On the maximal number of vertices covered by disjoint cycles

Hong Wang*

Department of Mathematics
The University of Idaho
Moscow, Idaho 83844, USA

Abstract

Let k, t and n be three integers with $t \geq 2$, $k \geq 2t$ and $n \geq 3t$. We conjecture that if G is a graph of order n with minimum degree at least k, then G contains t vertex-disjoint cycles covering at least $min(2k, n)$ vertices of G. We will show the conjecture to be true for $t = 2$.

1 Introduction

We discuss only finite simple graphs and use standard terminology and notation from [1] except as indicated. Let k be an integer with $k \geq 2$. Let G be a graph of order $n \geq 3$. P. Erdős and T. Gallai [5] showed that if G is 2-connected and every vertex of G with at most one exception has degree at least k, then G contains a cycle of length at least $min(2k, n)$. We wonder if G contains at least two vertex-disjoint cycles covering at least $min(2k, n)$ vertices of G. This is certainly true if $k \geq n/2$ with $k \geq 4$ and $n \geq 6$ by El-Zahar's result [4]. El-Zahar proved that if $n = n_1 + n_2$ is an integer partition of n with $n_1 \geq 3$ and $n_2 \geq 3$ and the minimum degree of G is at least $\lceil n_1/2 \rceil + \lceil n_2/2 \rceil$, then G contains two vertex-disjoint cycles of lengths n_1 and n_2, respectively. Corrádi and Hajnal [2] investigated the maximum number of vertex-disjoint cycles in a graph. They proved that if G is a graph of order at least $3t$ with minimum degree at least $2t$, then G contains t vertex-disjoint cycles. In particular, when the order of G is exactly $3t$, then G contains t vertex-disjoint triangles. Motivated by these results, we conjecture the following:

Conjecture *A Let k, t and n be three integers with $t \geq 2$, $k \geq 2t$ and $n \geq 3t$. Suppose that G is a graph of order n with minimum degree at least k. Then G contains t vertex-disjoint cycles covering at least $min(2k, n)$ vertices of G.*

Note that if this conjecture is true, then the condition on the degrees of G is sharp. This can be seen from the graph $K_{k-1,n-k+1}$ with $n > 2(k-1)$. By observing

*This research was supported by UIRC SEED GRANTS–KDY932.

$K_{k,n-k}$, we also see that when $n \geq 2k$, G may not contain t vertex-disjoint cycles covering more than $2k$ vertices of G.

Erdős and Faudree [6] conjectured that if G is a graph of order $4t$ with minimum degree at least $2t$, then G contains t vertex-disjoint cycles of length 4. With respect to this conjecture, we proved [10] that G contains t vertex-disjoint cycles such that $t-1$ of them are of length 4. It follows that G contains t vertex-disjoint cycles covering all the vertices of G such that at least $t-2$ of them are of length 4. Thus Conjecture A is true when $n = 2k = 4t$. In this paper, we will prove the following result.

Theorem B *Let k and n be two integers with $k \geq 4$ and $n \geq 6$. Let G be a graph of order n with minimum degree at least k. Then G contains two vertex-disjoint cycles covering at least $min(2k, n)$ vertices of G.*

We shall use the following terminology and notation. Let G be a graph. For a vertex $u \in V(G)$ and a subgraph H of G, $N(u, H)$ is the set of neighbors of u contained in H, i.e., $N(u, H) = N(u) \cap V(H)$. We let $d(u, H) = |N(u, H)|$. Thus $d(u, G)$ is the degree of u in G. For a subset U of $V(G)$, $G[U]$ denotes the subgraph of G induced by U. The length of a longest cycle of G is denoted by $c(G)$. We define $c_t(G)$ to be the maximal number of vertices of G covered by a set of t vertex-disjoint cycles of G. Thus $c_1(G) = c(G)$.

2 Lemmas

Let $G = (V, E)$ be a given graph in the following. Lemma 2.1 is an easy observation.

Lemma 2.1 *Let C be a cycle of length s in G. Let P be a path of length at least $\lfloor s/2 \rfloor - 1$ in $G - V(C)$. Suppose that x and y are the two endvertices of P with $d(x, C) \geq 1$ and $d(y, C) \geq 1$. Then either $G[V(C \cup P)]$ contains a cycle longer than C, or $N(x, C) = N(y, C) = \{u\}$ for some $u \in V(C)$.*

Lemma 2.2 *Let C be a cycle of length s in G. Let P be a path of length at least 2 in $G - V(C)$. Suppose that x and y are the two endvertices of P and $d(x, C) + d(y, C) > s/2$. Then $G[V(C \cup P)]$ contains a cycle longer than C.*

Proof. Let $C = u_1u_2 \ldots u_su_1$. The subscripts of the u_i's will be reduced modulo s in the following. Clearly, we have

$$2(d(x, C) + d(y, C)) = \sum_{i=1}^{s}(d(x, u_iu_{i+1}) + d(y, u_{i+2}u_{i+3})) > s.$$

This implies that there exists $i \in \{1, 2, \ldots, s\}$ such that $d(x, u_iu_{i+1}) + d(y, u_{i+2}u_{i+3}) \geq 2$. The lemma follows. □

Lemma 2.3 [5] *Let $C = u_1u_2 \ldots u_sx_1$ be a cycle of G. Let $i, j \in \{1, 2, \ldots, s\}$ with $i \neq j$. Suppose that $d(u_i, C) + d(u_j, C) \geq s + 1$. Then for each $\varepsilon \in \{-1, 1\}$, G has a path P from $u_{i+\varepsilon}$ to $u_{j+\varepsilon}$ such that $V(P) = V(C)$, where the subscripts are reduced modulo s.*

Lemma 2.4 [5] *Let $s \geq 2$ be an integer. Suppose that G is 2-connected and every vertex of G with at most one exception has degree at least s. Then G contains a cycle of length at least $min(2s, n)$.*

3 Proof of Theorem B

Let k and n be two integers with $k \geq 4$ and $n \geq 6$. Let $G = (V, E)$ be a graph of order n with $\delta(G) > k$. Suppose, for a contradiction, that G does not contain two vertex-disjoint cycles covering at least $min(2k, n)$ vertices of G, i.e., $c_2(G) < min(2k, n)$. By El-Zahar's result, $n > 2k$. Hence $c_2(G) < 2k$. Let C_0 be a smallest cycle of G, and subject to this, we choose C_0 such that the length of a longest cycle of $G - V(C_0)$ is maximal. Let C_1 be a longest cycle of $G - V(C_0)$. Subject to the choice of C_0 and C_1, we choose C_0 and C_1 such that the length of a longest path of $G - V(C_0 \cup C_1)$ is maximal. Set $H = G - V(C_0)$ and $D = H - V(C_1)$. Let P_0 be a longest path in D and set $D_0 = G[V(P_0)]$. We say that a block of H is an endblock if either the block contains exactly one cut-vertex of H or the block is a component of H.

We claim that C_0 is a triangle. If this is not true, then $d(x, C_0) \leq 2$ for all $x \in V(H)$ for otherwise G contains a smaller cycle than C_0. Hence $\delta(H) \geq k - 2$. Let $P = y_1 y_2 \ldots y_m$ be a longest path in H. Then $d(y_1, P) \geq k - 2$. As H does not contain a triangle, there exists y_i with $i \geq 2(k-2)$ such that $y_1 y_i \in E$. Hence $c(H) \geq 2(k-2)$ and therefore $c_2(G) \geq 2k$, a contradiction. Hence C_0 is a triangle. Then it is easy to see that C_1 exists.

Let $C_0 = u_1 u_2 u_3 u_1$. We divide our proof into the following two cases: $k = 4$ or $k \geq 5$.

Case 1. $k = 4$.

In this case, $c_2(G) \leq 7$. We break into the following two subcases according to whether H is 2-connected.

Case 1.1. H is 2-connected.

Clearly, $c(H) \geq 4$ as $|V(H)| = n - 3 > 4$. Thus $c_2(G) = 7$ and C_1 is of length 4. Let $C_1 = x_1 x_2 x_3 x_4 x_1$. As H is 2-connected, for each $x \in V(D)$, there exist two paths from x to two distinct vertices of C_1 such that x is the only common vertex of the two paths. Then we see that for each $x \in V(D)$, either $N(x, C_1) = \{x_1, x_3\}$ or $N(x, C_1) = \{x_2, x_4\}$ for otherwise $c(H) \geq 5$. Furthermore, D does not contain any edges. Let $x_0 \in V(D)$. Then $d(x_0, C_0) \geq 2$ and so $C_0 + x_0$ is hamiltonian. Consequently, $c_2(G) \geq 8$, a contradiction.

Case 1.2. H is not 2-connected.

Let H_1 and H_2 be two endblocks. Moreover, we choose H_1 and H_2 such that if H has a cut-vertex, then H_1 and H_2 are in the same component of H. For each $i \in \{1, 2\}$, let $x_i \in V(H_i)$ be such that if H_i contains a cut-vertex of H then it is x_i. We break into the following two situations.

Case 1.2(a). There exists $y_1 \in V(H_1 - x_1)$ such that $d(y_1, C_0) \geq 2$. Then $C_0 + y_1$ is hamiltonian. Hence $c(H_2) \leq 3$. This implies that $H_2 - x_2$ contains a vertex z_1 such that $d(z_1, C_0) \geq 2$. Therefore $c(H_1) \leq 3$. It follows that $H_i \cong K_2$ or K_3 for each $i \in \{1, 2\}$.

First, suppose that either $H_1 \cong K_2$ or $H_2 \cong K_2$. Say w.l.o.g. that $H_1 \cong K_2$. Then $d(y_1, C_0) = 3$. Assume that H has a third endblock H_3. Then we also have that $H_3 \cong K_2$ or K_3. Let $w_1 \in V(H_3)$ be such that w_1 is not a cut-vertex of H. Thus $d(w_1, C_0) \geq 2$ and $C_0 + y_1 + w_1$ is hamiltonian. Therefore any block of H other than H_1 and H_3 is of order 2. In particular, $H_2 \cong K_2$. Similarly, we can readily show that $H_3 \cong K_2$. If H_1 and H_2 are not in the same component of H, then by the choice of H_1 and H_2, H must consist of independent edges only, and we see that $c_2(G) \geq 9$ as $e(C_0, H_1 \cup H_2 \cup H_3) = 18$, a contradiction. Therefore H_1 and H_2 are in the same component of H. Notice that $d(w_1, C_0) = d(z_1, C_0) = 3$ where $H_2 = x_2z_1$. As $C_0 - u_1 + w_1$ is a triangle in G, it follows that $x_1 = x_2$ for otherwise $c(H - w_1 + u_1) \geq 5$. If H_3 is in a component D' of H which does not contain H_1, then we see that either $D' = H_3$ and so $G[V(H_3) \cup \{u_2, u_3\}] \cong K_4$, or $G[V(D' + u_2)]$ contains a cycle of length at least 4 by applying the above argument to H_3 and H_4 where H_4 is another endblock of D'. Thus $c_2(G) \geq 8$, a contradiction. This argument allows us to see that H is connected and conclude that $H \cong K_{1,n-4}$ with $d(x_1, H) = n - 4$. It follows that $d(x, C_0) = 3$ for all $x \in V(H) - \{x_1\}$, and consequently, we readily see that $c_2(G) \geq 8$. Therefore H does not have a third endblock. Then it is easy to see that H is a path and $c_2(G) \geq 8$.

Therefore $H_1 \cong K_3$. Similarly, $H_2 \cong K_3$. Let $H_1 = x_1y_1y_2x_1$. Then we see that $C_0 + y_1 + y_2$ is hamiltonian and so $c_2(G) \geq 8$, a contradiction.

Case 1.2(*b*). For each $y \in V(H_1 - x_1)$, $d(y, C_0) \leq 1$.

Similarly, we must have that $d(z, C_0) \leq 1$ for all $z \in V(H_2 - x_2)$. Thus for each $i \in \{1, 2\}$, $d(v, H_i) \geq 3$ for all $v \in V(H_i - x_i)$. Clearly, $c(H_1) \geq 4$ and $c(H_2) \geq 4$. On the other hand, we must have $c(H) \leq 4$ and so $c(H_1) = c(H_2) = 4$. Thus $x_1 = x_2$. Let $P = v_1v_2 \ldots v_m$ be a longest path of H_1 with $v_1 \neq x_1$. Then $N(v_1, H_1) = \{v_2, v_3, v_4\}$ and $d(v_1, C_0) = 1$. It is easy to see that $H_1 \cong K_4$ for otherwise we readily see that either $c(H_1) \geq 5$ or H_1 has a path longer than P. Similarly, $H_2 \cong K_4$. Clearly, $G[V(C_0 \cup H_1 - x_1)]$ contains a cycle of length at least 4. We obtain that $c_2(G) \geq 8$, a contradiction.

Case 2. $k \geq 5$.

Let $C_1 = x_1x_2 \ldots x_sx_1$. Then $s \leq 2k - 4$. We break into the following two cases: $s \geq 2k - 6$ or $s \leq 2k - 7$.

Case 2.1. $s \geq 2k - 6$.

Thus $s \in \{2k - 6, 2k - 5, 2k - 4\}$. Let $P_0 = y_1y_2 \ldots y_r$. As $s = c(H)$, we clearly have

$$d(y, C_1) \leq \lfloor s/2 \rfloor \textit{ for all } y \in V(D). \tag{1}$$

We claim

$$r \geq 4. \tag{2}$$

Proof of (2). On the contrary, suppose $r \leq 3$. First, assume $r = 1$. Then by (1), $d(y, C_0) \geq 2$ for all $y \in V(D)$. Thus $C_0 + y_1$ is hamiltonian and so $s \leq 2k - 5$. Then by (1) again, $d(y, C_0) \geq 3$ for all $y \in V(D)$. Clearly, adding any three vertices of D to C_0 will result in a hamiltonian subgraph of G. Consequently, $c_2(G) \geq 2k$, a contradiction.

Next, assume $r = 2$. If $d(y_1, C_0)+d(y_2, C_0) \leq 2$, then $d(y_1, C_1)+d(y_2, C_1) \geq 2k-4$. By (1), we must have that $d(y_1, C_1) = d(y_2, C_1) = k - 2$. It is easy to see that $C_1 + y_1 + y_2$ contains a cycle of length $s + 1$ or $s + 2$, a contradiction. Hence $d(y_1, C_0) + d(y_2, C_0) \geq 3$. Thus $C_0 + y_1 + y_2$ contains a cycle of length at least 4, and so $s \leq 2k - 5$. If $d(y_1, C_0) + d(y_2, C_0) = 3$, then $d(y_1, C_1) + d(y_2, C_1) \geq 2k - 5$, and consequently, either $d(y_1, C_1) \geq k - 2$ or $d(y_2, C_1) \geq k - 2$, contradicting (1). So $d(y_1, C_0) + d(y_2, C_0) \geq 4$. Thus $C_0 + y_1 + y_2$ is hamiltonian, and so $s = 2k - 6$. If $d(y_1, C_0) + d(y_2, C_0) = 4$, then we have, by (1), that $d(y_1, C_1) = d(y_2, C_1) = k - 3$. Again, we readily see that $C_1+y_1+y_2$ contains a cycle longer than C, a contradiction. Hence $d(y_1, C_0) + d(y_2, C_0) \geq 5$. Let y' be a third vertex of D. Then $d(y', D) \leq 1$ as $r = 2$. Thus $d(y', C_0) \geq 2$ by (1), and consequently, $C_0 + y_1 + y_2 + y'$ is hamiltonian. It follows that $c_2(G) \geq 2k$.

Finally, we assume that $r = 3$. By Lemma 2.2, $d(y_1, C_1) + d(y_3, C_1) \leq \lfloor s/2 \rfloor$. We must have that $d(y_1, C_0) + d(y_3, C_0) \leq 3$ for otherwise $C_0 + y_1 + y_2 + y_3$ is hamiltonian. This implies that $d(y_1) + d(y_3) \leq \lfloor s/2 \rfloor + 3 + 4$. Furthermore, if $d(y_1, C_0) + d(y_3, C_0) = 3$, then $C_0 + y_1 + y_3$ contains a cycle of length at least 4, and so we must have that $s \leq 2k - 5$. It follows that $d(y_1) + d(y_3) < 2k$, a contradiction. So (2) holds. □

By (2) and Lemma 2.2, we obtain

$$d(y_1, C_0) + d(y_r, C_0) \leq 3 \text{ and } d(y_1, C_1) + d(y_r, C_1) \leq \lfloor s/2 \rfloor. \tag{3}$$

Note that if $max(d(y_1, C_0), d(y_r, C_0)) \geq 2$, then $C_0 + y_1 + y_r$ contains a cycle of length at least 4 and so $s \leq 2k - 5$. Together with (3), we obtain

$$d(y_1, P_0) + d(y_r, P_0) \geq k. \tag{4}$$

By (4), we see that either $d(y_1, P_0) \geq \lceil k/2 \rceil$ or $d(y_r, P_0) \geq \lceil k/2 \rceil$, and so $c(D_0) \geq \lceil k/2 \rceil + 1$. As $c_2(H) < 2k$, $4 \leq \lceil k/2 \rceil + 1 \leq 5$. It follows

$$k \in \{5, 6, 7, 8\} \text{ and } s \in \{2k - 6, 2k - 5\}. \tag{5}$$

We now break into the following two situations.

Case 2.1(a). $s = 2k - 5$.

Then $c(G - V(C_1)) \leq 4$. W.l.o.g., say $d(y_1, P_0) \geq d(y_r, P_0)$. Then we must have that $k \in \{5, 6\}$ and $N(y_1, P_0) = \{y_2, y_3, y_4\}$. Then D_0 has a hamiltonian path from y_i to y_r for each $i \in \{1, 2, 3\}$. By Lemma 2.2, $d(y_1, C_1) + d(y_3, C_1) \leq k - 3$. First, suppose that $d(y_r, C_0) \geq 1$. Then we must have that $d(y_i, C_0) = 0$ for each $i \in \{1, 2, 3\}$. Consequently, $d(y_1, P_0) + d(y_3, P_0) \geq k + 3$. It follows that $c(D_0) \geq 5$, a contradiction. Therefore, we must have that $d(y_r, C_0) = 0$. By (1), $d(y_r, C_1) \leq k - 3$ and so $d(y_r, P_0) = 3$, too. Similarly, we can readily show that $d(y_1, C_0) = 0$, $d(y_1, P_0) + d(y_r, P_0) \geq k + 3$ and $c(D_0) \geq 5$, a contradiction.

Case 2.1(b). $s = 2k - 6$.

Note that $4 \le s \le 10$ by (5). First, suppose that $d(y_1, C_0) \ge 1$ and $d(y_r, C_0) \ge 1$. Then we must have that $N(y_1, C_0) = N(y_r, C_0) = \{u_i\}$ for some $i \in \{1,2,3\}$ and $r = 4$ for otherwise $c(G[V(C_0 \cup P_0)]) \ge 6$. If $d(y_1, C_1) = 0$, then $d(y_1, P_0) \ge k-1 \ge 4$ and so $r \ge 5$, a contradiction. Hence $d(y_1, C_1) \ge 1$, and similarly, $d(y_r, C_1) \ge 1$. Then we see that $c(H) \ge 5$ and so $k \ge 6$ by the maximality of s. It is easy to see that if either $max(d(y_1, C_1), d(y_r, C_1)) \ge 2$ or $N(y_1, C_1) \ne N(y_r, C_1)$, then $k = 8$ and $max(d(y_1, C_1), d(y_r, C_1)) \le 2$ for otherwise $c(H) > s$. Hence $d(y_1, C_1) = 1$ for otherwise $d(y_1, P_0) \ge 5$ and so $c(D_0) \ge 6$, a contradiction. It follows that $d(y_1, P_0) \ge k - 2 \ge 4$ and so $r \ge 5$, a contradiction.

Therefore we may assume w.l.o.g. that $d(y_r, C_0) = 0$. Then $d(y_r, C_1) \ge 1$ for otherwise we readily see that $c(D_0) \ge 6$. We claim that $d(y_1, C_1) = 0$. If this is not true, then $c(H) \ge 5$ and so $k \ge 6$. As $c(D_0) \le 5$, $d(y_r, P_0) \le 4$ and so $d(y_r, C_1) \ge 2$. Then again, we must have that $k = 8$ and $d(y_r, C_1) = 2$ for otherwise $c(H) > s$. Hence $d(y_r, P_0) \ge 6$ and so $c(D_0) \ge 7$, a contradiction. So $d(y_1, C_1) = 0$. Hence $d(y_1, C_0) \ge 1$ for otherwise $c(D_0) \ge 6$.

As $k \ge 5$ and $d(y_1, C_1) = 0$, $d(y_1, P_0) \ge 2$. Let $j + 1$ be the greatest integer in $\{2, 3, \ldots, r\}$ such that $y_1 y_{j+1} \in E$. Then D_0 has a hamiltonian path from y_j to y_r. Similarly, we must have that $d(y_j, C_1) = 0$ and $d(y_j, C_0) \ge 1$. As $y_1 y_{j+1} y_j$ is a path of G, we see that $d(y_1, C_0) = d(y_j, C_0) = 1$ for otherwise $c(G[V(C_0 \cup D_0)]) \ge 6$. This yields that $d(y_1, P_0) + d(y_j, P_0) \ge 2k - 2$, and consequently, $c(D_0) \ge k$. It follows that $k = 5$. But then $s = 4$, contradicting the maximality of s.

Case 2.2. $s \le 2k - 7$.

Clearly, we have that $\delta(H) \ge k - 3$. If H is 2-connected, then $c(H) \ge 2k - 6$ by Lemma 2.4, a contradiction. Hence H is not 2-connected. Let H_1 and H_2 be two arbitrary endblocks of H. Set $n_1 = |V(H_1)|$ and $n_2 = |V(H_2)|$. As $\delta(H) \ge k - 3$ and by Lemma 2.4, we must have

$$k - 2 \le n_1 \le 2k - 7 \textit{ and } k - 2 \le n_2 \le 2k - 7. \tag{6}$$

By Lemma 2.4, both H_1 and H_2 are hamiltonian. Let $Q_1 = z_1 z_2 \ldots z_{n_1} z_1$ and $Q_2 = y_1 y_2 \ldots y_{n_2} y_1$ be two hamiltonian cycles of H_1 and H_2, respectively such that every $v \in V(H_1 \cup H_2) - \{z_1, y_1\}$ is not a cut-vertex of H.

First, suppose that for each $i \in \{1, 2\}$, G does not have two independent edges between C_0 and H_i. As $\delta(G) \ge k$, this implies that $n_1 \ge k$ and $n_2 \ge k$. Therefore we must have that $z_1 = y_1$ for otherwise $c_2(H) \ge 2k$. As $2k - 7 \ge n_1 \ge k$, $k \ge 7$. As $\delta(H) \ge k - 3$, we have that $\delta(H_i - z_1) \ge k - 4 \ge (n_i - 1)/2$ for each $i \in \{1, 2\}$. Therefore both $H_1 - z_1$ and $H_2 - z_1$ are hamiltonian. Hence we must have that $n_1 = n_2 = k$ for otherwise $c_2(H) \ge 2k$. Therefore $d(z_i, C_0) \ge 1$ for all $i \in \{2, 3, \ldots, k\}$. As there exist no two independent edges between C_0 and H_1, we obtain that $d(z_i, C_0) = 1$ and $d(z_i, H_1) = k - 1$ for all $i \in \{2, 3, \ldots, k\}$. Consequently, $H_1 \cong K_k$, and we readily see that $c(G[V(C_0 \cup H_1 - z_1)]) \ge k$, and so $c_2(G) \ge 2k$, a contradiction.

Therefore we may assume w.l.o.g. that there exist two independent edges between C_0 and H_1. Say $\{u_1 z_i, u_2 z_j\} \subseteq E$ for some $1 \le i < j \le n_1$. If $\{z_i, z_j\} = \{z_2, z_{n_1}\}$, then $c(G[V(C_0 \cup H_1 - z_1)]) \geq k$. Then $n_2 \leq k-1$ for otherwise $c_2(G) \geq 2k$. Hence $d(y_i, C_0) \geq 2$ for all $i \in \{2, 3, \ldots, n_2\}$. As $\delta(H) \geq k-3$ and $n_2 \leq 2k-7$, it is easy to prove that H_2 contains a triangle. Therefore $2k-7 \geq k$ by the maximality of s, and so $k \geq 7$. It follows that there are two independent edges between C_0 and H_2 which are not incident with any of y_2 and y_{n_2}. Therefore by abusing notation, we may assume in the first place that $\{z_i, z_j\} \neq \{z_2, z_{n_1}\}$. Then either $z_1 \notin \{z_{i-1}, z_{j-1}\}$ or $z_1 \notin \{z_{i+1}, z_{j+1}\}$ where the subscripts are taken modulo n_1. We show $k \geq 7$ as follows. As $\delta(H) \geq k-3$, $n_1 \leq 2k-7$ and by Lemma 2.3, H_1 has a hamiltonian path from z_i to z_j and so $c(G[V(C_0 \cup H_1)]) \geq k+1$. As before, we readily see that if $H_2 - y_1$ contains a triangle, then $k \geq 8$. If $H_2 - y_1$ does not contain a triangle, then we must have that $d(y_2, C_0) = d(y_3, C_0) = 3$ and therefore $u_3 y_2 y_3 u_3$ is a triangle. Clearly, $c(H_1 + u_1 + u_2) \geq k$. Then we obtain $k \geq 7$ as $2k-7 \geq k$ by the maximality of s.

Suppose $z_1 \neq y_1$. Then we must have $n_2 = k-2$ by (6) for otherwise $c_2(G) \geq 2k$. Consequently, we see that $H_2 \cong K_{k-2}$ and $d(y_i, C_0) = 3$ for each $i \in \{2, 3, \ldots, k-2\}$. Similarly, we must have that $H_1 \cong K_{k-2}$ and $d(z_i, C_0) = 3$ for all $i \in \{2, 3, \ldots, k-2\}$. Then we see that H does not have a path of length at least 2 from z_1 to y_1 for otherwise $c_2(G) \geq 2k$. Thus H must have a third endblock H_3. Then we may assume that $H_1 \cap H_3 = \emptyset$ and repeat the above argument with H_3 replacing the role of H_2. Clearly, we see that $c_2(G) \geq 3(k-2) + 2 > 2k$, a contradiction.

Therefore $z_1 = y_1$. As $n > 2k$, H has a third endblock H_3, too. Set $n_3 = |V(H_3)|$. Similarly, we can show that $z_1 \in V(H_3)$, $k-2 \leq n_3 \leq k-1$ and $H_3 - z_1$ is hamiltonian. Clearly, $d(y_2, C_2) \geq 2$. As before, using Lemma 2.3, we see that H_1 has a hamiltonian path from z_1 to each $z \in V(H_1) - \{z_1\}$. In particular, H_1 has a hamiltonian path from z_1 to a vertex $z' \in \{z_i, z_j\}$. Then we see that $G[V(C_0 \cup H_1 \cup H_2)]$ is hamiltonian. Hence $c_2(G) \geq 3k-5 > 2k$, a contradiction. This proves the theorem.

4 References

[1] B. Bollobás, Extremal Graph Theory, Academic Press, London(1978).

[2] K. Corrádi and A. Hajnal, On the maximal number of independent circuits in a graph, Acta Math. Acad. Sci. Hungar. 14(1963), 423–439.

[3] G.A. Dirac, Some theorems on abstract graphs, Proc. London Math. Soc. 2(1952), 69–81.

[4] M.H. El-Zahar, On circuits in graphs, Discrete Math. 50(1984), 227–230.

[5] P. Erdős and T. Gallai, On maximal paths and circuits of graphs, Acta Math. Acad. Sci. Hungar. 10(1959), 337–356.

[6] P. Erdős, Some recent combinatorial problems, Technical Report, University of Bielefeld, Nov. 1990.

[7] H. Wang, Partition of a bipartite graph into cycles, Discrete Mathematics, 117 (1993), 287–291.

[8] H. Wang, Covering a graph with cycles, Journal of Graph Theory, Vol. 20, No.2 (1995), 203–211.

[9] H. Wang, Two vertex-disjoint cycles in a graph, Graphs and Combinatorics, 11(1995), 389–396.

[10] H. Wang, On quadrilaterals in a graph, manuscript.

(Received 4/5/99)