A characterization of uniquely colorable mixed hypergraphs of order n with upper chromatic numbers $n-1$ and $n-2$

Angela Niculitsa*

Department of Mathematical Cybernetics
Moldova State University, A.Mateevici str. 60, MD-2009, Chişinău, Moldova

Heinz-Juergen Voss

Institut für Algebra, Technische Universität Dresden
Mommsenstrasse 13, D-01062, Dresden, Germany

Abstract

A mixed hypergraph consists of two families of subsets of the vertex set: the $\mathcal{D}$-edges and the $\mathcal{C}$-edges. In a suitable coloring of a mixed hypergraph, every $\mathcal{C}$-edge has at least two vertices of the same color, and every $\mathcal{D}$-edge has at least two vertices colored differently. The largest and smallest possible numbers of colors in a coloring are called the upper and lower chromatic numbers, $\bar{\chi}$ and χ, respectively. A mixed hypergraph is uniquely colorable if it has just one coloring apart from permutations of colors.

We characterize all uniquely colorable mixed hypergraphs of order n with $\chi(\mathcal{H}) = \bar{\chi}(\mathcal{H}) = n-1$ and $n-2$.

1 Introduction

Throughout this paper we use terminology similar to that of [2,4]. The concepts not explained here are taken from [1].

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set, $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$ a family of subsets of X, where $n, m \geq 1$ are integers and $|S_i| \geq 2$ for $1 \leq i \leq m$. The pair $\mathcal{H} = (X, \mathcal{S})$ is called a *hypergraph* on X.

For any subset $Y \subseteq X$, the hypergraph $\mathcal{H}/Y = (Y, \mathcal{S}')$ is the *induced subhypergraph* of the hypergraph $\mathcal{H}$ if $\mathcal{S}'$ consists of all those sets in $\mathcal{S}$ which are entirely

*Partially supported by DAAD, TU-Dresden

contained in Y. On the other hand, a *subhypergraph* of $\mathcal{H} = (X, \mathcal{S})$ is a pair $(Y, \mathcal{S}'')$ where $Y \subseteq X$, $\mathcal{S}'' \subseteq \mathcal{S}$, each set $S \in \mathcal{S}''$ is entirely contained in Y, but $\mathcal{S}''$ is not supposed to contain every $S \in \mathcal{S}$ satisfying $S \subseteq Y$.

A *mixed hypergraph* $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is obtained from a hypergraph $(X, \mathcal{S})$ if $\mathcal{S}$ is split into $\mathcal{S} = \mathcal{C} \cup \mathcal{D}$. Here one of the subfamilies $\mathcal{D}$ and $\mathcal{C}$ may be empty, but on the other hand they need not be disjoint. We call every $D \in \mathcal{D}$ a *$\mathcal{D}$-edge*, and every $C \in \mathcal{C}$ a *$\mathcal{C}$-edge*. Further, it is convenient to use the term "$\mathcal{D}$-hypergraph" for the subhypergraph $\mathcal{H}_{\mathcal{D}} = (X, \emptyset, \mathcal{D})$ and "$\mathcal{C}$-hypergraph" for the subhypergraph $\mathcal{H}_{\mathcal{C}} = (X, \mathcal{C}, \emptyset)$. A $\mathcal{D}$-edge $\{x, y\} \in \mathcal{D}$ of size 2 is denoted by (x, y). In this paper $n = |X|$ always denotes the number of vertices of $\mathcal{H}$.

A $\mathcal{D}$-graph ($\mathcal{C}$-graph) is a subhypergraph of $\mathcal{H}_{\mathcal{D}}$ ($\mathcal{H}_{\mathcal{C}}$) containing only D-edges ($\mathcal{C}$-edges) of size 2. For $\mathcal{D}$-graphs all concepts of graph theory can be used as usual, for instance the concept of a complete graph.

An alternating sequence $z_0 S_0 z_1 S_1 z_2 \ldots z_t S_t z_{t+1}$ of vertices $z_0, z_1, \ldots, z_{t+1} \in X$ and $\mathcal{D}$- and $\mathcal{C}$-edges $S_0 S_1, \ldots, S_t \in \mathcal{C} \cup \mathcal{D}$, satisfying $z_j \neq z_{j+1}$, $j = 0, \ldots, t$, $z_0 \in S_0$, $z_{t+1} \in S_t$ and $z_i \in S_{i-1} \cap S_i$ $(i = 1, 2, \ldots, t)$ is called a *path*; and it is called a *$\mathcal{C}$-path* if $S_i \in \mathcal{C}$ and $|S_i| = 2$ for $i = 0, 1, \ldots, t$.

A mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is called *$\mathcal{C}$-connected* if any two different vertices are joined by a $\mathcal{C}$-path.

A *$\mathcal{C}$-component* $\mathcal{H}_1$ of a mixed hypergraph $\mathcal{H}$ is a maximal $\mathcal{C}$-connected mixed induced subhypergraph of $\mathcal{H}$.

A *suitable coloring* of a mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ with t colors is a mapping $c : X \to \{1, 2, \ldots, t\}$ such that any $\mathcal{C}$-edge has at least two vertices of the same color and any $\mathcal{D}$-edge has at least two vertices colored differently. A suitable t-coloring is *strict* if this mapping is surjective.

So, a suitable coloring exists only for $1 \leq t \leq n$. Two suitable colorings are called different if there exist two vertices of $\mathcal{H}$ that have the same color for one of these colorings and different colors for the other one. The maximum (minimum) t for which there exists a strict coloring of a mixed hypergraph $\mathcal{H}$ with t colors is called the upper (lower) chromatic number of $\mathcal{H}$ and is denoted by $\bar{\chi}(\mathcal{H})$ $(\chi(\mathcal{H}))$.

Let $r_i = r_i(\mathcal{H})$ be the number of strict colorings of a mixed hypergraph $\mathcal{H}$ with $i \geq 1$ colors. The vector $R(\mathcal{H}) = (r_1, r_2, \ldots, r_n)$ is said to be the *chromatic spectrum* of $\mathcal{H}$; obviously $R(\mathcal{H}) = (0, \ldots, 0, r_{\chi}, \ldots, r_{\bar{\chi}}, 0, \ldots, 0)$.

The *chromatic polynomial* $P(\mathcal{H}, \lambda)$ describes the number of all different suitable λ-coloring of $\mathcal{H}$.

If for a mixed hypergraph $\mathcal{H}$ there exists at least one suitable coloring, then it is called colorable. Otherwise $\mathcal{H}$ is called uncolorable.

A set Y of vertices is called $\mathcal{D}$-independent ($\mathcal{C}$-independent) if it does not contain a $\mathcal{D}$-edge ($\mathcal{C}$-edge). I_s always denotes a $\mathcal{D}$-independent set of s vertices.

A mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ with $\mathcal{C} \neq \emptyset$ is called a $\mathcal{C}$-bistar if there exist two $\mathcal{D}$-independent vertices x, y, such that each $\mathcal{C}$-edge contains x and y. The pair $\{x, y\}$ is called a $\mathcal{C}$-bitransversal of $\mathcal{H}$.

A mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ with $\mathcal{C} \neq \emptyset$ is called a hole if $\mathcal{H}$ is not a $\mathcal{C}$-bistar and there exist three $\mathcal{D}$-independent vertices x, y, z, such that each $\mathcal{C}$-edge contains at least two of the vertices x, y, z. The set $\{x, y, z\}$ is called a $\mathcal{C}$-bitransversal of the hole.

A *union of two mixed hypergraphs* $\mathcal{H}_1 = (X_1, \mathcal{C}_1, \mathcal{D}_1)$ and $\mathcal{H}_2 = (X_2, \mathcal{C}_2, \mathcal{D}_2)$ is a mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, with $X = X_1 \cup X_2$, $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ and $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$.

Definition 1 *A mixed hypergraph $\mathcal{H}$ is called uniquely colorable if it has just one strict coloring apart from permutations of colors.*

We will mention some results from [3].

(1) If $\mathcal{H}$ is a uniquely colorable hypergraph, then $\chi(\mathcal{H}) = \bar{\chi}(\mathcal{H})$, $r_\chi(\mathcal{H}) = r_{\bar{\chi}}(\mathcal{H}) = 1$, and $P(\mathcal{H}, \lambda) = \lambda(\lambda-1)(\lambda-2)\ldots(\lambda-\chi(\mathcal{H})+1) = \lambda^{(\chi)}$.

(2) A hypergraph $\mathcal{H} = (X, \mathcal{D})$ (with $\mathcal{C} = \emptyset$) is uniquely colorable if and only if $\mathcal{H}$ is a *complete graph* with $|X|$ vertices. Otherwise $\mathcal{H}$ has colorings with both $|X|-1$ and $|X|$ colors.

(3) A $\mathcal{C}$-hypergraph $\mathcal{H} = (X, \mathcal{C})$ (with $\mathcal{D} = \emptyset$) is uniquely colorable if and only if $\mathcal{H}$ is $\mathcal{C}$-connected. Indeed, it is obvious that if $\mathcal{H}$ is $\mathcal{C}$-connected then $\mathcal{H}$ is uniquely colorable by only one color. Assume now that $\mathcal{H}$ is not $\mathcal{C}$-connected. Let $\mathcal{H}' = \mathcal{H}/X'$, where $X' \subset X$, be a maximal $\mathcal{C}$-connected $\mathcal{C}$-subhypergraph of $\mathcal{H}$. Then we can color all vertices of X' with one color and all vertices of $X \setminus X'$ with a second color. Therefore, $\mathcal{H}$ is not uniquely colorable. (Recall that $\mathcal{H}$ is $\mathcal{C}$-connected if and only if for any two vertices x and y, there exists a $\mathcal{C}$-path from x to y, where each $\mathcal{C}$-edge in the $\mathcal{C}$-path has size 2.)

(4) If a mixed hypergraph $\mathcal{H}$ is colorable, then no $\mathcal{D}$-edge is contained in any $\mathcal{C}$-component of H.

Let $\mathcal{H}$ be a mixed hypergraph having included $\mathcal{C}$-($\mathcal{D}$-)edges and $\mathcal{C}$-edges of size 2. From the coloring point of view, the mixed hypergraph obtained from $\mathcal{H}$ by contracting $\mathcal{C}$-edges of size 2 and deleting $\mathcal{C}$-($\mathcal{D}$-) edges containing some other $\mathcal{C}$-($\mathcal{D}$-)edges has the same properties as $\mathcal{H}$.

A mixed hypergraph $H = (X, \mathcal{C}, \mathcal{D})$ is called *reduced* if all $\mathcal{C}$-edges have size at least 3 and all $\mathcal{D}$-edges have size at least 2, and no $\mathcal{D}$-edge ($\mathcal{C}$-edge) is included in another $\mathcal{D}$-edge ($\mathcal{C}$-edge).

The general problem to characterize the mixed hypergraphs such that $\chi = \overline{\chi} > 0$ was formulated in [4], (p. 43, Problem 4). Uniquely colorable hypergraphs were introduced in [3], in the simplest form they appeared in [2]. In [3] it has been shown that any colorable mixed hypergraph is an induced subhypergraph of some uniquely colorable mixed hypergraph. In general the upper chromatic number of $\mathcal{H}$ is different from $n-1$ and $n-2$. In this paper we shall characterize all reduced mixed hypergraphs uniquely colorable with $n-1$ and $n-2$ colors.

2 Uniquely colorable mixed hypergraphs with $\chi(\mathcal{H}) = \bar{\chi}(\mathcal{H}) = n-1$

In the following only reduced mixed hypergraphs will be considered.

It is an easy consequence of the definition of a $\mathcal{C}$-bistar: the upper chromatic number of a mixed hypergraph $\mathcal{H}$ is $n-1$ if and only if $\mathcal{H}$ is a $\mathcal{C}$-bistar. In this section we will characterize the uniquely colorable $\mathcal{C}$-bistars.

We remind the reader that a $\mathcal{D}$-graph has only $\mathcal{D}$-edges of size 2.

Theorem 1 *A mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ with $n \geq 3$ vertices is uniquely colorable with $n-1$ colors if and only if $\mathcal{H}$ is a $\mathcal{C}$-bistar satisfying the following conditions:*

(1) $\mathcal{H}$ has precisely one $\mathcal{C}$-bitransversal $T = \{x, y\}$;

(2) The set $X \setminus T$ induces a complete $\mathcal{D}$-graph of order $n-2$;

(3) Each vertex $v \notin T$ is joined to the $\mathcal{C}$-bitransversal T by a $\mathcal{D}$-edge of size 2 or 3, i.e., $T \cup \{v\}$ contains a $\mathcal{D}$-edge;

(4) If there are two vertices $v, w \notin T$ such that $(v, x), (w, y) \notin \mathcal{D}$ then there exists a $\mathcal{C}$-edge not containing v and w.

Proof.

$\Rightarrow$ Let $\mathbf{c}$ be the unique $(n-1)$–coloring of the mixed hypergraph $\mathcal{H}$. It will be shown that $\mathcal{H}$ satisfies all conditions of the theorem.

According to the definition of a suitable coloring of a mixed hypergraph with $n-1$ colors, in the coloring $\mathbf{c}$ there exist precisely two $\mathcal{D}$-independent vertices, say x and y, of the same color. Then the pair $\{x, y\}$ is contained in each $\mathcal{C}$-edge of the set $\mathcal{C}$. So, $\mathcal{H}$ is a $\mathcal{C}$-bistar with the $\mathcal{C}$-bitransversal $T = \{x, y\}$.

(1) Suppose $\mathcal{H}$ has two different $\mathcal{C}$-bitransversals T and T'. Color the vertices of T' by the same color and all other vertices by different colors. Then an $(n-1)$-coloring is obtained different from $\mathbf{c}$. This contradiction proves the condition (1) of the theorem.

(2) Assume there are two $\mathcal{D}$-independent vertices $u, v \in X \setminus T$. Replacing the color of vertex the u by the color of vertex the v results in a suitable coloring of $\mathcal{H}$ with $n-2$ colors, a contradiction. So, the condition (2) holds.

(3) Suppose a vertex $v \in X \setminus T$ forms a $\mathcal{D}$-independent set with the two vertices x, y of the $\mathcal{C}$-bitransversal T. In the coloring $\mathbf{c}$ recolor the vertices x and y with the color of v. The result is a suitable coloring of $\mathcal{H}$ with $n-2$ colors, a contradiction. Thus, the condition (3) is proved.

(4) Assume now that there exist two vertices $v, w \in X \setminus T$ with $(v, x), (w, y) \notin \mathcal{D}$ and each $\mathcal{C}$-edge of $\mathcal{H}$ contains v or w. Replace the color of vertex x by the color of v and the color of vertex y by the color of w. The obtained coloring is a suitable coloring of $\mathcal{H}$ with $n-2$ colors. This contradiction proves the condition (4) of the theorem.

Thus, if $\mathcal{H}$ is uniquely colorable with $n-1$ colors then it satisfies the conditions (1)-(4) of the theorem.

$\Leftarrow$ Let $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ have the structure described in (1)-(4) and let $\mathbf{c}$ be any of its suitable colorings.

The mixed hypergraph $\mathcal{H}$ has no n-coloring because, by condition (1), it is a $\mathcal{C}$-bistar with $\mathcal{C} \neq \emptyset$.

According to the condition (2) the set $X \setminus T$ induces a complete graph of order $n-2$. Then the coloring $\mathbf{c}$ has at least $n-2$ different colors. A further consequence is that each $\mathcal{D}$-independent set contains at most one vertex of $X \setminus T$.

Assume that $\mathcal{H}$ is colored with precisely $n-2$ colors. There are the following two possibilities:

1) there are three vertices of the same color forming a $\mathcal{D}$-independent set which contains x and y. This contradicts the condition (3);

2) there are two pair of vertices, say $\{x, u\}, \{y, v\}$, $u, v \in X \setminus T$, such that the vertices of each pair have the same color. According to condition (4) there is a $\mathcal{C}$-edge not containing u and v. Then this $\mathcal{C}$-edge is not suitably colored.

The mixed hypergraph $\mathcal{H}$ can be colored with $n-1$ colors by giving to the vertices x, y the same color.

Suppose that there is a second $(n-1)$-coloring of $\mathcal{H}$, where the vertex x has the color of the vertex $v \in X \setminus T$ and the vertex y has a color different from the colors of all other vertices of X. By condition (1) the pair $\{x, y\}$ is the unique $\mathcal{C}$-bitransversal of $\mathcal{H}$. Then there exists a $\mathcal{C}$-edge $C \in \mathcal{C}$ not containing the vertex v. So, all vertices of this $\mathcal{C}$-edge have different colors.

Consequently, $\mathbf{c}$ is a strict coloring with $n-1$ colors uniquely determined by the structure of the mixed hypergraph $\mathcal{H}$.

□

3 Uniquely colorable mixed hypergraphs with $\chi(\mathcal{H}) = \bar{\chi}(\mathcal{H}) = n-2$

It can easily be shown that the upper chromatic number of a mixed hypergraph $\mathcal{H}$ is $n-2$ if and only if $\mathcal{H}$ is not a $\mathcal{C}$-bistar and $\mathcal{H}$ is a hole or the union of two $\mathcal{C}$-bistars with two disjoint $\mathcal{C}$-bitransversals. Next the conditions are defined under which the union of two $\mathcal{C}$-bistars or a hole are uniquely colorable with $n-2$ colors.

Definition 2 *A union $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ of two $\mathcal{C}$-bistars with $n \geq 4$ vertices has property* **(A)** *if and only if the following three conditions are satisfied:*

(A1) *$\mathcal{H}$ is the union of two $\mathcal{C}$-bistars $S_1 = (X, \mathcal{C}_1, \mathcal{D})$ and $S_2 = (X, \mathcal{C}_2, \mathcal{D})$ with the $\mathcal{C}$-bitransversals T_1 and T_2, respectively, $T_1 \cap T_2 = \emptyset$; but $\mathcal{H} = S_1 \cup S_2$ is not a $\mathcal{C}$-bistar.*

(A2) *$\mathcal{H}$ does not contain:*

- **(A2.1)** *a pair P of $\mathcal{D}$-independent vertices, $P \cap (T_1 \cup T_2) = \emptyset$, i.e., the set $X \setminus (T_1 \cup T_2)$ induces a complete $\mathcal{D}$-graph of order $n-4$;*
- **(A2.2)** *a triple D of $\mathcal{D}$-independent vertices with $D \cap (T_1 \cup T_2) = T_1$ or $D \cap (T_1 \cup T_2) = T_2$.*

(A2.3) *a quadruple* $Q = T_1 \cup T_2$ *of* $\mathcal{D}$*-independent vertices.*

(A3) *If* $\mathcal{H}$ *contains a system* $\Sigma = \{I_1, \ldots, I_{p+q}\}$, $\Sigma \neq \{T_1, T_2\}$, *of* $p+q$ *disjoint sets of* $\mathcal{D}$*-independent vertices with* $|I_1| = \ldots = |I_p| = 2$ *and* $|I_{p+1}| = \ldots = |I_{p+q}| = 3$ *then there is a* $\mathcal{C}$*-edge meeting each of* $I_1, \ldots, I_{p+q}$ *in at most one vertex. For* (p, q) *there are eight cases:* $1 \leq p \leq 4$, *if* $q = 0$, *and* $0 \leq p \leq 2$, *if* $q = 1$, *and* $p = 0$, *if* $q = 2$ *(see Fig. 1).*

We remark that condition (A2.1) implies that at most one vertex of I_j is not in $T_1 \cup T_2$.

In Figures 1 and 2 the possible systems Σ are presented. Each $\mathcal{D}$-independent pair of vertices is depicted by a thin straight vertical line, and each $\mathcal{D}$-independent triple of vertices is depicted by a triangle of thin straight lines. Note that a pair or a triple of Σ can have all its vertices in $T_1 \cup T_2$ (these cases have not been depicted here).

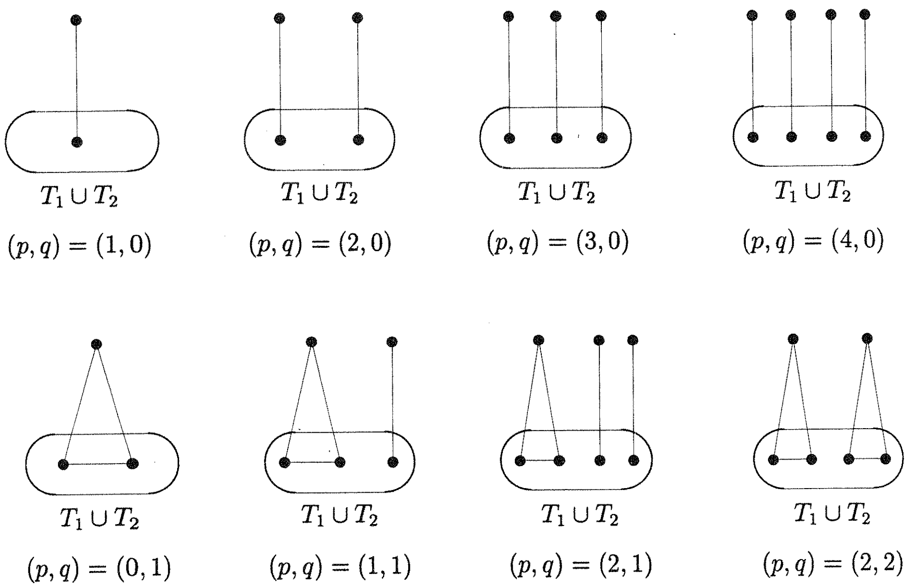

Fig. 1.

Definition 3 *A hole* $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ *with* $n \geq 3$ *vertices has property* **(B)** *if and only if the following conditions are satisfied:*

(B1) $\mathcal{H}$ *is a hole with* $\mathcal{C}$*-bitransversal* T.

(B2) $\mathcal{H}$ *does not contain:*

(B2.1) *a pair P of $\mathcal{D}$-independent vertices, $P \cap T = \emptyset$, i.e., the set $X \setminus T$ induces a complete $\mathcal{D}$-graph of order $n-3$;*

(B2.2) *a quadruple Q of $\mathcal{D}$-independent vertices containing all vertices of T.*

(B3) *If $\mathcal{H}$ contains a system $\Sigma = \{I_1, \dots, I_{p+q}\}$, $\Sigma \neq \{T\}$, of $p+q$ disjoint sets of $\mathcal{D}$-independent vertices with $|I_1| = \dots = |I_p| = 2$ and $|I_{p+1}| = \dots = |I_{p+q}| = 3$ then there is a $\mathcal{C}$-edge meeting each of $I_1, \dots, I_{p+q}$ in at most one vertex.*

For (p,q) there are five cases: $1 \leq p \leq 3$, if $q = 0$, and $0 \leq p \leq 1$, if $q = 1$ (see Fig. 2).

We remark that condition (B2.1) implies that at most one vertex of I_j is not in T.

In Figure 2 the posible systems Σ are presented with the agreements made above.

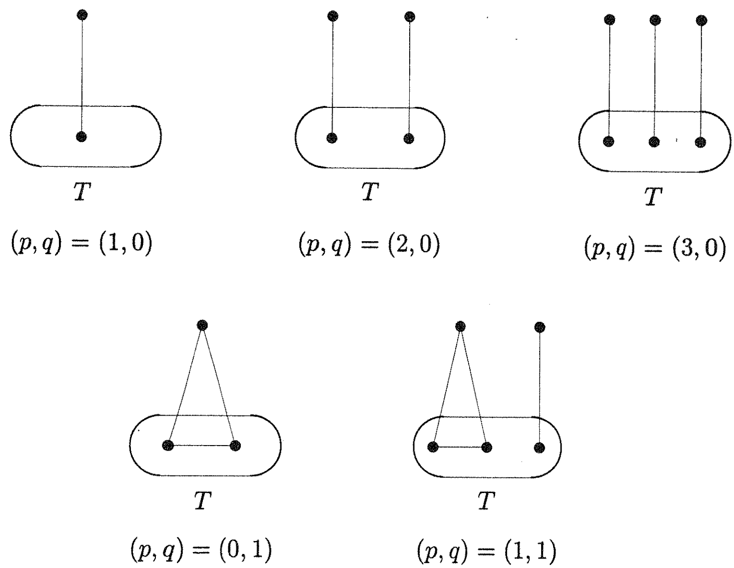

Fig. 2.

Theorem 2 *A mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ with $n \geq 7$ vertices is uniquely colorable with $n-2$ colors if and only if either $\mathcal{H}$ is the union of two $\mathcal{C}$-bistars satisfying* **(A)** *or $\mathcal{H}$ is a hole with property* **(B)**.

Proof.

$\Rightarrow$ Let $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ have a unique coloring **c** with $n-2$ colors.

There are two cases.

Case 1. In the unique coloring **c** let the vertices x_1, y_1 and x_2, y_2 have the same colors: $c(x_1) = c(y_1)$ and $c(x_2) = c(y_2)$, and $c(x_1) \neq c(x_2)$. Let T_i denote the set $\{x_i, y_i\}$, $i = 1, 2$. We shall show that the mixed hypergraph $\mathcal{H}$ is the union of two $\mathcal{C}$-bistars with property **(A)**.

(A1) According to the definition of the suitable coloring of a mixed hypergraph each $\mathcal{C}$-edge $C \in \mathcal{C}$ contains at least two vertices of the same color, i.e., $\{x_1, y_1\} \subseteq C$ or $\{x_2, y_2\} \subseteq C$. Hence $\mathcal{H}$ is the union of at most two $\mathcal{C}$-bistars $S_1 = (X, \mathcal{C}_1, \mathcal{D})$ and $S_2 = (X, \mathcal{C}_2, \mathcal{D})$, where $\mathcal{C}_i$ is the set of all $\mathcal{C}$-edges containing T_i, $i = 1, 2$.

Assume that all $\mathcal{C}$-edges contain precisely one pair of vertices, say $\{u, v\}$. Coloring the vertices u and v with the same color and giving all other vertices different colors results in a suitable coloring of $\mathcal{H}$ with $n-1$ colors. This contradiction shows that $\mathcal{H}$ is not a $\mathcal{C}$-bistar, and there are two $\mathcal{C}$-edges $C_1, C_2 \in \mathcal{C}$ such that $T_i \subseteq C_i$ and $T_i \not\subseteq C_{i+1}$ (indices mod 2). Consequently, $\mathcal{H}$ is a union of two $\mathcal{C}$-bistars S_1 and S_2 so that $C_1 \in \mathcal{C}_1$, $C_1 \notin \mathcal{C}_2$, $C_2 \in \mathcal{C}_2$, $C_2 \notin \mathcal{C}_1$ and $\mathcal{H}$ satisfies the condition (A1).

(A2.1) Suppose that in $\mathcal{H}$ there are two vertices $u, v \in X \setminus (T_1 \cup T_2)$ such that $(u, v) \notin \mathcal{D}$. Recolor u with the color of the vertex v. The obtained coloring is a suitable coloring with $n-3$ colors, a contradiction.

(A2.2) Assume for some i there is a vertex $v \in X \setminus (T_1 \cup T_2)$ forming with x_i and y_i a $\mathcal{D}$-independent triple $\{v, x_i, y_i\}$. Recoloring the vertices x_i, y_i by the color of the vertex v results in a suitable coloring with $n-3$ colors, a contradiction.

(A2.3) Suppose, that $T_1 \cup T_2$ is a quadruple of $\mathcal{D}$-independent vertices. Then the vertices of both $\mathcal{C}$-bitransversals can be recolored with the same color. Hence $\mathcal{H}$ has a suitable coloring with $n-3$ colors, a contradiction.

These contradictions prove the validity of condition (A2).

(A3) Let $S \neq \{T_1, T_2\}$ be a subgraph of $\mathcal{H}$ consisting of $1 \leq p \leq 4$ disjoint pairs of $\mathcal{D}$-independent vertices. By (A2.1) each pair contains at least one vertex of $T_1 \cup T_2$. Assume that every $\mathcal{C}$-edge of $\mathcal{H}$ contains at least one of the pairs. Color each pair with the same color, so that vertices not belonging to the same pair have different colors. The obtained coloring is a suitable coloring with $n-p$ colors different from the coloring **c**, a contradiction.

Assume there is a triple of $\mathcal{D}$-independent vertices and $0 \leq p \leq 2$ disjoint pairs of $\mathcal{D}$-independent vertices containing at most one of T_1 and T_2, such that each $\mathcal{C}$-edge contains at least one pair or at least two vertices from the triple. Color the triple and each pair with the same color, so that vertices not belonging to the triple or to the same pair have different colors. The obtained coloring is a suitable $(n-2-p)$-coloring of $\mathcal{H}$ different from the coloring **c**, a contradiction.

Suppose there are two disjoint triples of $\mathcal{D}$-independent vertices such that each $\mathcal{C}$-edge contains at least two vertices of the same triple. Color each triple with the same color, so that vertices not belonging to the same triple have different colors. The obtained coloring is a suitable $(n-4)$-coloring of $\mathcal{H}$, a contradiction. So, the condition (A3) holds.

Thus, if the mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is uniquely colorable with $n-2$ colors and in its unique coloring there are two disjoint pair of vertices of the same color then $\mathcal{H}$ is the union of two $\mathcal{C}$-bistars with property **(A)**.

Case 2. Let the coloring **c** have a set $T = \{x_1, x_2, x_3\}$ of three vertices with the same color. We shall show that $\mathcal{H}$ is a hole with property **(B)**.

(B1) By the definition of a suitable coloring each $\mathcal{C}$-edge of the hypergraph $\mathcal{H}$ contains at least two vertices of the same color.

Suppose first that all $\mathcal{C}$-edges of the set $\mathcal{C}$ contain one pair of vertices of the same color, say $\{x_1, x_2\}$. Recolor the vertex x_3 with a new color. The obtained coloring is a suitable coloring of $\mathcal{H}$ with $n-1$ colors.

Second, assume that all $\mathcal{C}$-edges of the set $\mathcal{C}$ contain the set $\{x_1, x_2, x_3\}$. Recolor one of vertices of this set with a new color. It results in a new suitable coloring of $\mathcal{H}$ with $n-1$ colors. These two suppositions contradict the unique colorability of $\mathcal{H}$.

So, there are at least two of three possible $\mathcal{C}$-edges $C_i \in \mathcal{C}, i = 1, 2, 3$ such that $\{x_i, x_{i+1}\} \subseteq C_i$ and $x_{i+2} \notin C_i$ (indices mod 3).

Consequently, $\mathcal{H}$ is a hole with $\mathcal{C}$-bitransversal $T = \{x_1, x_2, x_3\}$.

(B2.1) Assume there are two vertices $u, v \in X \backslash T$ such that $(u, v) \notin \mathcal{D}$. Replacing in the coloring **c** the color of the vertex u by the color of vertex v results in a suitable coloring of $\mathcal{H}$ with $n-3$ colors, a contradiction.

(B2.2) Suppose that there is a vertex $v \in X \setminus T$ forming with T a quadruple $\{v, x_1, x_2, x_3\}$ of $\mathcal{D}$-independent vertices. Recolor the vertices x_1, x_2, x_3 with the color of v. We obtain a suitable coloring of $\mathcal{H}$ with $n-3$ colors, a contradiction.

These contradictions prove the validity of condition (B2).

(B3) Assume that there is a vertex set S consisting of $1 \leq p \leq 3$ disjoint pairs of $\mathcal{D}$-independent vertices, such that each $\mathcal{C}$-edge contains at least one pair. Color the pairs with the same color, so that vertices not belonging to the same pair have different colors. The obtained coloring is a suitable $(n-p)$-coloring of $\mathcal{H}$ different from **c**, a contradiction.

Suppose that there is a vertex set $S \neq T$ consisting of a triple of $\mathcal{D}$-independent vertices and $0 \leq p \leq 1$ disjoint pairs of $\mathcal{D}$-independent vertices having no vertex of the triple, and each $\mathcal{C}$-edge contains at least one pair or at least two vertices from the triple. Color the triple and the pairs with the same color, so that vertices not belonging to the triple or to the same pair have different colors. The obtained coloring is a suitable $(n-2-p)$-coloring different from **c**, a contradiction. So, the condition (B3) holds.

Thus, if the mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is uniquely colorable with $n-2$ colors and in its unique coloring there are three vertices of the same color then $\mathcal{H}$ is a hole with property **(B)**.

$\Leftarrow$ Let $\mathcal{H}$ be either the union of two $\mathcal{C}$-bistars satisfying **(A)** or a hole with property **(B)**. Let **c** be any of its suitable colorings. It will be shown that **c** is a unique coloring with $n-2$ colors.

There are two cases.

Case 1. Let $\mathcal{H}$ be the union of two $\mathcal{C}$-bistars satisfying **(A)**.

According to the condition (A2.1) the vertices of the set $X \setminus (T_1 \cup T_2)$ induce a complete subgraph K_{n-4} which is colored by $\mathbf{c}$ with $n-4 \geq 3$ different colors.

Let $\mathbf{c}$ have the color classes $X_1, X_2, \ldots, X_s$, $s \geq 3$. By (A2.2) and (A2.3) each class has at most three vertices. Since $(X \setminus (T_1 \cup T_2), \mathcal{D})$ is a K_{n-4} the number p of pairs among $\{X_1, X_2, \ldots, X_s\}$ is $p \leq 4$ and the number q of triples is $q \leq 2$.

If $q = 2$ then $p = 0$ and $\mathbf{c}$ has precisely two disjoint triples of $\mathcal{D}$-independent vertices and no pair of $\mathcal{D}$-independent vertices. By (A2.2) and (A3) the coloring $\mathbf{c}$ is not suitable.

If $q = 1$ then $p \leq 2$ and $\mathbf{c}$ has precisely one triple of $\mathcal{D}$-independent vertices and at most two pairs of $\mathcal{D}$-independent vertices. By conditions (A2.2) and (A3) the coloring $\mathbf{c}$ is not suitable.

Next the case $q = 0$ and $0 \leq p \leq 4$ will be considered.

If $q = 0$ and $p = 0$ then $\mathbf{c}$ is an n-coloring of $\mathcal{H}$, which is not suitable by (A1).

By $q = 0$ the coloring $\mathbf{c}$ has no triple of $\mathcal{D}$-independent vertices. By $1 \leq p \leq 4$ the coloring $\mathbf{c}$ has a vertex set S of p disjoint pairs of $\mathcal{D}$-independent vertices. If $p = 2$ then let $S \neq \{T_1, T_2\}$. By condition (A3) the coloring $\mathbf{c}$ is not suitable.

Hence $\mathbf{c}$ has no triple of $\mathcal{D}$-independent vertices , and T_1 and T_2 are the only pairs of $\mathcal{D}$-independent vertices of $\mathbf{c}$. Consequently, $\mathbf{c}$ is the unique coloring of $\mathcal{H}$, which is suitable by (A1).

Thus, the coloring $\mathbf{c}$ of the mixed hypergraph $\mathcal{H}$ is a coloring with $n-2$ colors uniquely determined by its structure.

Case 2. Let $\mathcal{H}$ be a hole satisfying **(B)**.

According to the condition (B2.1) the set $X \setminus T$ induces a complete subgraph K_{n-3} which is colored with $n-3 \geq 4$ different colors.

Let $\mathbf{c}$ be a suitable coloring of $\mathcal{H}$ having the color classes $X_1, X_2, \ldots, X_s$, $s \geq 4$. According to condition (B2.2) each color class of the coloring $\mathbf{c}$ has a cardinality at most three. Since $\mathcal{H}$ contains a complete subgraph K_{n-3} the number p of pairs among $\{X_1, X_2, \ldots, X_s\}$ is $p \leq 3$ and the number q of triples is $q \leq 1$.

First the case $q = 0$ and $0 \leq p \leq 3$ is considered.

If $q = 0$ and $p = 0$ then $\mathbf{c}$ is an n-coloring of $\mathcal{H}$ which is not suitable by (B1).

If $q = 0$ and $1 \leq p \leq 3$ then $\mathbf{c}$ has precisely p pairs of $\mathcal{D}$-independent vertices. By (B3) the coloring c is not suitable.

Next the case $q = 1$ and $0 \leq p \leq 1$ is considered.

By $q = 1$ the coloring $\mathbf{c}$ has a triple D of $\mathcal{D}$-independent vertices. By $0 \leq p \leq 1$ the coloring $\mathbf{c}$ has p pairs of $\mathcal{D}$-independent vertices. If $p = 0$ then let $D \neq T$. According to condition (B3) the coloring $\mathbf{c}$ is not suitable.

If $\mathbf{c}$ has no pairs of $\mathcal{D}$-independent vertices and T is the only triple of $\mathcal{D}$-independent vertices of $\mathbf{c}$ then $\mathbf{c}$ is the unique coloring of $\mathcal{H}$, which is suitable by condition (B1).

Hence, the coloring $\mathbf{c}$ of the mixed hypergraph $\mathcal{H}$ is a coloring with $n-2$ colors uniquely determined by its structure.

□

References

[1] C.Berge, *Hypergraphs: combinatorics of finite sets* . North Holland, 1989.

[2] E.Bulgaru, V.Voloshin. *Mixed interval hypergraphs,* Discrete Applied Mathematics, **77(1)** (1997), 24–41.

[3] Zs.Tuza, V.Voloshin, H. Zhou. *Uniquely colorable mixed hypergraphs,* submitted to Discr. Math.

[4] V.Voloshin. *On the upper chromatic number of a hypergraph.* Australasian Journal of Combinatorics, 11 (1995), 25-45.

[5] A. Niculitsa, H.-J. Voss. *About uniquely colorable mixed hypergraphs.* Congressus Numerantium (accepted for publication in Vol. 136 - 141, 1999) .

(Received 30/3/99)