2$. Bondy [2] has shown it for $m = 3$ and Heinrich, Hořak, Wallis and Yu [4] have verified it for $m = 4$.

It follows from the results of this paper (see Section 4) that Conjecture 1.3 is true for bipartite graphs if $g \geq \dfrac{m+3}{2}$ and m is even.

Following the terminology of Bondy [2] define an *m-perfect path decomposition* (m-PPD) to be a decomposition of a graph G into paths of length m such that every vertex of G is an end of exactly two paths. We suppose that the following statement is true.

Conjecture 1.4 *Every $2m$-regular graph has an m-PPD.*

It is easily seen that the conjecture is true for $m = 1, 2$. In this paper we show it for any $m \leq 2g-3$. In particular the conjecture is true for $m = 3$.

2 Sequences of trails

In this paper by a trail we mean a sequence $v_0e_1v_1e_2v_2...e_kv_k$ whose terms are alternatively vertices and edges such that, for $1 \leq i \leq k$, the ends of e_i are v_{i-1} and v_i, and the edges $e_1, ..., e_k$ are pairwise distinct.

Let $\mathcal{D}$ be a family of edge-disjoint trails of length m such that each trail has one terminal edge colored red and the other one violet. The graph induced by the edges is simple. The terminal vertex of the trail incident with the red edge is called *r-terminal* (resp. *v-terminal*). The other end of the red (resp. violet) edge is called the *r-preterminal* (resp. *v-preterminal*) vertex of the trail. Let $\mathcal{D}'$ be another decomposition of the graph induced by the edges of the trails in $\mathcal{D}$. We call $\mathcal{D}'$ *terminal preserving* (resp. *r-preterminal preserving*) if every vertex is the same number of times terminal (resp. r-preterminal) in $\mathcal{D}'$ as in $\mathcal{D}$. We consider each of the above mentioned trails to be oriented from the r-terminal to the v-terminal vertex.

Lemma 2.1 *Suppose G is a graph of size m and girth g. If $m \leq 2g-2$ then any two cycles in G have at least two common edges.* □

We shall assume in the sequel that $m \leq 2g-3$.

By a B_r-*trail* in $\mathcal{D}$ we mean either a cycle or a trail such that the r-terminal vertex has degree 3 in the trail and after deleting the red edge we get a path.

Let P be a B_r-trail in $\mathcal{D}$. Denote by b the r-terminal vertex of P, by v_0 the r-preterminal vertex of P, and by v_1 the neighbour of b on the cycle of P not incident to the red edge of P. Let $P_0 = P$.

A B_r-*sequence* (see Figure 1) is a sequence of paths or B_r-trails $P_0, P_1, ..., P_k$ belonging to $\mathcal{D}$ for which there is a sequence of vertices $v_0, v_1, v_2, ..., v_k$ such that

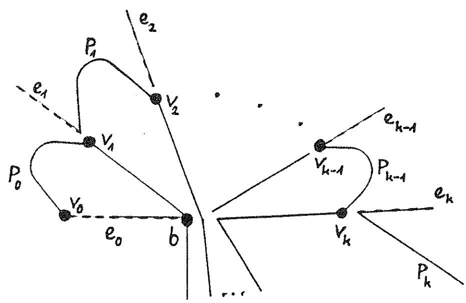


Figure 1:

(a) each of the trails $P_0, P_1, ..., P_{k-1}$ passes through b,

(a') P_k does not pass through b,

(b) v_j is the vertex preceding b on the trail P_{j-1}, for $j = 1, ..., k$,

(c) v_j is the r-preterminal vertex of P_j, for $j = 0, 1, ..., k$.

Any trail P_j, $j = 1, ..., k-1$, is said to be *internal* in the B_r-sequence and the vertex b is called the *central* vertex of the B_r-sequence.

Lemma 2.2 *If $m \leq 2g-3$ and the trails P_i, $i = 1, ..., k-1$, satisfy (a), (b) and (c), then each P_i is a path.*

Proof. Suppose P_j is a B_r-trail of length m, $0 < j < k$. The edge $v_j b$ belongs to P_{j-1} so $P_j \cup v_j b$ is a subgraph of G with $m+1$ edges in which two cycles have exactly one common edge. This is impossible by Lemma 2.1, so P_j is a path. □

Lemma 2.3 *Let G be a graph of girth g and $m \leq 2g-3$. Let $\mathcal{D}$ be a decomposition of the graph G into trails of length m and let the trails $P_0, P_1, ..., P_k \in \mathcal{D}$ form a B_r-sequence. Then*

(1) The trails $P_0, P_1, ..., P_k$ are pairwise different.

(2) The graph induced by the edges of the trails $P_0, P_1, ..., P_k$ has a decomposition into paths $P'_0, P'_1, ..., P'_k$ of length m which is terminal preserving and all r-preterminal vertices except that of P_0 are preserved.

Proof. 1. Assume that the lemma is not true. Let $P_j = P_i$ for some $i > j$ and suppose $i + j$ is as small as possible.

Case 1. $j = 0$

If $P_0 = P_i$ then obviously $i \neq 1$ (as by (b) and (c) $v_0 \neq v_1$). Suppose $i \geq 2$. Then, by the uniqueness of the r-preterminal vertices, $v_0 = v_i$, so $bv_0 = bv_i$. Since bv_0 is an edge of P_0 and bv_i is an edge of P_{i-1}, by (b) we get $P_0 = P_{i-1}$, a contradiction to the minimality of $i + j$.

Case 2. $j > 0$

Note that $v_j \neq v_i$. Otherwise, by (b), P_{i-1} and P_{j-1} have a common edge $bv_i = bv_j$ so $P_{i-1} = P_{j-1}$. We get a contradiction with the uniqueness of the r-preterminal vertex of P_i.

2. Let $e_0, e_1, ..., e_k$ be the red terminal edges of the trails $P_0, P_1, ..., P_k$, respectively. With the notation given in the definition of a B_r-sequence, let

$$P'_0 = (P_0 - bv_1) \cup e_1$$

$$P'_i = (P_i - bv_{i+1} - e_i) \cup bv_i \cup e_{i+1},$$

for $i = 1, 2, ..., k-1$ and

$$P'_k = (P_k - e_k) \cup bv_k.$$

Erase the red color from e_0, and color the edge bv_k with red (see Figure 2). Note that $\bigcup_i E(P_i) = \bigcup_i E(P'_i)$.

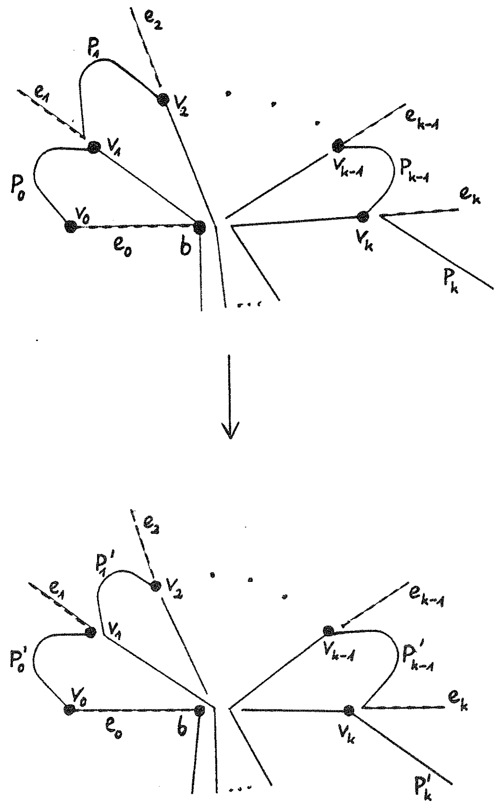


Figure 2:

Let us show that for each $i = 0, ..., k-1$, the r-terminal vertex of P_i' has degree 1 in P_i'. If the r-terminal vertex of some P_i' were of degree 3 or 2, then we would get two cycles with one common edge in the graph $P_i' \cup bv_{i+1}$, a contradiction to Lemma 2.1. For $i = k$ it follows by condition (a'). It is obvious now that $P_0', P_1', ..., P_k'$ are paths.

Suppose a vertex x is p times terminal in the family $(P_i)_i$. Then it is also p times terminal in the family $(P_i')_i$. It is clear for every vertex except b. The vertex b was terminal in P_0. If P_0 is not a cycle then b is no longer terminal in P_0' but it is terminal in P_k'. If P_0 is a cycle then b is terminal two times in P_0. In the family $(P_i')_i$, b is terminal one time in P_0' and one time in P_k'. We have shown that the family $(P_i')_i$ is terminal preserving.

It is routine to verify that all the r-preterminal vertices (except that of P_0) are preserved (i.e. are still r-preterminal). □

3 Decomposition of $2m$-regular graphs

In this section we prove the following theorem.

Theorem 3.1 *Let $m \leq 2g - 3$. Every $2m$-regular graph G_{2m} of girth g has an m-perfect path decomposition.*

To show this theorem by induction we use a decomposition into paths of $G - F$ (where F is a 4-factor) and then we add 2 edges to both ends of each path of the decomposition. We obtain a decomposition of G into trails of length m. Then we make some exchanges of edges to construct an m-perfect path decomposition.

First we need to prove several properties of B_r-sequences.

Lemma 3.1 *Let $m \leq 2g-3$. Suppose that every vertex is r-preterminal exactly once in some decomposition $\mathcal{D}$ of a graph G of girth g into m-edge paths and B_r-trails. Then each B_r-trail is the initial term of a B_r-sequence, and the sequence is unique.*

Proof. Let P_0 be a B_r-trail in $\mathcal{D}$, b the r-terminal and v_0 the r-preterminal vertex of P_0. Denote by v_1 the neighbour of b in the cycle of P_0 different from v_0. Let P_1 be the trail for which v_1 is the r-preterminal vertex.

Suppose we have already defined the trails $P_0, P_1, ..., P_i$ and the vertices $v_1, ..., v_i$ such that the conditions (a), (b) and (c) of the definition of a B_r-sequence are satisfied (for $k = i$).

Consider two cases:

- P_i does not pass through b.

Then the sequence $(P_0, ..., P_i)$ satisfies the conditions (a), (a'), (b) and (c) so it is a B_r-sequence.

- P_i passes through b.

Define v_{i+1} to be the vertex in P_i preceding b. Such a vertex is unique by Lemma 2.1. By our assumptions, v_{i+1} is r-preterminal for exactly one trail. Define P_{i+1} to be this trail. Note that (a), (b) and (c) are satisfied for $k = i + 1$. As the graph G is finite we shall finally obtain a path (say P_k) satisfying (a').

The uniqueness of the B_r-sequence is obvious. □

Lemma 3.2 *Let $m \leq 2g-3$. Suppose that every vertex is r-preterminal exactly once in some decomposition $\mathcal{D}$ of a graph G of girth g into m-edge paths and B_r-trails. Then for every trail Q in $\mathcal{D}$ there is at most one B_r-sequence in which Q is not the last term.*

Proof. If Q is a B_r-trail then, by Lemma 2.2, it is not an internal term of any B_r-sequence and by Lemma 3.1 it is an initial term for exactly one B_r-sequence. Now assume that Q is a path. Suppose there are two different sequences $S_0 =$

$(P_0, P_1, ..., P_k)$ and $S_1 = (Q_0, Q_1, ..., Q_l)$ such that P_i and Q_j are internal terms of S_0 and S_1, respectively and $P_i = Q_j$, for some $i, j \geq 1$. Assume that the pair is chosen such that i is as small as possible. By the definition of a $\mathrm{B_r}$-sequence there is an edge e in P_{i-1} (resp. an edge e' in Q_{j-1}) whose addition closes a cycle in $P_i = Q_j$. By Lemma 2.1, $e = e'$ so, as the trails in $\mathcal{D}$ are edge-disjoint, $P_{i-1} = Q_{j-1}$. If $i, j \geq 2$ then we get a contradiction with the choice of i. Otherwise $P_0 = Q_{j-1}$ or $P_{i-1} = Q_0$. By Lemmas 2.2 and 3.1, $S_0 = S_1$, a contradiction. □

Let us denote by $\mathcal{G}_\mathcal{D}$ an oriented graph whose vertices are the elements of some family $\mathcal{D}$ of edge-disjoint trails of length m and a pair (A, A') is an arc in $\mathcal{G}_\mathcal{D}$ if there exists a $\mathrm{B_r}$-sequence with $A = P_{i-1}$ and $A' = P_i$, for some $i = 1, ..., k$.

Lemma 3.2 implies the following corollary.

Corollary 3.1 *For any vertex P in $\mathcal{G}_\mathcal{D}$,*

$$d^+(P) \leq 1. \qquad \square$$

A graph satisfying the above condition is called an f-*graph* (c.f. Lipski [7]). Every component of this graph contains at most one cycle.

In the sequel we shall replace a $\mathrm{B_r}$-sequence $S_0 = (P_0, P_1, ..., P_k)$ of trails by a B_r-sequence $S_0' = (P_0', P_1', ..., P_k')$ of paths defined in the proof of Lemma 2.3. After this replacement the $\mathrm{B_r}$-sequences which intersected S_0 have to be modified.

Lemma 3.3 *Let G be a graph of girth g and let m be an integer such that $m \leq 2g-3$. Denote by $\mathcal{D}$ a decomposition of G into paths or $\mathrm{B_r}$-trails such that for every trail $Q \in \mathcal{D}$ there is at most one B_r-sequence for which Q is not the last term. Let $S_0 = (P_0, P_1, ..., P_k)$ and $S_1 = (Q_0, Q_1, ..., Q_l)$ be different B_r-sequences in $\mathcal{D}$. If $P_i = Q_l$, $k > i > 0$, then $S_2 = (Q_0, Q_1, ..., Q_{l-1}, P_{i-1}')$ is a B_r-sequence and if $P_k = Q_j$, $j > 0$, then $S_3 = (Q_0, Q_1, ..., Q_{j-1}, P_k', Q_{j+1}, ..., Q_l)$ is a B_r-sequence, where the paths P_t', $t = 0, 1, ..., k$, are defined in the proof of Lemma 2.3.*

Proof. Consider three cases (see Figure 3). Let b (resp. b') be the central vertex of the $\mathrm{B_r}$-sequence S_0 (resp. S_1).

Case 1. $P_i = Q_l$, $k > i > 0$.

By the definition of a $\mathrm{B_r}$-sequence, bv_i is an edge of P_{i-1} and $b'v_i$ an edge of Q_{l-1}. Thus $b \neq b'$ as $P_{i-1} \neq Q_{l-1}$. If b' belongs to P_{i-1}' then the graph induced by the set of edges $(E(P_{i-1}') - e_i) \cup v_i b \cup v_i b'$ contains two cycles with exactly one common edge, a contradiction with Lemma 2.1. Thus b' does not belong to P_{i-1}' so the condition (a') for the sequence S_2 is satisfied. Moreover the red edge of P_{i-1}' is incident to v_i, the r-preterminal vertex of Q_l, so the condition (c) of the definition of a $\mathrm{B_r}$-sequence is satisfied by S_2 too and consequently S_2 is a $\mathrm{B_r}$-sequence.

Case 2. $P_k = Q_l$.

As in the previous case we show that $b \neq b'$, so b' is not a vertex of P_k' because $P_k = Q_l$ does not contain b'. Hence the condition (a') is satisfied for S_3. The conditions (a), (b) and (c) are obviously satisfied.

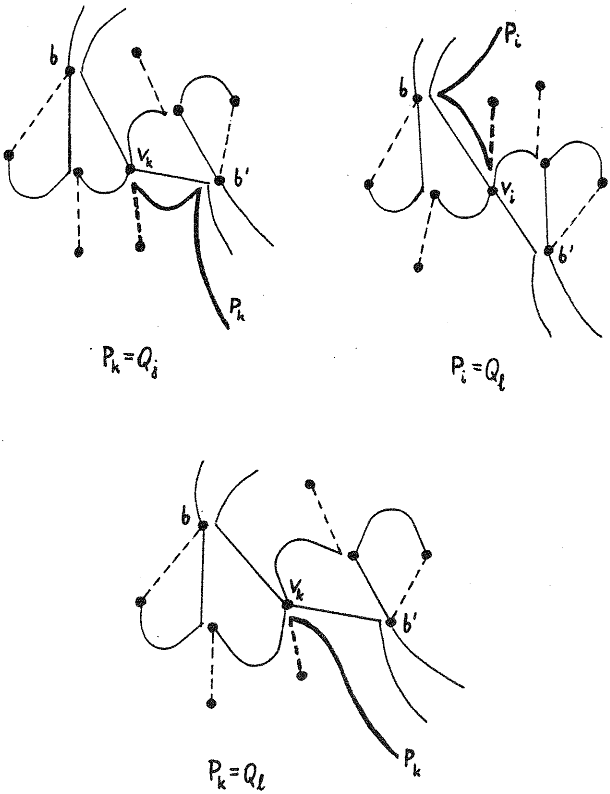


Figure 3:

Case 3. $P_k = Q_j$, $l > j > 0$.

Clearly the path P_k passes through b' but does not pass through b. Hence, $b \neq b'$. The vertex b' is not the end of e_k, the red edge of $P_k = Q_j$, because the edge $b'v_k$ (where v_k is the r-preterminal vertex of P_k) belongs to Q_{j-1}. Thus b' belongs to P'_k, so S_3 satisfies (a). The other conditions are obviously satisfied by S_3. □

Let the assumptions of Lemma 3.1 be satisfied and let P_0 be a B_r-trail. If $d^-{}_{\mathcal{G}_\mathcal{D}}(P_0) = 0$, then by applying Lemma 3.3 and replacing ths B_r-sequence S_0 starting at P_0 with the B_r-sequence S'_0, we decrease the number of the B_r-paths (and B_r-sequences). The conclusion of Lemma 3.1 still holds, but the r-preterminal vertex of P_0 is no longer r-preterminal in the new decomposition of G.

A cycle $\mathcal{C}$ in $\mathcal{G}_\mathcal{D}$ is called a *b-cycle* if $\mathcal{C}$ is the union of the terms of B_r-sequences $S_0, S_1, ..., S_p$ such that for each $i = 1, 2, ..., p$ the initial B_r-trail of S_i is equal to the terminal trail of S_{i-1} and the initial B_r-trail of S_0 is equal to the terminal trail of S_p.

Lemma 3.4 *Let G be a graph of girth g and let m be an integer such that $m \leq 2g-3$. Denote by $\mathcal{D}$ a decomposition of G into m-edge paths and B_r-trails. If*

(1) *for every trail* $Q \in \mathcal{D}$ *there is at most one* $\mathrm{B_r}$*-sequence for which* Q *is not the last term,*

(2) *each* $\mathrm{B_r}$*-trail is the initial term of some* $\mathrm{B_r}$*-sequence and*

(3) $\mathcal{G}_{\mathcal{D}}$ *contains no b-cycle*

then G *has a decomposition into paths of length* m *which is terminal preserving.*

Proof. Let us consider a terminal preserving decomposition $\mathcal{D}'$ satisfying the conditions (1), (2) and (3) with a minimum number β of $\mathrm{B_r}$-sequences. If $\beta \neq 0$ then let us consider a $\mathrm{B_r}$-sequence S_0 such that the initial term of it is not terminal of another one. It exists because there is no b-cycle in $\mathcal{G}_{\mathcal{D}'}$.We apply Lemma 3.3 to S_0 and to every sequence such that its terminal trail is internal in S_0. The family of trails $\mathcal{D}''$ obtained from $\mathcal{D}'$ by replacing the terms P_i of S_0 by the terms P_i' has less than β $\mathrm{B_r}$-sequences and satisfies (1), (2) and (3). By Lemma 2.3 $\mathcal{D}''$ is terminal preserving. We have got a contradiction with the minimality of $\mathcal{D}'$. Hence $\beta = 0$ so $\mathcal{D}'$ is a decomposition into paths. □

Lemma 3.5 *Let* G *be a graph of girth* g *and let* m *be an integer such that* $m \leq 2g-3$. *Denote by* $\mathcal{D}$ *a decomposition of* G *into* m*-edge paths and* B_r*-trails. Assume that every vertex of* G *is* r*-preterminal exactly once in the decomposition* $\mathcal{D}$. *If* $\mathcal{G}_{\mathcal{D}}$ *contains no b-cycle then* G *has a decomposition into paths of length* m *which is terminal preserving.*

Proof. The lemma follows by Lemmas 3.4, 3.2 and 3.1. □

We will also need to modify trails belonging to b-cycles. Intuitively, we shall reverse the orientation of these b-cycles.

Let $\mathcal{C}$ be a b-cycle in $\mathcal{G}_{\mathcal{D}}$ which is the union of terms of B_r-sequences $S_i = (P_0^i, ..., P_{n_i}^i)$, $i = 0, ..., p$. Denote by b_i the central vertex of S_i, by v_t^i the r-preterminal vertex of P_t^i and by e_t^i the red edge of P_t^i, $t = 0, ..., n_i$. By the definition of $\mathcal{C}$, $P_{n_i}^i = P_0^{i+1}$ and $v_{n_i}^i = v_0^{i+1}$, for $i = 0, 1, ..., p$, (where $v_0^{p+1} = v_0^0$). Let us define $\overline{P_t^i}$ as follows (see Figure 4).

We erase the red color from the edges $e_0^i = b_i v_0^i$ and color the edges $b_i v_{n_i}^i$ with red, for $i = 0, .., p$. Let

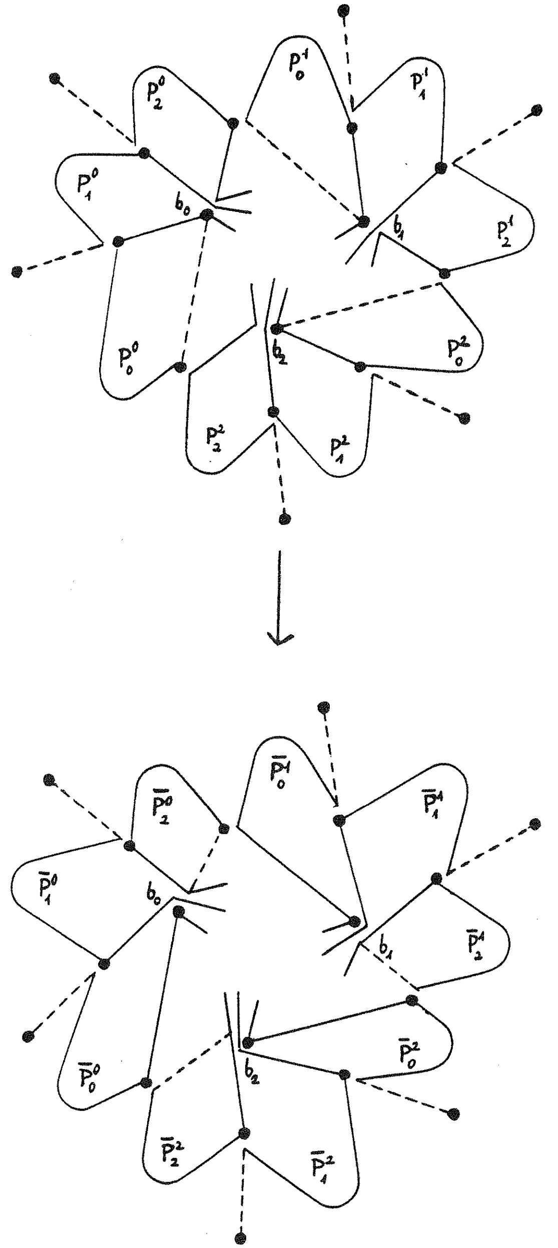
$$\overline{P_0^i} = (P_0^i - b_i v_1^i) \cup e_1^i,$$

$$\overline{P_t^i} = (P_t^i - b_i v_{t+1}^i - e_t^i) \cup b_i v_t^i \cup e_{t+1}^i,$$

for $t = 1, ..., (n_i - 2)$ and

$$\overline{P_{n_i-1}^i} = (P_{n_i-1}^i - e_{n_i-1}^i) \cup b_i v_{n_i-1}^i.$$

Denote $\overline{S_i} = (\overline{P_{n_i-1}^i}, \overline{P_{n_i-2}^i}, ..., \overline{P_0^i}, \overline{P_{n_{i-1}-1}^{i-1}})$. Let $\overline{\mathcal{D}}$ be the family of trails obtained from $\mathcal{D}$ by replacing all P_t^i's by $\overline{P_t^i}$'s.


P_0^1
P_1^1
P_2^0
P_1^0
b_0
b_1
P_2^1
P_0^0
b_2
P_0^2
P_2^2
P_1^2
$\bar{P}_0^1$
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b_0
b_1
$\bar{P}_2^1$
$\bar{P}_0^2$
$\bar{P}_0^0$
b_2
$\bar{P}_2^2$
$\bar{P}_1^2$

Figure 4:

Lemma 3.6 *Let G be a graph of girth g and let $m \leq 2g-3$. Denote by $\mathcal{D}$ a decomposition of G into trails of length m. Assume that $\mathcal{G}_{\mathcal{D}}$ contains a b-cycle C which is the union of terms of B_r-sequences S_i, $i = 0, 1, ..., p$. Then every $\overline{S}_i$ (defined above) is a B_r-sequence and $\overline{S}_0 \cup ... \cup \overline{S}_p$ is a b-cycle in $\mathcal{G}_{\overline{\mathcal{D}}}$ with the same number of B_r-trails as the b-cycle C. Moreover $\overline{\mathcal{D}}$ is terminal, r-preterminal and v-preterminal preserving.* □

We leave a routine proof of this lemma to the reader.

We shall show now the following lemma which is the induction step for the proof of Theorem 3.1.

Lemma 3.7 *Let $m \geq 3$. If every $2(m-2)$-regular graph $G_{2(m-2)}$ with girth at least $\frac{(m-2)+3}{2}$ has a decomposition into paths P_{m-2} such that every vertex is two times terminal then every $2m$-regular graph G_{2m} with girth at least $\frac{m+3}{2}$ has a decomposition $\mathcal{D}$ into paths and B_r-trails of length m such that*

(i) *there is a 2-coloring of the terminal edges of each trail such that one of the edges is red and the other one is violet,*

ii) *every vertex is two times terminal,*

(iii) *every vertex is one time r- and one time v-preterminal and*

(iv) *the graph $\mathcal{G}_{\mathcal{D}}$ does not contain b-cycles.*

Proof. Let F_1, F_2 be edge-disjoint 2-factors of G_{2m}. Color the edges of F_1 with red and the edges of F_2 with violet. The girth of the graph $G_{2(m-2)} = G_{2m} - F_1 - F_2$ is at least $\frac{m+3}{2} \geq \frac{(m-2)+3}{2}$. Decompose $G_{2(m-2)}$ into paths of length $m-2$ such that each vertex is exactly two times terminal. Call this decomposition $\mathcal{D}'$.

Assign to every vertex of G_{2m} an edge colored with red such that no edge is assigned to two different vertices. For each path P of the decomposition $\mathcal{D}'$ choose one of its ends $v(P)$, say such that no vertex is assigned to two different paths. It is possible as, by our assumption, every vertex of $G_{2(m-2)}$ is two times terminal. Extend each path by the red edge assigned to $v(P)$. Repeat this procedure for violet edges using the end of each path P different from $v(P)$.

We obtain a decomposition of G_{2m} into trails satisfying (i), (ii) and (iii) (some of them may be neither paths nor B_r-trails).

From the set of decompositions satisfying the conditions (i)-(iii) choose one minimizing the number $b_r + 2b_v$, where b_r (resp. b_v) is the number of r-terminal (resp. v-terminal) vertices of degree 2 or 3 (resp. of degree 3) in the trail in which they are r-terminal (resp. v-terminal). Denote this decomposition by $\mathcal{D}$.

Suppose there is a trail Q in $\mathcal{D}$ such that its v-terminal vertex has degree at least 3 (by Lemma 2.1 it is exactly 3) in Q. Let y be the v-preterminal vertex of Q and

yx the terminal violet edge in Q. By (iii) there exists a trail Q' in $\mathcal{D}$ such that an edge yx' is the red terminal edge of Q', where y is r-preterminal in Q'. Exchange the edges yx and yx' between Q and Q' and color yx with red and yx' with violet. We get a new decomposition of G_{2m} into trails satisfying (i)-(iii).

The trail $(Q - yx) \cup yx'$ is either a path or a B_r-trail by Lemma 2.1. If in the trail $(Q' - yx') \cup yx$ the v-terminal vertex has degree 3 then the same is true for Q'. In both cases the number $b_r + 2b_v$ in the new decomposition is smaller than in $\mathcal{D}$, contradicting to the definition of $\mathcal{D}$. Thus $\mathcal{D}$ consists of paths and $\mathrm{B_r}$-trails only. Hence $b_v = 0$.

Suppose $\mathcal{G}_\mathcal{D}$ contains a b-cycle $\mathcal{C}$. Let v_0 be the r-preterminal vertex in some $\mathrm{B_r}$-trail P_0 in the b-cycle and let b be the central vertex of the $\mathrm{B_r}$-sequence starting at P_0. Denote by Q the trail in $\mathcal{D}$ for which v_0 is v-preterminal. If Q does not pass through b then we exchange the edge v_0b with the violet edge e of Q and recolor v_0b violet and the edge e red. By Lemma 2.1 $(P_0 - v_0b) \cup e$ is a path and since Q does not pass through b, if $(Q - e) \cup v_0b$ is a $\mathrm{B_r}$-trail then Q is a $\mathrm{B_r}$-trail too. In the new decomposition the conditions (i)-(iii) are satisfied and the number $b_r + 2b_v = b_r$ is smaller, a contradiction.

Hence Q passes through b. Let b' be the central vertex of the $\mathrm{B_r}$-sequence in $\mathcal{C}$ terminating at P_0. Note that, if v_0b' belongs to Q, then necessarily Q is the trail which precedes P_0 in the b-cycle $\mathcal{C}$ and, by Lemma 2.1, it can not pass through b, a contradiction. As v_0b and v_0b' are not edges of Q, by Lemma 2.1 applied to $(Q - e) \cup v_0b \cup v_0b'$ we get $b' \notin Q$. Let $\overline{\mathcal{D}}$ be the decomposition of G obtained from $\mathcal{D}$ by substituting all the trails P_j^i belonging to $\mathcal{C}$ by the trails $\overline{P}_j^i$. By Lemma 3.6 and the definition of $\overline{\mathcal{D}}$, the decomposition $\overline{\mathcal{D}}$ satisfies the conditions (i)-(iii) and the number $b_r + 2b_v = b_r$ is the same as for $\mathcal{D}$. In $\overline{\mathcal{D}}$ the vertex v_0 is the r-preterminal vertex of some $\mathrm{B_r}$-trail $\overline{P}$ such that b' is the central vertex for the $\mathrm{B_r}$-sequence starting at $\overline{P}$. We exchange the red edge v_0b' of $\overline{P}$ with the violet edge e of Q and recolor v_0b' to violet and e to red. As in the previous paragraph, we get a contradiction. Thus $\mathcal{G}_\mathcal{D}$ does not contain a b-cycle so (iv) is satisfied. □

Proof of Theorem 3.1. We show the theorem by induction. For $m = 1$ this is trivial. For $m = 2$ it follows from the reasoning in the first two paragraphs of the proof of Lemma 3.7. Suppose the theorem is true for $m - 2$, $m \geq 3$. By Lemma 3.7 and Lemma 3.5, G_{2m} has the required decomposition. □

4 Bipartite case, m even

We will prove the following result.

Theorem 4.1 *Let m be an even positive integer, and g an integer such that $m \leq 2g - 3$. Every bipartite m-regular graph G_m of girth g with vertex classes X and Y can be decomposed into paths of length m. In this decomposition each vertex of X is exactly two times terminal and no vertex of Y is terminal.*

By a reasoning similar to that in the previous section we show the following statement (which is a bipartite analogy of Lemma 3.6).

Lemma 4.1 *Let G be a bipartite graph of girth g with vertex classes X and Y and let m be an integer such that $m \leq 2g-3$. Denote by $\mathcal{D}$ a decomposition of G into m-edge paths and $\mathrm{B_r}$-trails. Assume that every vertex of Y is r-preterminal exactly once in $\mathcal{D}$ and no vertex of X is r-preterminal. If $\mathcal{G}_{\mathcal{D}}$ has no b-cycle then the graph G has a decomposition into paths of length m which is terminal preserving.*

Sketch of proof of Lemma 4.1. The proof of this lemma is the same as the proof of Lemma 3.6 except instead of using Lemmas 3.2 and Lemma 3.3, we use their bipartite analogies: Lemmas 3.2' and Lemma 3.3'.

Lemma 3.2' *Let G be a bipartite graph of girth g with vertex classes X and Y and let m be an integer such that $m \leq 2g-3$. Suppose every vertex of Y is r-preterminal exactly once and no vertex of X is r-preterminal in some decomposition $\mathcal{D}$ of the graph G into m-edge paths and B_r-trails. Then each B_r-trail is the initial term of a B_r-sequence, and the sequence is unique.*

To proof Lemma 3.2' we proceed as in the proof of Lemma 3.2. The statement that each B_r-trail is the initial term of some B_r- sequence follows from the fact that each vertex v_i is r-preterminal. In the proof of Lemma 3.2', v_0 is r-preterminal so v_0 belongs to Y. The distance between v_0 and each v_i is equal to 2, hence v_i belongs to Y and consequently v_i is r-preterminal.

Lemma 3.3' *Let G be a bipartite graph of girth g with vertex classes X and Y and let m be an integer such that $m \leq 2g-3$. Suppose every vertex of Y is r-preterminal exactly once and no vertex of X is r-preterminal in some decomposition $\mathcal{D}$ of the graph G into m-edge paths and B_r-trails. Then for each trail Q in $\mathcal{D}$ there is at most one B_r-sequence in which Q is not the last term.*

The proof of this lemma is the same as that of Lemma 3.3 except instead of using Lemma 3.2 we use Lemma 3.2'. □

An analogy of Lemma 3.7 for the bipartite case can also be proved.

Lemma 4.2 *Let $m > 2$ be even. If every bipartite $(m-2)$-regular graph G_{m-2} with girth at least $\frac{(m-2)+3}{2}$, has a decomposition into paths P_{m-2} such that every vertex of Y is two times terminal then every bipartite m-regular graph G_m with girth at least $\frac{m+3}{2}$, has a decomposition $\mathcal{D}$ into paths and B_r-trails of length m such that*

(i) *there is a 2-coloring of the terminal edges of each trail such that one of the edges is red and the other one is violet,*

(ii) *every vertex in X is two times terminal,*

(iii) *every vertex in Y is one time r-preterminal and one time v-preterminal and*

(iv) *the graph $\mathcal{G}_{\mathcal{D}}$ does not contain b-cycles.*

Proof of Lemma 4.2. The proof is analogous to that of Lemma 3.7 except that here F_1 and F_2 are perfect matchings instead of 2-factors. We color the edges of F_1 red and the edges of F_2 in violet. We get easily the properties (i), (ii), and (iii). Then the property (iv) follows from the property (iii) by a reasoning analogous to the one applied in the proof of Lemma 3.7. □

It is easily seen now that Theorem 4.1 follows by induction from Lemma 4.2 and Lemma 4.1.

The statement in Theorem 4.1 is still true if we exchange X and Y. This way we prove the following result.

Corollary 4.1 *If $m \leq 2g - 3$ and m is even then every m-regular bipartite graph with girth g has an m-PPDC.* □

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