Classification of simple 2-(6,3) and 2-(7,3) trades*

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Abstract

In this paper, we present a complete classification of the simple 2-$(v, 3)$ trades, for $v = 6$ and 7. For $v = 6$, up to isomorphism, there are unique trades with volumes 4, 6, and 10 and trades with volumes 7-9 do not exist; for $v = 7$, up to isomorphism, there exist two trades with volume 6, two trades with volume 7, two trades with volume 9, five trades with volume 10, and only one trade with volume 12. For $v = 7$, trades with volumes 8 and 11 do not exist.

1. Introduction

Let $X = \{1, 2, \cdots, v\}$ and let $P_k(X)$ be the set of all k-subsets of X. The elements of $P_k(X)$ are usually called *blocks*. A t-(v, k, λ) design is a collection of blocks in which every element of $P_t(X)$ is contained in exactly λ blocks. A t-(v, k) *trade* T consists of two disjoint collections of the elements of $P_k(X)$, T_1 and T_2, such that every t-subset of X which appears in T_1 (T_2) appears in T_2 (T_1) with the same frequency. The set $\{x \in X | \exists B \in T, x \in B\}$ is called the *foundation* of T and is denoted by $found(T)$. To avoid any confusion, we have made the assumption that the foundation size of a 2-$(v, 3)$ trade is equal to v. From the definition of a trade, we can conclude that

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$|T_1| = |T_2|$; the number $|T_1|$ is called the *volume* of T, and is denoted by $vol(T)$. We will denote a trade T by $\{T_1, T_2\}$. A trade without repeated blocks is called a *simple trade.*

The following basic information on trades is obtained from early literature on the subject.

Theorem [2]. Let T be a t-(v, k) trade, then
(i) $|\text{found}(T)| \geq k + t + 1$,
(ii) $\text{vol}(T) \geq 2^t$.

A trade T with $|\text{found}(T)| = k + t + 1$ and $\text{vol}(T) = 2^t$ is called a *minimal trade.*

Trades are used in the following: constructing t-designs and signed t-designs; defining sets of designs; block intersection problem of designs; construction of nonisomorphic designs from a given design; and the problem of support sizes of designs [1,4,6]. Therefore, we believe that the study of existence, structure, and construction methods of trades has a great significance in combinatorial design theory.

Hwang [2] has classified t-(v, k) trades with $\text{vol}(T) = 2^t$. In [3], 2-$(v, 3)$ trades with $6 \leq \text{vol}(T) \leq 9$, in which every pair appears at most once in T_1 (T_2), have been studied. In this paper we completely classify simple 2-$(v, 3)$ trades for $v = 6$ and 7. Note that if T is a 2-$(v, 3)$ trade then clearly $\text{vol}(T) \leq \binom{v}{3}/2$. In [5], it has been shown that 2-$(7, 3)$ trades with volumes ≥ 13 do not exist.

A 2-$(6, 3)$ trade with volume 4 is a minimal trade and up to isomorphism has a unique structure. Also (see [2]) there is no 2-$(7, 3)$ trade with volume 4, and no 2-$(v, 3)$ trade with volume 5 (for any v).

The following theorems summarize the results of the paper.

Theorem 1. Up to isomorphism, there are unique 2-(6,3) trades with volumes 4, 6, and 10. Also 2-(6,3) trades with volumes 7-9 do not exist.

Theorem 2. For 2-$(7, 3)$ trades, we have the following results:

(i) Up to isomorphism, there exist two trades with volume 6; two trades with volume 7; two trades with volume 9; five trades with volume 10; and a unique trade with volume 12;

(ii) Trades with volumes 8 and 11 do not exist.

2. Some definitions and some elementary lemmas

Throughout, a trade $T = \{T_1, T_2\}$ will be a simple 2-$(v, 3)$ trade with $v \in \{6, 7\}$. For simplicity we will use $x_1 x_2 \cdots x_k$ for $\{x_1, x_2, \cdots, x_k\}$. The following notations will be adopted

$$r_x = |\{B | B \in T_1, x \in B\}|, \qquad \lambda_{xy} = |\{B | B \in T_1, xy \subset B\}|,$$
$$E(i) = \{x | x \in \text{found}(T), r_x = i\}, \quad S(x) = |\{y | y \in \text{found}(T), \lambda_{xy} = 2\}|.$$

Lemma 1. Let T be a trade and $x, y \in \text{found}(T)$. Then $2 \leq r_x \leq 6$, and $0 \leq \lambda_{xy} \leq 2$.

Proof. Since $|\{B|B \in T_1, xy \subset B\}| + |\{B|B \in T_2, xy \subset B\}|$ is even, and every pair appears in the blocks of $P_3(X)$ at most five times, we can conclude that $\lambda_{xy} \leq 2$. Consider the set $C_x = \{(y, B)|xy \subseteq B, B \in T_1\}$. We can compute $|C_x|$ in two ways. For any block B containing x, there exist two pairs in C_x. Hence $|C_x| = 2r_x$. On the other hand, for every $y \in \text{found}(T)$ and $y \neq x$, there are at most two blocks containing x and y. Hence $|C_x| \leq 2 \times 6$, and we conclude that $r_x \leq 6$. □

With each $x \in \text{found}(T)$ we associate a graph $G_x = (V, E)$, where $V = \{y \in \text{found}(T)|\lambda_{xy} \neq 0\}$ and $yz \in E \Leftrightarrow xyz \in T_1$ (T_2). Using Lemma 1 and the definition of T the possible forms for G_x are seen to be those listed in Table 1. Note that for elements of $E(2)$ there is only one possible form, H, for G_x. In all other cases G_x can take one of two possible forms. We use the notation $G^i_j, 1 \leq i \leq 2$ and $j \in \text{found}(T)$, for the graph of element j in T_i.

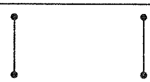
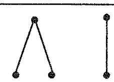
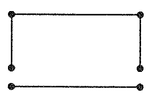
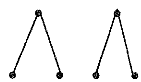
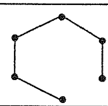
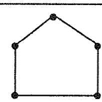
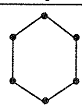
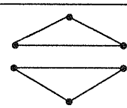
r_x	H	H'
2		
3		
4		
5		
6		

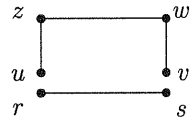
Table 1. The graphs of the elements of found(T).

Lemma 2. Let T be a trade and $x \in \text{found}(T)$, then

(i) if $r_x = 2$, then $S(x) = 0$, and $G^1_x \approx G^2_x$;
(ii) if $r_x = 3$, then $S(x) \leq 1$, and $G^1_x \approx G^2_x$;
(iii) if $r_x = 4$, then $S(x) = 2$, and if $G^1_x \approx H$, then $G^2_x \approx H'$;
(iv) if $r_x = 5$, then $4 \leq S(x) \leq 5$, and $G^1_x \approx G^2_x$;
(v) if $r_x = 6$, then $S(x) = 6$, and if $G^1_x \approx H'$, then $G^2_x \approx H$.

Proof. The proofs of the different cases of the theorem are similar, and we only give the proof for the case (iii). We can conclude from Table 1 that $S(x) = 2$. Now let

G_x^1 be as follows:



Then, $xzw \in T_1$, and hence $xzw \notin T_2$. Thus the vertices of degree 2 in G_x^2 are not adjacent, and we conclude that $G_x^2 \approx H'$. □

Lemma 3. Let T be a trade. Then
(i) for $1 \le m \le 3$, if $|E(6)| \ge m$, then $E(m+1) = \varnothing$;
(ii) if $|E(2)| \ge 3$, then $E(5) = \varnothing$;
(iii) if $E(4) \ne \varnothing$ or $E(6) \ne \varnothing$, then $|\text{found}(T)| = 7$;
(iv) if $|E(5)| \ge 3$, then $\text{vol}(T) \ge 9$;
(v) if $|E(4)| \ge 3$, then there exist elements x and y in $E(4)$ such that $\lambda_{xy} = 1$.

Proof.

(i) Since for any $x \in \text{found}(T)$, and $y \in E(6)$, $\lambda_{xy} = 2$, thus if $|E(6)| \ge m$, then $S(x) \ge m$, and we obtain the result by Lemma 2(i-iii).

The proofs of the remaining parts are similar. □

Let $\text{found}(T) = \{x_1, x_2, \cdots, x_m\}$. We associate with every trade T, a decreasing sequence $(r_{x_1}, \cdots, r_{x_m})$ such that $2 \le r_{x_m} \le \cdots \le r_{x_1} \le 6$ and we call it the *element occurrence sequence of* T (abbreviated to EOS(T)). We denote $\sum_{i=1}^{m} r_{x_i}$ by $N(T)$. Clearly the following inequality holds:

$$2|\text{found}(T)| \le N(T) = 3\text{vol}(T).$$

In subsequent sections, for simplicity we use i for x_i.

3. Trades with volume 6

We consider two cases:

(i) $|\text{found}(T)| = 6$. In this case, if $E(2) \ne 0$ then by noting that $N(T) = 18$, we conclude that $E(4) \ne \varnothing$, and this contradicts Lemma 3(iii). So the only possible EOS(T) is (3,3,3,3,3,3). Clearly for any $x \in \text{found}(T)$, $G_x^1 \approx G_x^2 \approx H'$. Let $\lambda_{12} = 2$, so the blocks of T are as follows:

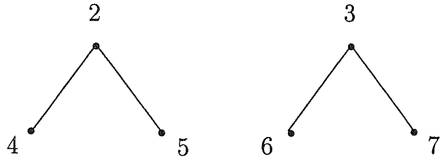
$$T = \{T_1, T_2\} = \{\{123, 124, 156, 256, 345, 346\}, \{125, 126, 134, 234, 356, 456\}\}.$$

(ii) $|\text{found}(T)| = 7$. Clearly $|E(2)| \ge 3$. By Lemma 3(ii), $E(5) = \varnothing$, and we have the following sequences:

(1) $(4,4,2,2,2,2,2)$ (2) $(4,3,3,2,2,2,2)$ (3) $(3,3,3,3,2,2,2)$

(1) Since $1 \in E(4)$, so $S(1) = 2$, and there exists $x \in E(2)$ such that $\lambda_{1x} = 2$. But this contradicts Lemma 2(i).

(2) Let $\lambda_{13} = \lambda_{12} = 2$, and $123 \notin T_1$. Hence the graph G_1^1 is as follows:

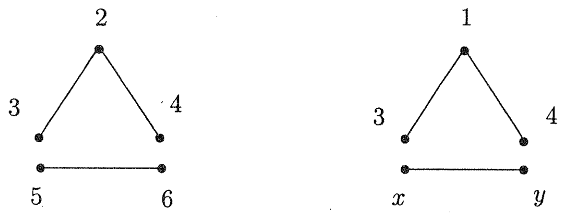


Therefore, $\lambda_{23} = 0$, since otherwise $23x \in T_1$, and $x \in \{4,5,6,7\}$. So $S(x) \geq 1$, and this contradicts Lemma 2(i). With no loss of generality, the blocks of T are

$$T = \{T_1, T_2\} = \{\{124, 125, 136, 137, 267, 345\}, \{126, 127, 134, 135, 245, 367\}\}.$$

(3) To verify this case, first we need the following lemma.

Lemma 4. For any $x, y \in E(3)$, $\lambda_{xy} = 1$.
Proof. Clearly, for any $x, y \in E(3)$, $\lambda_{xy} \geq 1$. Let $\lambda_{12} = 2$. Hence the graphs G_1^1 and G_2^1 are as follows:



G_1^1 G_2^1

Therefore, $126, 125, 134, 234 \in T_2$. Thus $\lambda_{34} = 2$. By a similar argument $\lambda_{56} = 2$, and hence $\{3,4,5,6\} \subset E(3)$, but this is impossible. □

By the above lemma, the blocks of T are

$$T = \{T_1, T_2\} = \{\{125, 136, 147, 237, 246, 345\}, \{126, 137, 145, 235, 247, 346\}\}.$$

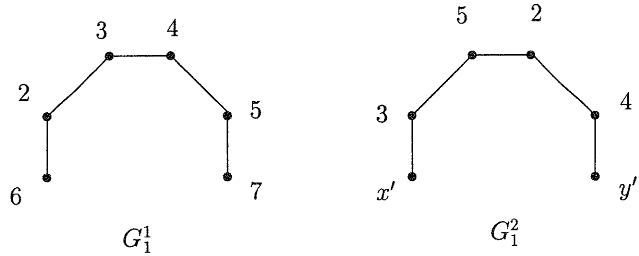
The results are summarized as follows:

Theorem 5. There exist a 2-(6,3) trade and two 2-(7,3) trades with $\mathrm{vol}(T) = 6$.

4. Trades with volume 7

For any trade T with $\mathrm{vol}(T) = 7$, $|\mathrm{found}(T)| = 6$ is impossible. To see this, let $|\mathrm{found}(T)| = 6$. Then, for any $x \in \mathrm{found}(T)$, we have $r_x = 5$ or 3. If $E(5) \neq \varnothing$, then $E(2) = \varnothing$, and there can not exist any EOS in this case. Hence, $E(5) = \varnothing$. Now, suppose that $r_x = 3$. Again by looking at $N(T)$, this case is easily ruled out. Therefore, we consider trades T with $\mathrm{vol}(T) = 7$ and $|\mathrm{found}(T)| = 7$.

Lemma 6. For any $x \in \text{found}(T)$, $r_x \leq 4$.
Proof. By Lemma 3(i), for any $x \in \text{found}(T)$, $r_x \leq 5$. Now suppose that $x \in \text{found}(T)$ and $r_x = 5$. So $\text{EOS}(T) = (5,3,3,3,3,2,2)$. Therefore, the graphs G_1^1 and G_1^2 are the following:



where $\{x', y'\} = \{6, 7\}$. Because $2 \in E(3)$, we can conclude that $245 \in T_1$. So $\lambda_{45} = 2$, and therefore $S(4) \geq 2$. But $4 \in E(3)$, and $S(4) = 1$. □

By Lemma 6, the possible EOS(T)'s for the above trades are
(1) $(4,4,4,3,2,2,2)$
(2) $(4,3,3,3,3,3,2)$
(3) $(4,4,3,3,3,2,2)$
(4) $(3,3,3,3,3,3,3)$.

(1) By Lemma 3(v), there exist two elements $x, y \in E(4)$ such that $\lambda_{xy} = 1$. Let $\lambda_{12} = 1$. Since $S(1) = S(2) = 2$, hence $\lambda_{14} = \lambda_{24} = 2$. Therefore, $S(4) = 2$ and this contradicts Lemma 2(ii).

(2) By Lemma 3(i), we can assume that $G_1^1 \approx H'$. Suppose that $\lambda_{12} = \lambda_{13} = 2$, and $123 \notin T_1$. Therefore we can show that the blocks of T_1 are as follows:

$$T_1 = \{124, 125, 136, 137, 2ab, 45c, 45d\}.$$

Thus $c = 3$, $d = 6$, and hence $267 \in T_1$. Since $G_3^1 \approx H'$, so $134, 135, 367 \in T_2$, and we note that the blocks of T_2 are as follows:

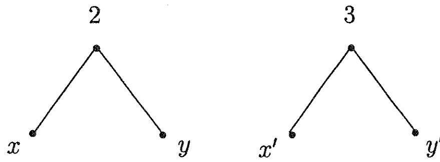
$$T_2 = \{12a', 12b', 134, 135, 2c'd', 367, e'f'g'\}.$$

To determine the unknown entries, we should have the block $456 \in T_2$. Hence $T_1 \cap T_2 \neq \varnothing$, and this contradicts the definition of trade.

(3) To construct a trade of kind (3), we need the following lemma.

Lemma 7. $\lambda_{12} = 2$.
Proof. Suppose that $\lambda_{12} = 1$ and $123 \in T_1$. Hence, $\lambda_{13} = \lambda_{23} = 1$. But this is impossible since $2 \in E(3)$. □

Now suppose that $\lambda_{12} = \lambda_{13} = 2$. By Lemma 2(iii), the graph G_1^1 is as follows:



G_1^1

Lemma 8. $x \in E(3)$ and $y \in E(2)$.
Proof. Clearly, $|E(3) \cap \{x,y\}| \leq 1$. Since otherwise by considering the blocks of T_1, we have $\lambda_{2x} = \lambda_{2y} = 2$. Since $\lambda_{12} = 2$ and therefore $S(2) = 3$, this is a contradiction. On the other hand, $23u \in T_1$ and $u \notin \{x', y'\}$ (since $3 \in E(3)$). Then $u = x$ or $u = y$, and therefore $\{x,y\} \cap E(3) \neq \varnothing$. □

By the above lemmas, the blocks of T are as follows:

$$\begin{aligned} T &= \{T_1, T_2\} \\ &= \{\{124, 126, 135, 137, 234, 257, 456\}, \{123, 127, 134, 156, 246, 247, 357\}\}. \end{aligned}$$

(4) To handle this case the following lemma is clear and helpful.

Lemma 9. For any $x, y \in \text{found}(T)$, $\lambda_{xy} = 1$.

By the above lemma, T is composed of two disjoint Fano planes [3].

The results of this section establish the following theorem.

Theorem 10. There are two nonisomorphic 2-(7,3) trades with volume 7.

5. Trades with volume 8

If $|\text{found}(T)| = 6$, then for any $x \in \text{found}(T)$, $r_x = 3$ or 5 and $\text{EOS}(T) = (5,5,5,3,3,3)$. This case is ruled out by Lemma 3(iv).

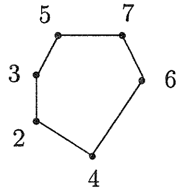
Now we study the trades with $|\text{found}(T)| = 7$.

Lemma 11.

(i) $E(6) = \varnothing$,
(ii) $|E(5)| \leq 2$,
(iii) if $|E(2)| = 2$ and $|E(5)| \geq 2$, then $E(3) = \varnothing$,
(iv) if $E(2) = \varnothing$, then $|E(4)| \leq 4$,
(v) if $E(2) \neq \varnothing$ and $|E(5)| = 2$, then $E(3) = \varnothing$.

Proof.

(i) Suppose that $E(6) \neq \varnothing$. Then $|E(6)| = 1$ or 2. The case $|E(6)| = 2$ is ruled out by Lemma 3(i). Let $|E(6)| = 1$. Then the only possibility for $\mathrm{EOS}(T)$ is $(6,3,3,3,3,3,3)$. By Lemma 2(v), the graph G_1^1 is as follows:



Therefore, $234, 235 \in T_2$ and since $\lambda_{12} = 2$, then $r_2 = 4$. By considering $\mathrm{EOS}(T)$ we have $r_2 = 3$. Hence (i) is proven.

The proofs of the remaining parts are similar. □

By the above lemma the possible $\mathrm{EOS}(T)$'s for these kinds of trades are as follows:

(1) $(5,4,4,4,3,2,2)$ (2) $(5,4,4,3,3,3,2)$ (3) $(5,4,3,3,3,3,3)$
(4) $(4,4,4,4,3,3,2)$ (5) $(4,4,4,3,3,3,3)$.

(1) Since $|E(2)| = 2$, then $G_1^1 \approx H$. So $\lambda_{16} = \lambda_{17} = 1$. By considering the graph of the elements of $E(2)$, we can conclude that $\lambda_{56} = \lambda_{57} = 0$. Therefore, G_5^1 contains at most 4 vertices. Hence there does not exist any trade with this $\mathrm{EOS}(T)$.

(2) Clearly $\lambda_{12} = \lambda_{13} = \lambda_{23} = 2$. With no loss of generality suppose that $123 \notin T_1$. Let $23v, 23u, 12x, 12y, 13z, 13w \in T$. But since the elements u, v, x, y, z, w are distinct, therefore $|\mathrm{found}(T)| \geq 9$. Hence this case is also ruled out.

(3) Since $S(1) \geq 4$ and $S(2) = 2$, hence there exists $x \in E(3)$ such that $\lambda_{1x} = \lambda_{2x} = 2$, and this contradicts Lemma 2(ii).

(4) Clearly $\lambda_{57} = \lambda_{67} = 0$ and there exist $x, y, z \in E(4)$, such that $\lambda_{xy} = \lambda_{xz} = 2$. Suppose that $\lambda_{12} = \lambda_{13} = 2$, and $G_1^1 \approx H$ (Lemma 2(iii)). Clearly $\lambda_{23} = 1$. With no loss of generality the blocks of T_1 are as follows:

$$T_1 = \{124, 125, 136, 137, 23a, 2bc, 3de, fgh\}, \quad (*)$$

or

$$T_1 = \{124, 127, 136, 135, 23a, 2bc, 3de, fgh\}. \quad (**)$$

In $(*)$, $7 \in E(2)$ and hence $247 \in T_1$. Therefore, $a = 6$ and $345 \in T_1$. So $\lambda_{47} = 2$ and hence $S(7) = 1$. But this contradicts Lemma 2(i). The case $(**)$ can be similarly ruled out.

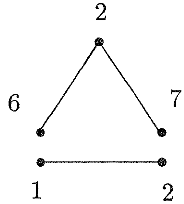
(5) By considering Lemma 3(v), there exist $x, y \in E(4)$ such that $\lambda_{xy} = 1$.

In the process of constructing trades in this case, the following lemma is useful.

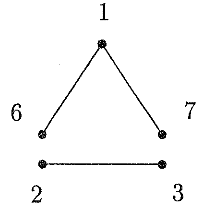
Lemma 12. If $\lambda_{12} = 1$ and $12x \in T_1$, then $x \in E(3)$.
Proof. $x \in E(4)$ implies that $x = 3$. Therefore, $\lambda_{13} = \lambda_{23} = 2$. By Lemma 2(iii), we can show that the blocks of T_1 are as follows:

$$T_1 = \{123, 134, 156, 157, 23a, 2bc, 2de, 3fg\}.$$

Since $5 \in E(3)$, and $\lambda_{25}, \lambda_{35} \neq 0$, we can conclude that $a = 5$, and $\{f, g\} = \{6, 7\}$. We also have $\{4, 6, 7\} \subseteq E(3)$ and hence $b = d = 4$, $c = 7$, and $e = 6$. Now the graphs G_4^1 and G_5^1 are as follows:



G_4^1



G_5^1

Therefore, $467, 567 \in T_2$ and hence $\lambda_{67} = 2$. The pair 67 appears in T_1 only once. Hence, $x \in E(3)$. □

By the above lemma, $126 \in T_1$. If $16x \in T_1$ then $x \in E(3)$, otherwise we face a contradiction similar to the argument stated above. The blocks of T_1 are

$$T_1 = \{126, 167, 1ab, 1ac, 2de, 2fg, 2hk, 6mn\}.$$

Clearly $\{a, b, c\}$ and $\{m, n\}$ are subsets of $\{3, 4, 5\}$. If $\{m, n\} = \{4, 5\}$, then $a = 3$, $b = 4$, and $c = 5$. So $245, 237, 236 \in T_1$. Hence we have

$$T_1 = \{126, 167, 134, 135, 236, 237, 245, 456\}.$$

Therefore, $7 \in E(2)$. The cases $\{m, n\} = \{3, 4\}$ and $\{m, n\} = \{3, 5\}$ are similarly ruled out.

The results of this section establish the following theorem.

Theorem 13. There does not exist any 2-(7,3) trade with volume 8.

6. Trades with volume 9

It is easy to show that simple 2-(6,3) trades with volume 9 do not exist. Therefore, we let T be a simple 2-(7,3) trade with $\text{vol}(T) = 9$. Then we can make the following assertions.

Lemma 14. For any $x \in \text{found}(T)$, $r_x \leq 5$.
Proof. Suppose that $E(6) \neq \varnothing$. By Lemma 3(i), $E(2) = \varnothing$ and the possible EOS(T)'s are

(1) $(6,5,4,3,3,3,3)$ (2) $(6,4,4,4,3,3,3)$.

In (1), since $S(1) = 6$, and $S(2) \geq 4$, we can conclude that there exists an $x \in E(3)$ such that $\lambda_{1x} = \lambda_{2x} = 2$. Hence $S(x) \geq 2$ and this contradicts Lemma 2(ii).

In (2), the fact that for any $x \in \text{found}(T)$, $\lambda_{1x} = 2$, leads us to conclude that the graphs of the elements of $E(3)$ are isomorphic to H'. Since $|\{\{x,y\}|\lambda_{xy} = 0\}|$ is even, this case is also ruled out. □

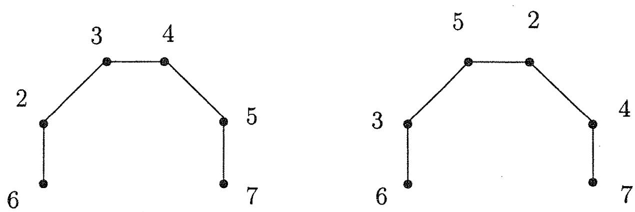
By the above lemma, we have the following possible EOS(T)'s.

(1)	$(5,4,4,4,4,4,2)$	(2)	$(5,4,4,4,4,3,3)$
(3)	$(5,5,4,4,4,3,2)$	(4)	$(5,5,4,4,3,3,3)$
(5)	$(5,5,5,4,4,2,2)$	(6)	$(5,5,5,4,3,3,2)$
(7)	$(5,5,5,3,3,3,3)$	(8)	$(4,4,4,4,4,4,3)$

(1) For any $x \in E(4)$, $\lambda_{7x} \neq 0$, thus this case is ruled out.

(2) For this case the following lemma is needed.

Lemma 15. $\lambda_{67} = 0$.
Proof. Suppose $\lambda_{67} = 1$. Then $G_7^1 \approx G_6^1 \approx H$. Therefore, $G_1^1 \approx H$. The graphs G_1^1 and G_1^2 are as follows:



G_1^1 G_1^2

Since $67x \in T_1$ and $67x' \in T_2$, we can conclude that $x \in \{3,4\}$ and $x' \in \{2,5\}$. If $x = 3$ and $x' = 2$, then $456 \in T_1 \cap T_2$ which is a contradiction. The other cases are similarly ruled out. □

Now, by the above lemma, we will show that there does not exist any trade satisfying (2). Let the blocks of T_1 be as follows:

$$T_1 = \{123, 126, 134, 145, 157, 2ab, 2cd, 3ef, ghk\}.$$

Let the graphs G_1^1 and G_1^2 be as the graphs in Lemma 15. Clearly $\lambda_{26} = 2$, since otherwise the pair 67 appears in a block of T_1, and this contradicts the above lemma. By considering the blocks of T_2 we have $\lambda_{36} = 2$, and so $S(6) \geq 2$. This contradicts Lemma 2(ii).

Lemma 16. There does not exist any trade with EOS(T)'s equal to (3), (5), (6) and (7).

Proof. In the case (3), $7 \in E(2)$ hence $\lambda_{17} = 1$, and $\lambda_{67} = \lambda_{27} = 0$. If $123, 124 \in T_1$ and $125, 126 \in T_2$, then by considering the blocks of T_2, we have $\lambda_{67} \neq 0$, which is a contradiction. Cases (5) and (6) can be easily ruled out. For the case (7), we define a set $M_i (1 \leq i \leq 3)$ as follows:

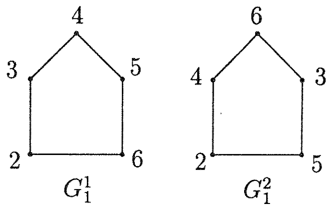
$$M_i = \{x | x \in E(3), \lambda_{ix} = 2\}.$$

Clearly $|M_i| \geq 2$. So there exists an $x \in E(3)$, contained in at least two of the M_i's, and this is a contradiction to Lemma 2(ii). □

(4) To establish this case, we need the following lemma.

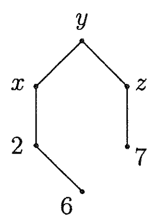
Lemma 17. $G_1^1 \approx G_2^1 \approx H$

Proof. Suppose that $G_1^1 \approx H'$. There exists $x \in E(3)$ such that $\lambda_{1x} = 0$ and let $x = 7$. Therefore, $\lambda_{15} = \lambda_{16} = 2$ and $G_5^1 \approx G_6^1 \approx H'$. The graphs G_1^1 and G_2^1 are as follows:



Since $\lambda_{67} = \lambda_{57} = 1$, then for any $x \in$ found(T), $\lambda_{6x} \neq 0$. Hence $G_6^1 \approx H$ and this is also a contradiction. □

Using the above lemma, we can construct T. By the above lemma, there is an element x of $E(3)$ such that $G_x^1 \approx H$. Let $x = 7$. Hence $\lambda_{17} = \lambda_{27} = 1$ and $127 \notin T_1$ (or T_2), otherwise we have vol(T) ≥ 10. Also $\lambda_{16} = 1$ or $\lambda_{26} = 1$. Suppose that $\lambda_{16} = 1$. Therefore, G_1^1 is as follows:



Since $\lambda_{16} = 1$ and $\lambda_{15} = 2$, hence $\lambda_{26} = 2$ and $\lambda_{25} = 1$. If $x = 5$ or $y = 5$, we can not complete the blocks of T_1. Therefore $z = 5$ and $\{x, y\} = \{3, 4\}$, and so we have

$$T = \{T_1, T_2\} = \{\{123, 126, 134, 145, 157, 235, 246, 247, 367\}, \{124, 125, 135, 137, 146, 234, 236, 267, 457\}\}.$$

(8) Clearly $G_7^1 \approx H$, and so for any $x \in \text{found}(T)\lambda_{x7} = 1$. Let $\lambda_{12} = \lambda_{13} = 2$. By Lemma 3(v), $\lambda_{23} = 1$. Suppose that

$$T_1 = \{124, 127, 135, 136, 23a, 2bc, 37g, 7hk, 4lm\}.$$

Since at most one of the elements of the set $\{g, h, k\}$ is 4, we can conclude that one of the elements of the set $\{a, b, c\}$ is 4. Hence, $\lambda_{24} = 2$. If $g = 4$ then $\{l, m\} = \{h, k\} = \{5, 6\}$, $a = 6$, and $\{b, c\} = \{4, 5\}$. But we can not complete the blocks of T_2. Hence $g \neq 4$, and $g \in \{5, 6\}$. With no loss of generality, let $g = 5$. Therefore, $\{h, k\} = \{4, 6\}$ and $\{b, c\} = \{l, m\} = \{5, 6\}$. So the blocks of T_2 are as follows:

$$T_2 = \{123, 125, 137, 146, 247, 246, 345, 356, 567\}.$$

The results of this section establish the following theorem.

Theorem 18. There are two nonisomorphic 2-$(7, 3)$ trades with volume 9.

7. Trades with volume 10

First, we assume that $|\text{found}(T)| = 6$. For any $x, y \in \text{found}(T)$, $\lambda_{xy} = 2$. Hence $T = \{T_1, T_2\}$ is composed of two disjoint 2-$(6, 3, 2)$ designs.

Now let T be a 2-$(7, 3)$ trade with volume 10. The following lemma is clear.

Lemma 19.
(i) $|E(6)| \leq 1$,
(ii) if $E(6) = \varnothing$, then $|E(5)| \geq 2$.

By the above lemma and Lemma 4, we have the following possible EOS(T)'s:
(1) (6,5,5,4,4,3,3) (2) (6,5,4,4,4,4,3) (3) (6,4,4,4,4,4,4)
(4) (5,5,5,5,5,3,2) (5) (5,5,5,5,4,3,2) (6) (5,5,5,5,4,4,2)
(7) (5,5,5,4,4,4,3) (8) (5,5,4,4,4,4,4).

Lemma 20. There does not exist any trade with EOS(T) equal to (1), (3), (5) or (6).
Proof. In (1), since $\lambda_{17} = \lambda_{16} = 2$, therefore $\lambda_{37}, \lambda_{36}, \lambda_{26}, \lambda_{27} \leq 1$ and hence, $\lambda_{34} = \lambda_{24} = 2$. If $\lambda_{14} = 2$, then $S(4) \geq 3$, which contradicts Lemma 2(iii). In (3),

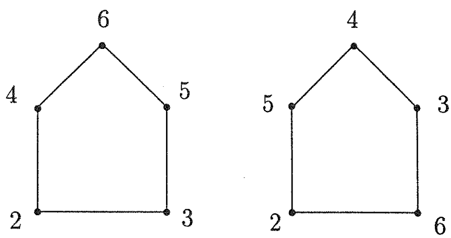
let $123, 124 \in T_1$. With no loss of generality, suppose that $\lambda_{23} = \lambda_{24} = 1$. Therefore, $\lambda_{34} = 2$, and $34x, 34x' \in T_1$. Since $13y', 14y \in T_1$, and $S(3) = S(4) = 2$, we can conclude that the elements x', y', x and y are distinct. Hence $|\text{found}(T)| \geq 8$, which is impossible. In (6), clearly there exist two elements 1 and 2 in $E(5)$ such that $\lambda_{17} = \lambda_{27} = 0$. Therefore, $\lambda_{26} = \lambda_{25} = \lambda_{15} = \lambda_{16} = 2$. But $\lambda_{35} = 2$ or $\lambda_{36} = 2$. Both of these cases are contradictory. In (5), there exist two elements 1 and 2 in $E(5)$ such that $\lambda_{15} = \lambda_{25} = 1$ (since $S(5) = 2$). Clearly $125 \notin T_1, T_2$, since otherwise, we have $\text{vol}(T) > 10$. Also $\lambda_{12} = 2$. If $12x, 12x' \in T_1$, then $|\{x, x'\} \cap \{3, 4\}| \leq 1$, otherwise $345 \in T_1$ appears twice. Suppose that $123 \in T_1$ and $124 \in T_2$. So $345 \in T_1 \cap T_2$ which is impossible. □

Now we study the other cases.

(2) Since $\lambda_{17} = 2$, so $G_7^1 \approx H'$. Therefore, there exists $x \in \text{found}(T)$ such that $\lambda_{7x} = 0$. Clearly $x = 2$. If $123, 124 \in T_1$ and $125, 126 \in T_2$, then $567 \in T_1$, $347 \in T_2$, and the blocks of T are as follows:

$$T = \{T_1, T_2\} = \{\{123, 124, 135, 146, 157, 167, 236, 245, 256, 347\},\\ \{125, 126, 136, 145, 137, 147, 234, 235, 246, 567\}\}.$$

(4) There exists $1 \in E(5)$ such that $\lambda_{17} = 0$. So $G_1^1 \approx H'$. Suppose that the graphs G_1^1, and G_1^2 are as follows:



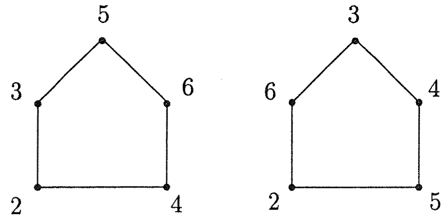
G_1^1 G_1^2

Clearly $\lambda_{34} = 2$ (otherwise $\lambda_{46} = 2$ or $\lambda_{36} = 2$, and hence $S(6) = 2$). By the same argument $\lambda_{23} = 2$. Thus $257, 347 \in T_1$ and the blocks of T are as follows:

$$T = \{T_1, T_2\} = \{\{123, 124, 135, 136, 156, 236, 245, 257, 345, 347\},\\ \{125, 126, 136, 134, 145, 234, 235, 247, 357, 456\}\}.$$

For (7), first we prove the following lemma.

Lemma 21. For any $x, y \in \text{found}(T)$, $\lambda_{xy} \geq 1$.
Proof. Let $\lambda_{17} = 0$. Therefore $G_1^1 \approx H'$. Suppose that the graphs G_1^1 and G_1^2 are as follows:

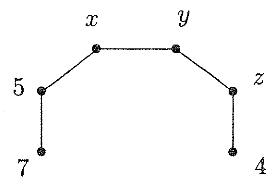


G_1^1 G_1^2

If $\lambda_{24} = 2$, then $247, 234 \in T_2$. But $234 \in T_1$ and we can conclude that $T_1 \cap T_2 \neq \varnothing$. Therefore, $\lambda_{24} = 1$. Similarly $\lambda_{26} = \lambda_{25} = 1$. Thus $S(2) \leq 3$, this contradicts Lemma 2(iv). □

Now suppose that for any $x, y \in \text{found}(T)$, $\lambda_{xy} \geq 1$ and $123 \in T_1$. Since $S(1) = S(2) = S(3) = 4$, therefore, with no loss of generality, we can assume that $\lambda_{36} = \lambda_{25} = \lambda_{14} = 1$. Hence $\lambda_{15} = \lambda_{16} = \lambda_{26} = \lambda_{24} = \lambda_{34} = \lambda_{35} = 2$.

If $7ab \in T_1$, then $a \in E(3)$ and $b \in E(2)$. Therefore, with no loss of generality, we suppose that $157, 267, 347 \in T_1$. Let G_1^1 be the following graph:



We have the following cases to consider:

(*a*) $(x, y, z) = (6, 3, 2)$ (*b*) $(x, y, z) = (6, 2, 3)$

(*c*) $(x, y, z) = (2, 3, 6)$ (*d*) $(x, y, z) = (3, 2, 6)$

The cases (*b*), (*c*), and (*d*) are easily ruled out. In (*a*) we have the following trade:

$$T = \{\{123, 124, 136, 156, 157, 235, 246, 267, 345, 347\},$$
$$\{125, 126, 135, 137, 146, 234, 236, 247, 345, 657\}\}.$$

For (8), first we prove the following lemma.

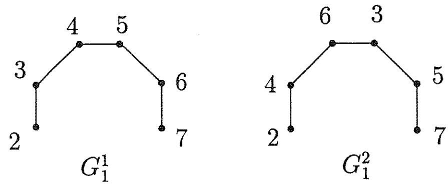
Lemma 22. $\lambda_{12} \leq 1$.

Proof. Suppose $\lambda_{12} = 2$. Let $124, 123 \in T_1$ and $125, 126 \in T_2$. By considering the graphs of the elements of $E(5)$, we can conclude that $\lambda_{17} = \lambda_{27} = 1$. If $\lambda_{13} = \lambda_{25} = 1$, then $357 \in T_1 \cap T_2$, this contradicts the definition of trade. Therefore, $\lambda_{12} \leq 1$. □

Below we can construct two trades with $\lambda_{12} = 0$ and 1, respectively. With no loss of generality, the blocks of T could be as follows:

$$T = \{T_1, T_2\} = \{\{134, 145, 156, 167, 137, 246, 247, 235, 236, 257\},$$
$$\{146, 147, 135, 136, 157, 234, 237, 245, 256, 267\}\}.$$

Let $\lambda_{12} = 1$. Let G_1^1 and G_1^2 be as follows:



So, the trade will be

$$T = \{T_1, T_2\} = \{\{123, 134, 145, 156, 167, 235, 246, 247, 257, 367\},$$
$$\{124, 135, 136, 146, 157, 234, 237, 245, 256, 267\}\}.$$

The results of this section establish the following theorem.

Theorem 23. There is one 2-$(6,3)$ trade and, up to isomorphism, there are five 2-$(7,3)$ trades with volume 10.

8. Trades with volume 11

Let T be a 2-(7,3) trade with $\text{vol}(T) = 12$. Then we have the following lemma.

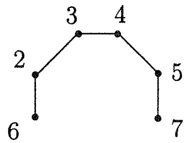
Lemma 24.
(i) $|E(6)| \leq 1$. (ii) $|E(5)| \geq 3$.

Proof. (i) Suppose $|E(6)| = 2$. Therefore, for any $x \in \text{found}(T), r_x \geq 4$. Thus the only possible $\text{EOS}(T)$ is (6,6,5,4,4,4,4). Hence there exists $x \in E(4)$ such that $\lambda_{1x} = \lambda_{2x} = \lambda_{3x} = 2$. So $S(x) = 3$, and this contradicts Lemma 2(iii). The case (ii) is easily ruled out. □

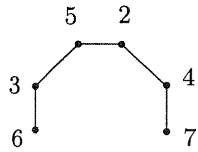
By the above lemma we have the following possible $\text{EOS}(T)$'s:
(1) (6,5,5,5,4,4,4) (2) (5,5,5,5,5,4,4).

(1) Let $\lambda_{23} = 1$. With no loss of generality, $124, 125 \in T_1$ and $126, 127 \in T_2$. Hence $\lambda_{24} = 2$, since otherwise $\text{vol}(T) \geq 12$. Similarly $\lambda_{25} = 2$, since otherwise, the block 345 appears in T_1 twice. By the same argument $\lambda_{26} = \lambda_{27} = 2$. Hence $S(2) \geq 5$, and this contradicts Lemma 2(iii). Thus for any $x, y \in E(5)$, $\lambda_{xy} = 2$. So for any $x \in E(5)$, there exists only one element $y \in E(4)$ such that $\lambda_{xy} = 1$. Let $12x, 12y \in T_1$. Thus $x, y \in E(4)$, otherwise $\text{vol}(T) \geq 12$. With the same argument, if $12x', 12y' \in T_2$, then we can conclude that $x', y' \in E(4)$. Therefore, $\{x, y, x', y'\} \subset E(4)$ and so $|E(4)| \geq 4$, and this contradicts $\text{EOS}(T)$.

(2) Clearly there exists element $x \in E(5)$ such that $\lambda_{6x} = \lambda_{7x} = 1$. Let $x = 1$. Suppose that G_1^1 and G_1^2 are as follows:



G_1^1



G_1^2

If $\lambda_{23} = 1$, then $\lambda_{26} = \lambda_{36} = \lambda_{34} = \lambda_{24} = 2$. By considering the blocks of T_2, we have $\lambda_{25} = 2$ (since $\lambda_{23} = 1$). Now, if $\lambda_{25} = 1$, then $\lambda_{35} = 2$, and therefore $\lambda_{27} = \lambda_{37} = 1$. Thus we can conclude that $r_7 = 3$, this is a contradiction. Hence $\lambda_{45} = 2$ and $\lambda_{35} = 2$, and again we have $r_7 = 3$, this is a contradiction. Hence $\lambda_{23} = 2$. By the same argument $\lambda_{24} = \lambda_{25} = \lambda_{35} = \lambda_{45} = \lambda_{34} = 2$. Therefore, for any $x \in E(5)$, $\lambda_{6x} = \lambda_{7x} = 1$. Thus $S(7), S(6) \leq 1$. But this contradicts Lemma 2(iii).

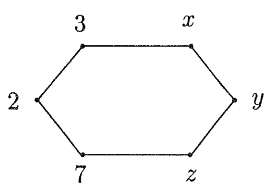
Based on the above arguments we have the following theorem.

Theorem 25. There are no 2-(7,3) trades with volume 11.

9. Trades with volume 12

Since $|\text{found}(T)| = 7$ and hence $N(T) = 36$, therefore $E(6) \neq \varnothing$.

Lemma 26. $|E(6)| = 1$.
Proof. Let $|E(6)| \geq 2$. Therefore the only possible EOS(T) is (6,6,5,5,5,4,4). Since we have only two blocks without the elements 1 and 2, we can conclude that $\{x, y\} \cap E(5) = \varnothing$. With no loss of generality we assume that $x \in E(5)$, and $y \in E(4)$. Let G_1^1 be as follows:



where $\{x, y, z\} = \{4, 5, 6\}$. If $y, z \in E(5)$, then $x = 6$. Since $\lambda_{34} = \lambda_{35} = 2$, therefore $236, 235 \in T_1$. Thus $\lambda_{23} = 3$. But this contradicts Lemma 1. The cases $x, y \in E(5)$, and $x, z \in E(5)$ are similarly ruled out. □

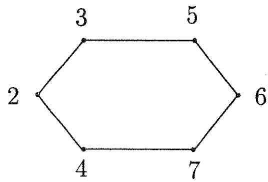
By the above lemma we have (6,5,5,5,5,5,5) as a candidate for a possible EOS(T).

Lemma 27. For any $x, y \in \text{found}(T)$, $\lambda_{xy} \geq 1$.
Proof. Let $\lambda_{23} = 0$. We can show that the blocks of T are as follows:

$$T = \{T_1, T_2\} = \{\{124, 125, 136, 137, 146, 157, 267, 26x, 27y, 345, 34u, 35v\},$$
$$\{126, 127, 134, 135, 14a, 15b, 245, 24x', 25y', 367, 36u', 37v'\}\}.$$

Clearly $x \in \{4,5\}$. If $x=4$, then $\lambda_{46}=2$. Thus two elements of the set $\{u', x', a\}$ are equal to 4 or 6. In both cases $T_1 \cap T_2 \neq \varnothing$. The case $x=5$, is similarly ruled out. So we have no choices for x. □

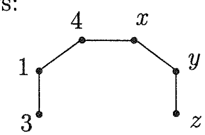
Now suppose that $124, 123 \in T_1$. With no loss of generality $\lambda_{23}=1$. By the Lemma 2(v), G_1^1 is as follows:



Since $\lambda_{23}=1$, clearly $\lambda_{34}=\lambda_{25}=\lambda_{35}=\lambda_{24}=2$. Therefore the blocks of T_1 are

$$T=\{123, 124, 135, 147, 156, 167, 24a, 2bc, 2de, 34f, 34g, 3hk\}.$$

So the graph G_2^1 is as follows:



where $\{x,y,z\}=\{4,5,6\}$. Since $\lambda_{25}=2$, therefore one of the elements of the set $\{x,y\}$ is 5, and so we have the following cases

(a) $(x,y,z)=(5,6,7)$ (c) $(x,y,z)=(6,5,7)$

(b) $(x,y,z)=(5,7,6)$ (d) $(x,y,z)=(7,5,6)$.

The cases (a), (b) and (c) are easily ruled out, and now there remains only the case (d). In this case we can construct a unique trade as follows:

$$T=\{T_1,T_2\}=\{\{123, 124, 135, 147, 156, 167, 247, 256, 257, 345, 346, 367\},$$
$$\{125, 127, 134, 136, 146, 157, 234, 245, 267, 356, 357, 467\}\}.$$

Therefore, we have the following result.

Theorem 28. There is a unique 2-(7, 3) trade with $\text{vol}(T)=12$.

In Tables 2 and 3, we present a summary of the results.

Table 2.

vol(T) \ |found(T)|	6	7
4	1	0
6	1	2
7	0	2
9	0	2
10	1	5
12	0	1

Table 3.

|found(T)|	6	6	7	7	7	7	7	7	6	7	7	7	7	7	7
T_1	123	123	123	123	123	123	123	123	123	123	123	123	123	123	123
	145	124	124	167	124	146	124	126	124	124	124	124	124	124	124
	246	156	156	247	157	157	157	134	135	157	157	156	135	157	147
	356	256	157	256	167	247	167	145	146	167	167	167	156	167	156
		345	267	346	267	256	247	157	156	237	237	235	167	235	167
		346	345	357	347	345	256	235	236	256	246	246	236	246	237
					456	367	345	246	245	267	256	257	245	256	246
							367	247	256	346	345	267	256	347	256
							456	367	345	356	356	347	346	367	345
									346	457	457	456	357	457	346
															367
															457
T_2	124	125	126	127	126	127	126	124	125	126	126	126	125	125	126
	135	126	127	136	127	136	127	125	126	127	127	127	126	126	127
	236	134	135	235	137	145	137	135	134	135	135	135	136	137	137
	456	234	145	246	145	235	145	137	136	147	147	146	137	147	145
		356	234	347	234	246	234	146	145	234	234	234	145	234	146
		456	567	567	467	347	245	234	234	236	236	237	234	236	234
					567	567	356	236	235	257	245	245	235	245	236
							467	267	246	367	357	256	246	357	245
							567	457	356	456	456	467	356	467	347
									456	567	567	567	567	567	356
															467
															567

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