New Lower Bounds of Fifteen Classical Ramsey Numbers*

Su Wenlong†

Wuzhou First Middle School
Wuzhou, Guangxi 543002, P. R. China

Luo Haipeng

Guangxi Academy of Sciences
Nanning, Guangxi 530031, P. R. China
E-mail: gxkxylhp@public. nn. gx. cn

Zhang Zhengyou

Commission of Sci. and Tech. of Guangxi
Nanning, Guangxi 530012, P. R. China

Li Guiqing

CAD lab. of Institute of Computing Tech.
The Chinese Academy of Science, Beijing 100080, P. R. China
E-mail: gqli@gxu. edu. cn

Abstract

An algorithm to compute lower bounds for the classical Ramsey numbers $R(q_1, q_2, \cdots, q_n)$ is developed. The decompositions of each of fifteen complete graphs of prime order into n circulant graphs are constructed and their properties are studied by computer search. This leads to new lower bounds of eight multicolor and seven 2-color Ramsey numbers, namely, $R(3,3,6) \geq 54$, $R(3,3,7) \geq 72$, $R(3,3,9) \geq 110$, $R(3,3,11) \geq 138$, $R(3,3,3,5) \geq 110$, $R(3,3,3,6) \geq 138$, $R(3,3,3,7) \geq 194$, $R(3,3,4,4) \geq 114$ and $R(4,19) \geq 194$, $R(4,20) \geq 230$, $R(4,21) \geq 242$, $R(5,17) \geq 282$, $R(5,19) \geq 338$, $R(6,17) \geq 500$, and $R(7,17) \geq 548$.

*Research supported by Guangxi Natural Science Foundation
†Visiting professor in Guangxi Computing Center

The classical Ramsey number $R(q_1, q_2, \cdots, q_n)$ (for $n \geq 2$) is the smallest integer r such that if the edges of K_r, the complete graph of order r, are colored with n colors, then for some $i \in \{1, 2, \cdots, n\}$ there is a monochromatic K_{q_i}.

Only nine exact values of 2-color Ramsey numbers have been found up to now. The number of best lower bounds is no more than 40 (see [11]). Prior to our results, no non-trivial lower bounds for $R(q_1, q_2)$ when $q_1 \geq 4$ and $q_2 \geq 16$ were known. The following are two of the results we obtained previously:

$$\begin{array}{ll} R(4,12) \geq 128 & [13] \\ R(6,12) \geq 224 & [7]. \end{array}$$

When $n \geq 3$, less is known about $R(q_1, q_2, \cdots, q_n)$. So far, only one exact value of a multicolor Ramsey number has been found: $R(3,3,3) = 17$ [4]. Some non-trivial lower and upper bounds have been found, mostly by computer, for example:

$$\begin{array}{ll} 30 \leq R(3,3,4) \leq 31 & [5,10] \\ 45 \leq R(3,3,5) \leq 57 & [1,6,2] \\ 55 \leq R(3,4,4) \leq 79 & [6,2] \\ 87 \leq R(3,3,3,4) \leq 155 & [2,3] \\ 80 \leq R(3,4,5) \leq 161 & [2,3]. \end{array}$$

We ourselves obtained the following:

$$\begin{array}{ll} 458 \leq R(4,4,4,4) & [12] \\ 90 \leq R(3,3,9) & [8] \\ 108 \leq R(3,3,11) & [9]. \end{array}$$

For details, please refer to the authoritative dynamic survey [11], which also provides many references.

In this paper, we present a new algorithm to compute bounds on classical Ramsey numbers. The algorithm is based on circulant graphs of prime order. It is so efficient that the following fifteen new lower bounds are obtained, of which eight are for multicolor Ramsey numbers and the other seven (which have been quoted by Radziszowski [11]) are for 2-color ones.

Theorem 1 $R(3,3,6) \geq 54$, $R(3,3,7) \geq 72$, $R(3,3,9) \geq 110$, $R(3,3,11) \geq 138$, $R(3,3,3,5) \geq 110$, $R(3,3,3,6) \geq 138$, $R(3,3,3,7) \geq 194$, $R(3,3,4,4) \geq 114$; $R(4,19) \geq 194$, $R(4,20) \geq 230$, $R(4,21) \geq 242$, $R(5,17) \geq 282$, $R(5,19) \geq 338$, $R(6,17) \geq 500$, $R(7,17) \geq 548$.

1 Circulant graphs of prime order

For a given prime integer $p = 2m+1$, let $Z_p = \{-m, \cdots, -1, 0, 1, \cdots, m\} = [-m, m]$. (For $s \leq t$, we denote the set $\{s, s+1, \cdots, t\}$ by $[s, t]$.) In the rest of the paper, unless the contrary is implied, all integers will be considered to be members of Z_p, all arithmetic will be modulo p, and "=" will be used to denote "congruent modulo p". In addition, $\pi_n(S)$ will denote an n-partition $\{S_1, S_2, \cdots, S_n\}$ of the set S.

Definition 1 Let n be an integer, $n \geq 2$; put $S = [1, m]$, and let $\pi_n(S)$ be an n-partition of S. Suppose that $V = Z_p$ is the vertex set of the complete graph K_p, so the edge set E of K_p consists of all the 2-element subsets of Z_p. The partition $\pi_n(S)$ induces the following partition of E:

$$\pi_n(E) = \{E_1, E_2, \cdots, E_n\}, \text{ where } E_i = \{\{x, y\} \in E : |x - y| \in S_i\}.$$

The edges in E_i are said to be colored with *color* S_i. This gives a coloring of K_p with n colors. Denote by $G_p(S_i) = (Z_p, E_i)$ the subgraph of K_p whose edges are the edges of color S_i.

Lemma 1 The map $f : x \mapsto -x$ on Z_p is a graph isomorphism of $G_p(S_i)$ onto itself. //

The graphs $G_p(S_i)$ (where $i \in [1, n]$), are a special type of Cayley graph, and are called circulant graphs.

Denote the clique number of $G_p(S_i)$ by $c_i = c(G_p(S_i))$. Immediately from Definition 1, we have:

Theorem 2 $R(c_1 + 1, c_2 + 1, \cdots, c_n + 1) \geq p + 1$. //

2 The cliques and clique number of $G_p(S_i)$

Now we consider the cliques and clique number of $G_p(S_i)$. It is well known that circulant graphs are vertex-transitive, and hence the clique number of $G_p(S_i)$ is equal to the maximum order of those cliques which include the vertex 0. Thus we only need study the cliques including vertex 0. From Definition 1 we know that the other non-zero vertices of such a clique are the elements of the set $A_i = \{x : |x| \in S_i\}$. Let $G_p[A_i]$ be the subgraph of $G_p(S_i)$ induced by the set A_i, and write $[A_i]$ for the clique number of $G_p[A_i]$. We have:

Lemma 2 $c_i = [A_i] + 1.$//

Hence, the clique number of $G_p(S_i)$ can be obtained by finding the clique number of $G_p[A_i]$. For this purpose, we introduce a total order on A_i.

Definition 2 For $x \in A_i$, let

$$d_i(x) = |\{y \in A_i : |y - x| \in S_i\}|.$$

We define the order $\prec$ on A_i as follows:

(i) The pair $\{a, -a\}$ of A_i constitutes an interval for $\prec$ and $a \prec -a$ *iff* $a \in S_i$.

(ii) For two different pairs $\{a, -a\}$ and $\{b, -b\}$ of A_i, with $x \in \{a, -a\}$ and $y \in \{b, -b\}$, we stipulate $x \prec y$ *iff* $d_i(x) < d_i(y)$, or $d_i(x) = d_i(y)$ and the representative elements a and b satisfy $a < b$ as integers, $(a, b \in S_i)$.

Notice that exactly one of a, $-a$ is in S_i for each $a \in A_i$, and

$$y \in A_i, |y-a| \in S_i \Leftrightarrow -y \in A_i, |-y+a| \in S_i.$$

Hence $d_i(a) = d_i(-a)$. This means that $\prec$ is well defined, and $(A_i, \prec)$ is a totally ordered set. $x \prec y$ is read as "x precedes y" or "y succeeds x".

Definition 3 A chain $x_0 \prec x_1 \prec \cdots \prec x_k$ (where $k \geq 1$) in the totally ordered set $(A_i, \prec)$ is said to have *length* k, and to *start at the point* x_0. The chain is said to have *color* S_i provided $|x_h - x_j| \in S_i$ for all h, j with $0 \leq h < j \leq k$. The maximum length of the chains starting at the point x_0 and having color S_i is denoted by $l_i(x_0)$. If there is no chain starting at the point x_0 with length $k \geq 1$ and having color S_i, we put $l_i(x_0) = 0$.

Lemma 3 $[A_i] = 1 + \max\{l_i(a) : a \in S_i\}$.

Proof This is obvious if $[A_i] = 1$.

Suppose $[A_i] = 1 + k$, where $k \geq 1$. If $x_0 \prec x_1 \prec \cdots \prec x_k$ is a chain in A_i having color S_i, then the $k+1$ elements $x_0, x_1, \cdots, x_k$ constitute a clique of $G_p[A_i]$. Thus $[A_i] \geq 1 + \max\{l_i(a) : a \in S_i\}$. Conversely, suppose $[A_i] \leq 1 + \max\{l_i(a) : a \in S_i\}$. Since $[A_i] = 1 + k$, there exists a $(k+1)$-element clique in $G_p[A_i]$. By arranging these $k+1$ elements in their $\prec$-order, we obtain a chain of length k in $(A_i, \prec)$ having color S_i, starting at the element x_0 say. Hence $l_i(x_0) \geq k$. Let $x_0 \prec x_1 \prec \cdots \prec x_r$ be any chain in A_i starting at x_0 and having color S_i. By Lemma 1, $f : x \mapsto -x$ is an isomorphism of $G_p(S_i)$ to itself. Thus f maps the clique $x_0, x_1, \cdots, x_r$ to the clique $-x_0, -x_1, \cdots, -x_r$ in A_i, and it follows from Definition 2 that $-x_0 \prec -x_1 \prec \cdots \prec -x_r$ is a chain in $(A_i, \prec)$ having color S_i. Hence $l_i(x_0) = l_i(-x_0)$. Since either $x_0 \in S_i$ or $-x_0 \in S_i$, we get that $k \leq \max\{l_i(a) : a \in A_i\}$. Hence $[A_i] = 1 + k \leq 1 + l_i(x_0) \leq 1 + max\{l_i(a) : a \in S_i\}$. The result follows. //

Notice that for $a \in S_i$, if $|y - a| \notin S_i$ for all $y \in A_i$ then by the definition of $d_i(a)$ and $l_i(a)$ we have $d_i(a) = 0$ and $l_i(a) = 0$. By Lemma 3, we can easily show:

Lemma 4 $\max\{d_i(a) : a \in S_i\} = 0 \Leftrightarrow [A_i] = 1.$//

3 Description of the algorithm

The above theory provides a new method for computing the clique number of $G_p(S_i)$. Usually, a depth-first search algorithm is employed to calculate the clique number, but its computing time is an exponential function of the number of nodes. Because $|A_i|$ is far less than p, the computation of the clique number of $G_p[A_i]$ is far easier than that of $G_p(S_i)$. In addition, the way of ordering $(A_i, \prec)$ can also save backtracking time. Assume $t = |S_i|$ and

$$(A_i, \prec) = \{a_1, -a_1, a_2, -a_2, \cdots, a_t, -a_t\}, \tag{1}$$

where $d_i(a_1) \leq d_i(a_2) \leq \cdots \leq d_i(a_t)$. Except in the special case that $d_i(a_t) = 0$, $d_i(a_1)$ is generally far less than $d_i(a_t)$. Thus, computational efficiency is raised through successively computing $l_i(a_1), l_i(a_2), \cdots, l_i(a_t)$ according to Definition 3.

Now, we describe the algorithm which we have developed to calculate lower bounds of the classical Ramsey numbers $R(q_1, q_2, \cdots, q_n)$.

1° For n (with $n \geq 2$) ordered integers $q_1, q_2, \cdots, q_n$ (all $q_i \geq 3$), select an odd prime number $p = 2m + 1$, put $S = [1, m]$, then construct a partition $\pi_n(S) = \{S_1, S_2, \cdots, S_n\}$, and generate the graphs $G_p(S_i)$ as given by Definition 1. Initialize i at $i = 1$.

2° Put $A_i = \{x : |x| \in S_i\}$. For each $a \in S_i$ calculate $d_i(a) = |\{y \in A_i : |y - a| \in S_i\}|$. If $\max\{d_i(a) : a \in S_i\} = 0$, go to 6° (and by Lemma 4 we have $[A_i] = 1$).

3° Construct the ordered set $(A_i, \prec)$ according to Definition 2 and arrange the elements of A_i as in formula (1). Put $j = 1$.

4° Find all the chains starting at the point $a_j \in S_i$ having color S_i. If there exists one of length $k \geq q_i - 2$, the algorithm stops. Otherwise, by Definition 3 we have $l_i(a_j) \leq q_i - 3$.

5° Put $j = j + 1$. If $j \leq |S_i|$, go to 4°. Otherwise, by Lemma 3 we have

$$[A_i] = 1 + \max\{l_i(a) : a \in S_i\} \leq q_i - 2.$$

6° Put $i = i + 1$. If $i \leq n$, go to 2°. Otherwise, $[A_i] \leq q_i - 2$ holds for all $i \in [1, n]$.

7° By Lemma 2, for the clique number of $G_p(S_i)$, we have $c_i = [A_i] + 1 \leq q_i - 1$, $i \in [1, n]$. By Theorem 2, we have

$$R([A_1] + 2, [A_2] + 2, \cdots, [A_n] + 2) \geq p + 1.$$

Thus,

$$R(q_1, q_2, \cdots, q_n) \geq p + 1.$$

At this stage, the algorithm has successfully completed.

The initial data at step 1° of the above algorithm are set by ourselves. The other jobs are all completed by computer. It is obvious that the initial data have to be elaborately designed if the algorithm is to reach step 7° and give a lower bound of a classical Ramsey number. Otherwise, the algorithm will stop at step 4°. In the latter case, all that we can do is to select new initial data and excute the algorithm again.

Though the selection of proper initial data is difficult, the above algorithm is still rather effective. The following example gives the lower bound $R(3, 3, 4) \geq 30$, which is the best known at present.

Example. $R(3, 3, 4) \geq 30$.

Proof Let $n = 3$, $q_1 = q_2 = 3$, $q_3 = 4$. Select the prime number $p = 29$. Take the partition $\pi_3(S) = \{S_1, S_2, S_3\}$ of $S = [1, 14]$ where $S_1 = \{1, 4, 10, 12\}$, $S_2 = \{2, 5, 6, 14\}$, $S_3 = \{3, 7, 8, 9, 11, 13\}$.

At step 2° of the algorithm, when $i = 1, 2$, we find

$$max\{d_i(a) : a \in S_i\} = 0,$$

and hence $[A_1] = [A_2] = 1$. When $i = 3$, we construct the totally ordered set $(A_i, \prec)$ by Definition 2. We have $d_3(3) = d_3(7) = 3,\ d_3(8) = d_3(9) = 4,\ d_3(11) = d_3(13) = 5$, and thus

$$(A_3, \prec) = \{3, -3, 7, -7, 8, -8, 9, -9, 11, -11, 13, -13\}.$$

Then, by steps 4° and 5°, we obtain $\max\{l_3(a) : a \in S_3\} = 1$. So, from step 5° we have $[A_3] = 2$. Then from steps 6° and 7°, we finally obtain $R(3, 3, 4) \geq 30$.

This is the simplest example in computing lower bounds of multicolor Ramsey numbers. With the above efficient algorithm, we can even obtain the result by hand. It takes far less than 1 second to complete the algorithm in this example on a Pentium 300 computer.

4 Proof of theorem 1

In section 3, we described the algorithm concretely and give a detailed example. Once the initial data are determined the algorithm can be automatically completed on a computer. Thus, we will only give the prime p used and the partition $\pi_n(S) = \{S_1, S_2, \cdots, S_n\}$. (Because $S = [1, m] = \bigcup \pi_n(S)$, we will only write down $S_1, S_2, \cdots, S_{n-1}$.)

Proof of theorem 1

In cases (1) to (4), we select $n = 3$, and thus the partition of S has three elements, $\pi_3(S) = \{S_1, S_2, S_3\}$.

(1) Let $p = 53, S = [1, 26]$, where
$S_1 = \{1, 6, 8, 11, 15, 20, 24\}$,
$S_2 = \{12, 17, 21, 22, 23, 25, 26\}$.
By the algorithm, $[A_1] = 1$, $[A_2] = 1$ and $[A_3] = 4$ and thus $R(3, 3, 6) \geq 54$.

(2) Let $p = 71, S = [1, 35]$, where
$S_1 = \{1, 5, 8, 11, 15, 18, 25, 32\}$,
$S_2 = \{2, 6, 7, 10, 21, 22, 26, 34\}$.
By the algorithm, $[A_1] = 1$, $[A_2] = 1$ and $[A_3] = 5$ and thus $R(3, 3, 7) \geq 72$.

(3) Let $p = 109, S = [1, 54]$, where
$S_1 = \{1, 6, 8, 11, 15, 20, 24, 29, 33, 36, 46, 50\}$,
$S_2 = \{2, 3, 9, 10, 14, 22, 26, 30, 37, 38, 43, 54\}$.
By the algorithm, $[A_1] = 1$, $[A_2] = 1$ and $[A_3] = 7$ and thus $R(3, 3, 9) \geq 110$.

(4) Let $p = 137, S = [1, 68]$, where
$S_1 = \{1, 7, 15, 17, 21, 27, 31, 37, 40, 45, 51, 56\}$,
$S_2 = \{4, 9, 10, 11, 16, 23, 24, 29, 41, 44, 59, 66\}$.
By the algorithm, $[A_1] = 1$, $[A_2] = 1$ and $[A_3] = 9$ and thus $R(3, 3, 11) \geq 138$.

In cases (5) to (8), we select $n = 4$, and thus the partition of S has four elements, $\pi_4(S) = \{S_1, S_2, S_3, S_4\}$.

(5) Let $p = 109$, $S = [1, 54]$, where
$S_1 = \{18, 28, 35, 38, 41, 44, 45, 48, 49, 51, 52, 54\}$,
$S_2 = \{3, 8, 9, 10, 14, 26, 30, 31, 42, 46\}$,
$S_3 = \{1, 5, 7, 11, 13, 15, 19, 27, 33, 36, 50, 53\}$.
By the algorithm, $[A_1] = 1$, $[A_2] = 1$, $[A_3] = 1$ and $[A_4] = 3$ and thus $R(3, 3, 3, 5) \geq 110$.

(6) Let $p = 137$, $S = [1, 68]$, where
$S_1 = \{8, 10, 21, 22, 25, 28, 34, 41, 45, 54, 57, 60\}$,
$S_2 = \{4, 9, 11, 12, 14, 17, 30, 33, 35, 36, 38, 56, 59, 62\}$,
$S_3 = \{1, 3, 7, 15, 20, 24, 26, 32, 37, 43, 49, 53, 55, 66\}$.
By the algorithm, $[A_1] = 1$, $[A_2] = 1$, $[A_3] = 1$ and $[A_4] = 4$ and thus $R(3, 3, 3, 6) \geq 138$.

(7) Let $p = 193$, $S = [1, 96]$, where
$S_1 = \{1, 4, 11, 14, 21, 24, 27, 34, 36, 39, 46, 49, 52, 59, 62, 64, 71, 74, 81, 84, 87, 94\}$,
$S_2 = \{5, 9, 13, 15, 19, 23, 25, 29, 33, 43, 51, 57, 61, 67, 77, 78, 88, 95\}$,
$S_3 = \{8, 10, 17, 26, 32, 35, 38, 44, 56, 60, 63, 69, 83, 85, 90, 96\}$.
By the algorithm, $[A_1] = 1$, $[A_2] = 1$, $[A_3] = 1$ and $[A_4] = 5$ and thus $R(3, 3, 3, 7) \geq 194$.

(8) Let $p = 113$, $S = [1, 56]$, where
$S_1 = \{2, 10, 17, 18, 29, 30, 37, 41, 44, 49, 50, 56\}$,
$S_2 = \{1, 3, 9, 15, 20, 22, 26, 33, 39, 43, 45, 51\}$,
$S_3 = \{4, 6, 7, 8, 14, 16, 23, 25, 27, 34, 35, 36, 40, 47, 53, 55\}$.
By the algorithm, $[A_1] = 1$, $[A_2] = 1$, $[A_3] = 2$ and $[A_4] = 2$ and thus $R(3, 3, 4, 4) \geq 114$.

Cases (9) to (15) are the 2-color Ramsey numbers, so for them we have $n = 2$ and the partition of S has two elements, $\pi_2(S) = \{S_1, S_2\}$.

(9) Let $p = 193$, $S = [1, 96]$, where
$S_1 = \{1, 2, 5, 6, 9, 18, 19, 22, 28, 31, 43, 45, 46, 48, 54, 56, 59, 65, 75, 81, 86, 92, 93, 96\}$.
By the algorithm, $[A_1] = 2$ and $[A_2] = 17$, and thus $R(4, 19) \geq 194$.

(10) Let $p = 229$, $S = [1, 114]$, where
$S_1 = \{1, 3, 4, 6, 10, 13, 15, 18, 24, 26, 34, 36, 47, 49, 51, 53, 56, 67, 75, 77, 78, 82, 89, 90, 94, 95, 106, 107, 112, 114\}$.
By the algorithm, $[A_1] = 2$ and $[A_2] = 18$, and thus $R(4, 20) \geq 230$.

(11) Let $p = 241$, $S = [1, 120]$, where
$S_1 = \{1, 2, 4, 7, 12, 15, 16, 25, 30, 32, 36, 38, 41, 42, 49, 51, 58, 60, 61, 64, 67, 82, 88, 93, 94, 105, 107, 108, 112, 115\}$.
By the algorithm, $[A_1] = 2$ and $[A_2] = 19$, and thus $R(4, 21) \geq 242$.

(12) Let $p = 281, S = [1, 140]$, where
$S_1 = \{1, 3, 15, 19, 21, 22, 23, 27, 32, 37, 41, 42, 44, 46, 47, 48, 51, 52, 54, 59, 60, 61, 65, 67, 77, 79, 82, 93, 95, 96, 99, 100, 104, 107, 109, 110, 113, 115, 116, 131, 133, 139\}$.
By the algorithm, $[A_1] = 3$ and $[A_2] = 15$, and thus $R(5, 17) \geq 282$.

(13) Let $p = 337, S = [1, 168]$, where
$S_1 = \{1, 6, 7, 8, 9, 10, 11, 12, 16, 19, 21, 25, 28, 29, 31, 33, 44, 47, 49, 57, 61, 65, 70, 71, 75, 76, 79, 85, 87, 89, 91, 99, 100, 103, 106, 109, 111, 116, 121, 123, 127, 130, 132, 133, 138, 148, 151, 153, 161, 162, 164, 166\}$.
By the algorithm, $[A_1] = 3$ and $[A_2] = 17$, and thus $R(5, 19) \geq 338$.

(14) Let $p = 499, S = [1, 249]$, where
$S_1 = \{1, 2, 3, 4, 6, 8, 9, 12, 13, 16, 18, 24, 26, 27, 31, 32, 36, 37, 39, 48, 52, 54, 55, 59, 62, 64, 67, 72, 74, 77, 78, 81, 83, 93, 96, 97, 104, 108, 110, 111, 115, 117, 118, 124, 125, 127, 128, 134, 144, 145, 148, 149, 154, 156, 161, 162, 165, 166, 167, 169, 175, 177, 186, 187, 191, 192, 194, 201, 203, 208, 209, 211, 216, 220, 222, 230, 231, 234, 236, 243, 245, 248, 249\}$.
By the algorithm, $[A_1] = 4$ and $[A_2] = 15$, and thus $R(6, 17) \geq 500$.

(15) Let $p = 547, S = [1, 273]$, where
$S_1 = \{1, 2, 5, 6, 8, 12, 13, 16, 17, 20, 25, 26, 28, 29, 30, 37, 38, 39, 40, 46, 47, 48, 51, 53, 56, 58, 60, 61, 62, 65, 67, 70, 71, 72, 73, 75, 76, 88, 89, 91, 92, 95, 96, 99, 100, 102, 106, 107, 109, 111, 115, 117, 121, 124, 130, 140, 144, 145, 150, 153, 156, 157, 158, 159, 168, 172, 174, 176, 178, 179, 180, 181, 183, 187, 189, 190, 194, 197, 199, 200, 201, 203, 211, 213, 214, 219, 222, 224, 228, 230, 231, 233, 234, 235, 237, 238, 240, 243, 250, 254, 255, 261, 265, 271\}$.
By the algorithm, $[A_1] = 5$ and $[A_2] = 15$, and thus $R(7, 17) \geq 548$. //

For the given input data in the various cases of the proof of Theorem 1, the execution time of the algorithm on our Pentium 300 computer was less than one second in cases (1) to (8) for the multicolor Ramsey numbers, whereas for the two color numbers, cases (9), (10) and (12) took from just over an hour to just over three hours, and the remaining cases took from nine and a half hours to nearly fifteen hours.

Acknowledgement
The authors would like to thank the referee and the Managing Editor who carefully read our paper and put forward many valuable comments which have been adopted by us.

References

[1] Exoo G. Constructing Ramsey graphs with a computer. *Congressus Numerantium*, 1987, 59:31–36.

[2] Exoo G. On two classical Ramsey numbers of the form $R(3, n)$. *SIAM Journal of Discrete Mathematics*, 1989, 2:488–490.

[3] Exoo G. Some new Ramsey Coloring. *The Electronic Journal of Combinatorics*, #R29, 1998. 5.

[4] Greenwood R E and Gleason A M. Combinational relations and chromatic graphs. *Canadian Journal of Mathematics*, 1955, 7:1–7.

[5] Kalbfleisch J G. Chromatic graphs and Ramsey theorem. *Ph. D. thesis, University of Waterloo, January 1966.*

[6] Kreher D L, Wei Li and Radziszowski S P. Lower bounds for multi-colored Ramsey numbers from group orbits. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 1988, 4:87–95.

[7] Luo Haipeng, Su Wenlong and Li Qiao. New Lower Bounds of Classical Ramsey Numbers $R(6, 12), R(6, 14)$ and $R(6, 15)$ (English). *Chinese Science Bulletin*, 1998, 43(10):817–818.

[8] Luo Haipeng and Su Wenlong. New lower bound of classical three-color Ramsey number $R(3, 3, 9)$ (Chinese). *Guangxi Computer Application*, 1998, 1:17–19.

[9] Luo Haipeng and Su Wenlong. New lower bound of classical three-color Ramsey number $R(3, 3, 11)$ (Chinese). *Journal of Guangxi Academy of Sciences*, 1998, 14(3):1–3.

[10] Piwakowski K and Radziszowski S P. New upper bound for the Ramsey number $30 \leq R(3, 3, 4) \leq 31$. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 1998, 27:135–141.

[11] Radziszowski S P. Small Ramsey numbers. *The Electronic Journal of Combinatorics*, 1994, DS1, updated on 7/9/1998.

[12] Su Wenlong. The estimation of lower bounds about some Ramsey numbers $R_3(n)$ and $R_4(n)$ (Chinese). *Guangxi Sciences*, 1996, 3(3):3–7.

[13] Su Wenlong, Luo Haipeng and Li Qiao. New Lower Bounds of Classical Ramsey Numbers $R(4, 12), R(5, 11)$ and $R(5, 12)$ (English). *Chinese Science Bulletin*, 1998, 43(6):528.

(Received 27/1/98; revised 5/10/98)