Near α-labelings of bipartite graphs*

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Abstract

An α-labeling of a bipartite graph G with n edges easily yields both a cyclic G-decomposition of $K_{n,n}$ and of K_{2nx+1} for all positive integers x. A β-labeling (or graceful labeling) of G yields a cyclic decomposition of K_{2n+1} only. It is well-known that certain classes of trees do not have α-labelings. In this article, we introduce the concept of a *near α-labeling* of a bipartite graph, and prove that if a graph G with n edges has a near α-labeling, then there is a cyclic G-decomposition of both $K_{n,n}$ and K_{2nx+1} for all positive integers x. We conjecture that all trees have a near α-labeling and show that certain classes of trees which are known not to have an α-labeling have a near α-labeling.

1 Introduction

Only graphs without loops and without multiple edges will be considered herein. Undefined graph-theoretic terminology can be found in the textbook by Chartrand and Lesniak [1]. If m and n are integers with $m \leq n$ we denote $\{m, m+1, \ldots, n\}$ by $[m, n]$. Let N denote the set of nonnegative integers and Z_n the group of integers modulo n. If we consider K_m to have the vertex set Z_m, by *clicking* we mean applying the isomorphism $i \rightarrow i+1$. Likewise if we consider $K_{m,m}$ to have the vertex set $Z_m \times Z_2$, with the obvious vertex bipartition, by *clicking* we mean applying the isomorphism $(i, j) \rightarrow (i+1, j)$.

Let K and G be graphs with G a subgraph of K. A *G-decomposition* of K is a set $\Gamma = \{G_1, G_2, \ldots, G_t\}$ of subgraphs of K each of which is isomorphic to G and such that the edge sets of the graphs G_i form a partition of the edge set of K. In this case, we say G *divides* K. If K is K_m or $K_{m,m}$, a G-decomposition Γ is *cyclic* (*purely cyclic*) if clicking is a permutation (t-cycle) of Γ.

A *labeling* or *valuation* of a graph G is one-to-one function from $V(G)$ into N. In 1967, Rosa [7] introduced several types of graph labelings as tools for decomposing

*Research supported by a research grant from Illinois State University

complete graphs into isomorphic subgraphs. These labelings are particularly useful in attacking the following conjectures.

Conjecture 1 (Ringel [6], 1964) *Every tree with n edges divides the complete graph* K_{2n+1}.

Conjecture 2 *Every tree with n edges divides the complete bipartite graph* $K_{n,n}$.

Conjecture 2, which is part of the folklore of the subject, is a special case of the conjecture by Häggkvist that every tree with n edges divides every n-regular bipartite graph [5]. Since every tree with n edges divides a tree with nx edges for all positive integers x, Conjecture 1 implies the following.

Conjecture 3 *Every tree with n edges divides* K_p *for all* $p \equiv 1 \pmod{2n}$.

Let G be a graph with n edges. In 1967, Alex Rosa [7] called a function γ a *ρ-labeling* of G if γ is an injection from $V(G)$ into $\{0,1,\ldots,2n\}$ such that $\{\min\{|\gamma(u)-\gamma(v)|, 2n+1-|\gamma(u)-\gamma(v)|\} : \{u,v\} \in E(G)\}\} = \{1,2,\ldots,n\}$. Rosa proved the following result.

Theorem 1 (Rosa [7], 1967) *Let G be a graph with n edges. A purely cyclic G-decomposition of* K_{2n+1} *exists if and only if G has a ρ-labeling.*

The above result does not necessarily extend to G-decompositions of K_{2nx+1}. Also, if G is bipartite, then a ρ-labeling of G does not necessarily yield a G-decomposition of $K_{n,n}$.

Conjecture 4 (Rosa [7], 1967) *Every tree has a ρ-labeling.*

Rosa [7] also introduced β-labelings. A *β-labeling* of a graph G with n edges is an injection γ from $V(G)$ into $\{0,1,\ldots,n\}$ such that $\{|\gamma(u)-\gamma(v)| : \{u,v\} \in E(G)\} = \{1,2,\ldots,n\}$. Golomb [4] subsequently called such a labeling a *graceful* labeling and that is now the popular term.

Since a β-labeling is also a ρ-labeling, Theorem 1 also applies to "graceful" graphs. Unfortunately, from a graph decomposition point of view, a graceful labeling, which is far more restrictive than a ρ-labeling, offers no additional applications.

Theorem 2 *Let G be a graph with n edges that has a β-labeling. Then there exists a purely cyclic G-decomposition of the complete graph* K_{2n+1}.

Again, Theorem 2 does not necessarily extend to G-decompositions of K_{2nx+1} nor does it necessarily yield a G-decomposition of $K_{n,n}$ when G is bipartite.

The following conjecture is attributed to both Ringel and Kotzig.

Conjecture 5 *Every tree has a β-labeling.*

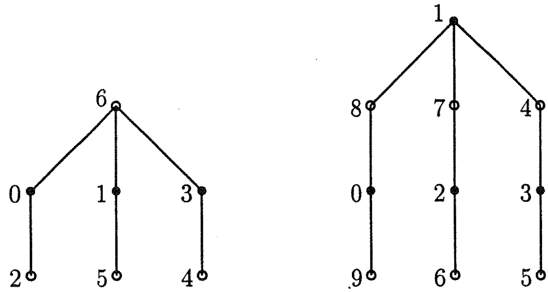

Figure 1: A near α-labeling of $S_{3,2}$ and an α-labeling of $S_{3,3}$

Conjecture 5 is known as *the graceful tree conjecture.* It is one of the best-known problems in the theory of graphs. Since Rosa's 1967 article [7], there have been over 200 research papers related to this conjecture (see Gallian [3]). In spite of many partial results, the conjecture remains open.

A more restrictive labeling than either ρ or β was also introduced by Rosa [7]. An *α-labeling* of G is a β-labeling having the additional property that there exists an integer λ such that if $\{u,v\} \in E(G)$, then $\min\{\gamma(u),\gamma(v)\} \leq \lambda < \max\{\gamma(u),\gamma(v)\}$. Note that if G admits an α-labeling then G is bipartite with parts A and B, where $A = \{u \in V(G) : \gamma(u) \leq \lambda\}$, and $B = \{u \in V(G) : \gamma(u) > \lambda\}$. Rosa proved the following result.

Theorem 3 (Rosa [7], 1967) *Let G be a graph with n edges that has an α-labeling. Then there exists a cyclic G-decomposition of K_{2nx+1} for all positive integers x.*

It can also be easily shown (see [2]) that α-labelings are useful in finding purely cyclic G-decompositions of $K_{n,n}$.

Theorem 4 *If a graph G with n edges has an α-labeling, then there exists a purely cyclic decomposition of the complete bipartite graph $K_{n,n}$ into isomorphic copies of G.*

The condition of having an α-labeling is the most restrictive applied by Rosa, and there are trees which do not admit α-labelings. In particular, he points out [7] that trees of diameter four that contain the comet $S_{3,2}$ as a subtree (See Figure 1) do not admit α-labelings. The *comet* $S_{k,n}$ is the graph obtained from the k-star $K_{1,k}$ by replacing each edge by a path with n edges. We note that not all comets fail to admit an α-labeling (see Figure 1 for an α-labeling of $S_{3,3}$).

In this article we introduce the concept of a near α-labeling of a bipartite graph, and prove that if a graph G with n edges has a near α-labeling, then G divides both $K_{n,n}$ and K_{2nx+1} for all positive integers x. We conjecture that all trees have a near α-labeling and show that certain classes of trees which are known not to have an α-labeling have a near α-labeling. We also show that a result of Snevily [8] on the weak tensor product of graphs with α-labelings extends to graphs with near α-labelings.

2 Near α-labelings

We call γ a *near α-labeling* of a graph G if γ is a graceful labeling of G such that $V(G)$ has a partition V_1, V_2 with the property that each edge of G has the form $\{v_1, v_2\}$, where $v_1 \in V_1$, $v_2 \in V_2$, and $\gamma(v_1) < \gamma(v_2)$. Note that necessarily G is bipartite. The proofs of the following two theorems are essentially the same as those of Theorems 3 and 4.

Theorem 5 *Let G be a graph with n edges that has a near α-labeling. Then there exists a cyclic G-decomposition of K_{2nx+1} for all positive integers x.*

Proof. Let G have the near α-labeling γ, with V_1 and V_2 as in the definition. Consider the bipartite graph G^* with bipartition $V_1^* = V_1$ and $V_2^* = V_2 \times \{0, 1, \ldots, x-1\}$, with the nx edges $\{v_1, (v_2, i)\}$ for $v_1 \in V_1$, $\{v_1, v_2\} \in E(G)$, and $0 \leq i < x$. Note that G divides G^*. Define the labeling γ^* on $V(G^*)$ to be γ on V_1 and $\gamma + in$ on $V_2 \times \{i\}$, $0 \leq i < x$. Then γ^* can be seen to be a near α-labeling of G^*. Thus K_{2nx+1} has a purely cyclic G^*-decomposition by Theorem 2. □

Theorem 6 *If a graph G with n edges has a near α-labeling, then there exists a purely cyclic decomposition of the complete bipartite graph $K_{n,n}$ into isomorphic copies of G.*

Proof. Let G have the near α-labeling γ, with V_1 and V_2 as in the definition. Define $\delta : V(G) \to Z_2$ to be 0 on V_1 and 1 on V_2. Take the vertex set of $K_{n,n}$ to be $Z_n \times Z_2$, with the obvious bipartition. We denote the edge $\{(u, 0), (v, 1)\}$ of $K_{n,n}$ by (u, v). Consider the isomorphism $\psi : G \to K_{n,n}$ given by $\psi(v) = (\gamma(v), \delta(v))$. This is one-to-one since $n \notin V_1$ and $0 \notin V_2$. We claim that clicking $\psi(G)$ gives a purely cyclic G-decomposition of $K_{n,n}$.

Suppose that $(\gamma(u), \gamma(v))$ and $(\gamma(r), \gamma(s))$ are edges of $\psi(G)$ and clicking the first i times gives the same edge as clicking the second j times. Then $\gamma(u) + i = \gamma(r) + j$ and $\gamma(v) + i = \gamma(s) + j$ in Z_n. Then $\gamma(v) - \gamma(u) = \gamma(s) - \gamma(r)$, and since γ is a near α-labeling, $u = r$ and $v = s$. Thus $i = j$ in Z_n. Thus $\psi(G)$ and the graphs formed by clicking it $n - 1$ times are edge disjoint and give a cyclic decomposition of $K_{n,n}$. □

We append to the list of longstanding conjectures in the area by proposing the following.

Conjecture 6 *Every tree has a near α-labeling.*

Let G and H be bipartite graphs with vertex bipartions V_1, V_2 and W_1, W_2, respectively. Snevily [8] defines their *weak tensor product* (with respect to the given bipartitions) to be the bipartite graph $G \bar{\otimes} H$ with vertex bipartition $V_1 \times W_1, V_2 \times W_2$ and with (v_1, w_1) and (v_2, w_2) adjacent if and only if v_1 is adjacent to v_2 in G and w_1 is adjacent to w_2 in H. He proves that if G and H have α-labelings then so does $G \bar{\otimes} H$. Snevily's result considerably enlarges the class of graphs known to have α-labelings. We show a similar result for near α-labelings.

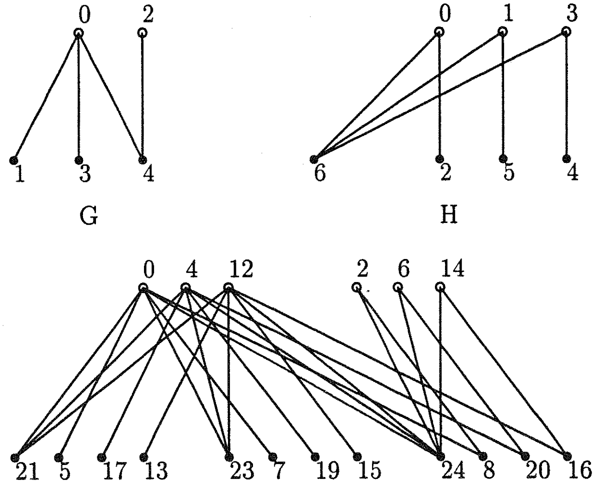

Figure 2: Near α-labelings of G, H and $G\bar{\otimes}H$

Theorem 7 *If the graphs G and H have near α-labelings, then so does their weak tensor product with respect to the corresponding vertex partitions.*

Proof. Let G and H have the near α-labelings γ and δ with corresponding bipartitions V_1, V_2 and W_1, W_2, respectively. Suppose G has m edges and H has n edges. Define σ on $V(G\bar{\otimes}H)$ by

$$\begin{array}{ll} \sigma(v_1, w_1) = m\delta(w_1) + \gamma(v_1) & \text{for } (v_1, w_1) \in V_1 \times W_1, \\ \sigma(v_2, w_2) = m(\delta(w_2) - 1) + \gamma(v_2) & \text{for } (v_2, w_2) \in V_2 \times W_2. \end{array}$$

(See Figure 2 for an example.) It is easily seen that the range of σ is a subset of $[0, mn]$. Now we show that σ is injective. First suppose $\sigma(v_1, w_1) = \sigma(v_1^*, w_1^*)$ with $(v_1, w_1), (v_1^*, w_1^*) \in V_1 \times W_1$. Then $m\delta(w_1) + \gamma(v_1) = m\delta(w_1^*) + \gamma(v_1^*)$ and so $m|\gamma(v_1) - \gamma(v_1^*)$. Thus $v_1 = v_1^*$, and so $w_1 = w_1^*$ also. The proof that if $\sigma(v_2, w_2) = \sigma(v_2^*, w_2^*)$ with $(v_2, w_2), (v_2^*, w_2^*) \in V_2 \times W_2$, then $v_2 = v_2^*$ and $w_2 = w_2^*$ is similar.

Now suppose $\sigma(v_1, w_1) = \sigma(v_2, w_2)$ with $(v_1, w_1) \in V_1 \times W_1$ and $(v_2, w_2) \in V_2 \times W_2$. Then $m\delta(w_1) + \gamma(v_1) = m(\delta(w_2) - 1) + \gamma(v_2)$, and so $m|\gamma(v_2) - \gamma(v_1)$. We conclude that $\gamma(v_2) = m$ and $\gamma(v_1) = 0$. This implies $\delta(w_1) = \delta(w_2)$, which is impossible.

Finally suppose $\{(v_1, w_1), (v_2, w_2)\}$ and $\{(v_1^*, w_1^*), (v_2^*, w_2^*)\}$ are edges of $G\bar{\otimes}H$, with $(v_1, w_1), (v_1^*, w_1^*) \in V_1 \times W_1$ and $(v_2, w_2), (v_2^*, w_2^*) \in V_2 \times W_2$. Note that $\sigma(v_2, w_2) - \sigma(v_1, w_1) = m(\delta(w_2) - 1) + \gamma(v_2) - (m\delta(w_1) + \gamma(v_1)) = m(\delta(w_2) - \delta(w_1) - 1) + \gamma(v_2) - \gamma(v_1) > 0$. Since the weak tensor product has mn edges with labels in $[1, mn]$, it suffices to show they are distinct. Suppose $\sigma(v_2, w_2) - \sigma(v_1, w_1) = \sigma(v_2^*, w_2^*) - \sigma(v_1^*, w_1^*)$. Then

$$m(\delta(w_2) - \delta(w_1) - 1) + \gamma(v_2) - \gamma(v_1) = m(\delta(w_2^*) - \delta(w_1^*) - 1) + \gamma(v_2^*) - \gamma(v_1^*).$$

Thus $m|\gamma(v_2) - \gamma(v_1) - (\gamma(v_2^*) - \gamma(v_1^*))$, and so $v_1 = v_1^*$ and $v_2 = v_2^*$, since γ is a β-labeling. It follows in the same way that $w_1 = w_1^*$ and $w_2 = w_2^*$. □

3 Trees with near α- but no α-labelings

Let $S_{m,2}$ denote the graph with vertices $x, y_1, y_2, \ldots, y_m, z_1, z_2, \ldots, z_m$ and edges $\{x, y_i\}$ and $\{y_i, z_i\}$ for $1 \le i \le m$. If $m > 2$ then $S_{m,2}$ has no α-labeling by the previously mentioned result of Rosa. However, we shall show that $S_{m,2}$ has a near α-labeling for all m.

Lemma 1 *If m is a positive integer congruent to 0 or 1 modulo 3, then $S_{m,2}$ has a near α-labeling γ with $\gamma(x) = 2m$.*

Proof. Note that $S_{m,2}$ has $2m$ edges and $2m+1$ vertices. Set $k = 1 + \lfloor 3m/4 \rfloor$. We define sets

$$\begin{aligned} A &= \{i : 1 \le i < k, i \equiv 1 \pmod 2\}, \\ B &= \{i : k \le i < m, i \equiv m \pmod 2\}, \\ C &= \{i : 1 \le i < k, i \equiv 0 \pmod 2\}, \\ D &= \{i : k \le i < m, i \not\equiv m \pmod 2\}. \end{aligned}$$

Notice that $A \cup B \cup C \cup D = [1, m-1]$. We define a labeling γ by

$$\begin{aligned} &\gamma(x) = 2m, \\ &\gamma(y_i) = i + \lfloor i/3 \rfloor - 1 \text{ for } 1 \le i \le m, \end{aligned}$$

and

$$\gamma(z_i) = \begin{cases} 2i & \text{if } i \in A \cup B, \\ 2m - i + 1 + \lfloor i/3 \rfloor & \text{if } i \in C \cup D, \\ \lfloor 4m/3 \rfloor & \text{if } i = m. \end{cases}$$

We will show that γ is a near α-labeling with $V_1 = \{y_1, y_2, \ldots, y_m\}$ and $V_2 = \{x, z_1, z_2, \ldots, z_m\}$. First we will show that γ maps the vertices of $S_{m,2}$ onto $[0, 2m]$. Since there are $2m+1$ vertices it suffices to show that γ is one-to-one. Note that as a function of i, $\gamma(y_i)$ is increasing on $[1, m]$, while $\gamma(z_i)$ is increasing on $A \cup B$ and decreasing on $C \cup D$ except for the possibility that $\gamma(z_{k-1}) = \gamma(z_k)$ with $k - 1 \in C$ and $k \in D$. This could only happen when m is even and $3|k$. But then $m = 4Q + R$, where R is 0 or 2, which gives $k = 1 + 3Q + \lfloor 3R/4 \rfloor \not\equiv 0 \pmod 3$, a contradiction.

Note that the largest possible element of B is $m - 2$ and the smallest possible element of C is 2. Then we have

$$\begin{aligned} i \in [1, m] &\Rightarrow \gamma(y_i) \in [0, m + \lfloor \tfrac{m}{3} \rfloor - 1] = [0, \lfloor \tfrac{4m}{3} \rfloor - 1], \\ i \in A &\Rightarrow \gamma(z_i) \in [2, 2(k-1)], \\ i \in B &\Rightarrow \gamma(z_i) \in [2k, 2(m-2)], \\ i \in C &\Rightarrow \gamma(z_i) \in [2m - (k-1) + 1 + \lfloor \tfrac{k-1}{3} \rfloor, 2m - 1] \\ i \in D &\Rightarrow \gamma(z_i) \in [2m - (m-1) + 1 + \lfloor \tfrac{m-1}{3} \rfloor, 2m - k + 1 + \lfloor \tfrac{k}{3} \rfloor]. \end{aligned}$$

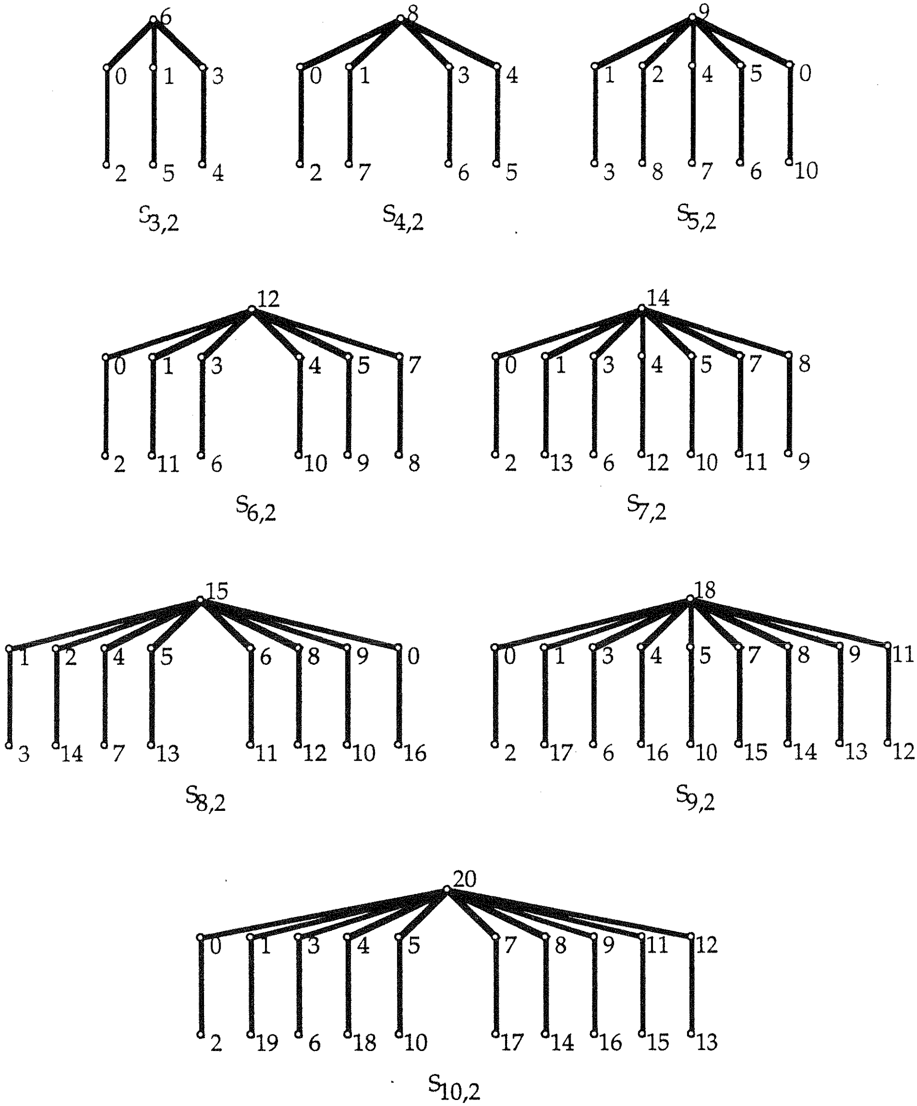

Figure 3: Near α-labelings of $S_{3,2}, S_{4,2}, \ldots, S_{10,2}$.

Now we will establish some congruences for the values of γ. Let $i = 3q + r$, $0 \le r < 3$. Then $\gamma(y_i) = 3q+r+q-1 \equiv r-1 \not\equiv 2 \pmod 4$. Now if $i \in C$, then i is even and $q \equiv r \pmod 2$. Then $\gamma(z_i) = 2m-2q-r+1$. By considering the possible values of r we conclude that $\gamma(z_i) \not\equiv 2m \pmod 4$. Likewise if $i \in D$ then $i \equiv m+1 \pmod 2$. Then $\gamma(z_i) = 2m - i + 1 + \lfloor i/3 \rfloor \equiv 2(i-1) - i + 1 + q \equiv 4q + r - 1 \not\equiv 2 \pmod 4$. Thus we have

$$\begin{aligned} i \in [1,m] &\Rightarrow \gamma(y_i) \not\equiv 2 \pmod 4, \\ i \in A &\Rightarrow \gamma(z_i) \equiv 2 \pmod 4, \\ i \in B &\Rightarrow \gamma(z_i) \equiv 2m \pmod 4, \\ i \in C &\Rightarrow \gamma(z_i) \not\equiv 2m \pmod 4, \\ i \in D &\Rightarrow \gamma(z_i) \not\equiv 2 \pmod 4. \end{aligned}$$

Now we will show that $\gamma(y_i) = \gamma(z_j)$ leads to a contradiction, using the intervals established above. If $j \in C \cup D$, then $\gamma(z_j) \ge m + 2 + \lfloor (m-1)/3 \rfloor > m + \lfloor m/3 \rfloor - 1 \ge \gamma(y_i)$. Now suppose $j \in B$. Then $m + \lfloor m/3 \rfloor - 1 \ge \gamma(y_i) = \gamma(z_j) \ge 2k = 2(1 + \lfloor 3m/4 \rfloor) > 2(3m/4)$. This yields $\lfloor m/3 \rfloor > m/2 + 1$, which is impossible. The congruences above rule out $i \in A$. Clearly $\gamma(z_m) = \lfloor 4m/3 \rfloor > \gamma(y_i)$.

Next we show that the values of $\gamma(z_i)$ are distinct for $i \in A \cup B \cup C \cup D$. Since $\gamma(z_i)$ is strictly monotonic on $A \cup B$ and on $C \cup D$, it suffices to show that $\gamma(z_i) = \gamma(z_j)$ is impossible for $i \in A \cup B$ and $j \in C \cup D$. By the congruences above the only cases to consider are $i \in A$ and $j \in C$ and $i \in B$ and $j \in D$, with m even in the latter case. Now if $i \in A$ and $j \in C$, then $2(k-1) \ge \gamma(z_i) = \gamma(z_j) \ge 2m - k + 2 + \lfloor (k-1)/3 \rfloor > 2m - k + 1 + (k-1)/3$. Then $3m/4 + 1 < k = 1 + \lfloor 3m/4 \rfloor$, a contradiction. Likewise if $i \in B$ and $j \in D$ with m even, then $2k \le \gamma(z_i) = \gamma(z_j) \le 2m - k + 1 + \lfloor k/3 \rfloor \le 2m - k + 1 + k/3$. Then $3m/4 + 3/8 \ge k = \lfloor 3m/4 \rfloor + 1$. This gives $3m/4 - \lfloor 3m/4 \rfloor \ge 5/8$, impossible for m even.

We must also exclude $\gamma(z_i) = \gamma(z_m) = \lfloor 4m/3 \rfloor$. If $i \in C \cup D$, then $\gamma(z_i) \ge m + 2 + \lfloor (m-1)/3 \rfloor > m + 1 + (m-1)/3 = 4m/3 + 2/3 > \gamma(z_m)$. Likewise if $i \in B$ then $\gamma(z_i) \ge 2k = 2(1 + \lfloor 3m/4 \rfloor) > 2(3m/4) = 3m/2 > \gamma(z_m)$. Finally let $m = 3q + r$, where by the hypothesis of this lemma, $r = 0$ or $r = 1$. Then $\gamma(z_m) = \lfloor 4(3q+r)/3 \rfloor = 4q + \lfloor 4r/3 \rfloor \equiv 0$ or $1 \pmod 4$. Thus $\gamma(z_m) = \gamma(z_i)$ for $i \in A$ is impossible.

We have already seen the $\gamma(x) = 2m$ exceeds the value of γ at any other vertex. This concludes the proof that γ is one-to-one.

Now we will show that the edge labels are exactly the set $[1, 2m]$. Since there are $2m$ edges it suffices to show they have distinct labels in this set. Note that

$$\gamma(x) - \gamma(y_i) = 2m - i - \lfloor \frac{i}{3} \rfloor + 1 \in [m - \lfloor \frac{m}{3} \rfloor + 1, 2m] \text{ for } 1 \le i \le m. \tag{1}$$

Also since $i \in B$ implies $i \le m - 2$ we have

$$\gamma(z_i) - \gamma(y_i) = i - \lfloor \frac{i}{3} \rfloor + 1 \in [2, m - \lfloor \frac{m-2}{3} \rfloor - 1] \text{ for } i \in A \cup B. \tag{2}$$

Note that $i - \lfloor i/3 \rfloor + 1$ is an increasing function of i on $A \cup B$ unless

$$(k-1) - \lfloor (k-1)/3 \rfloor + 1 = k - \lfloor k/3 \rfloor + 1,$$

where 3 divides k, $k - 1 \equiv 1 \pmod 2$, and $k \equiv m \pmod 2$. But then m is even, say $m = 4q + r$ with $r = 0$ or 2. Then $k = 1 + \lfloor 3m/4 \rfloor = 1 + 3q + \lfloor 3r/4 \rfloor \not\equiv 0 \pmod 3$, a contradiction.

A third type of edge label is

$$\gamma(z_i) - \gamma(y_i) = 2m - 2i + 2 \in [4, 2m-2] \text{ for } i \in C \cup D. \tag{3}$$

The labels in (1) and (3) are clearly decreasing functions of i. Also $\gamma(z_m) - \gamma(y_m) = 1$. It is easy to see that the intervals in (1) and (2) do not intersect. Thus it suffices to show that no label appearing in (3) appears in (1) or (2).

Let $i = 3q + r$, $0 \le r < 3$. Then the label in (1) is $2m - 4q - r + 1 \not\equiv 2m + 2 \pmod 4$. If this equals the label $2m - 2j + 2$ in (3), then j must be odd. Then by the definition of C we must have $j \in D$ and m even. But if $2m - 2j + 2$ is a label in (1) we must have $m - \lfloor m/3 \rfloor + 1 \le 2m - 2j + 2 \le 2m - 2k + 2$. This yields $2k \le m + \lfloor m/3 \rfloor + 1 \le 4m/3 + 1$. But then $4m/3 + 1 \ge 2(1 + \lfloor 3m/4 \rfloor) \ge 2(1/2 + 3m/4)$, since m is even, and this is impossible.

It remains to show that no label occurs in both (2) and (3). Let $i = 3q + r$, $0 \le r < 3$. Then the edge label in (2) is $2q + r + 1$. Now if i is odd, then $q \not\equiv r \pmod 2$. Examining the possibilities for r we see that the label $2q + r + 1 \not\equiv 0 \pmod 4$. This applies if $i \in A$, and, if m is odd, to $i \in B$ also. If m is even and $i \in B$, then i is even and $q \equiv r \pmod 2$. In this case we see that the edge label $2q + r + 1 \not\equiv 2 \pmod 4$. We see that

$$\gamma(z_i) - \gamma(y_i) \not\equiv \begin{cases} 0 \pmod 4 & \text{if } i \in A, \\ 2m + 2 \pmod 4 & \text{if } i \in B. \end{cases}$$

From the expression in (3) and the definitions of C and D it is easy to see that

$$\gamma(z_j) - \gamma(y_j) \equiv \begin{cases} 2m + 2 \pmod 4 & \text{if } j \in C, \\ 0 \pmod 4 & \text{if } j \in D. \end{cases}$$

From the above congruences we see that the only way we could have $\gamma(z_i) - \gamma(y_i) = \gamma(z_j) - \gamma(y_j)$ would be if m is even and either $i \in A$ and $j \in C$, or else $i \in B$ and $j \in D$. Then if $i \in A$ and $j \in C$ we have $\gamma(z_i) - \gamma(y_i) = i - \lfloor i/3 \rfloor + 1 \le k - 1 - \lfloor (k-1)/3 \rfloor + 1 < k - 1 - (\frac{k-1}{3} - 1) + 1 = \frac{8}{3}(k-1) - 2(k-1) + 2 = \frac{8}{3}\lfloor 3m/4 \rfloor - 2(k-1) + 2 \le 2m - 2(k-1) + 2 \le \gamma(z_j) - \gamma(y_j)$.

Finally, if $i \in B$ and $j \in D$, then $\gamma(z_i) - \gamma(y_i) = i - \lfloor i/3 \rfloor + 1 \ge k - \lfloor k/3 \rfloor + 1 \ge 2k/3 + 1$. Likewise $\gamma(z_j) - \gamma(y_j) = 2m - 2j + 2 \le 2m - 2k + 2$. We will finish this miserable proof by showing that $2k/3 + 1 \le 2m - 2k + 2$ is impossible. This inequality reduces to $8k \le 6m + 3$. Substituting $k = 1 + \lfloor 3m/4 \rfloor$ leads to $3m/4 - \lfloor 3m/4 \rfloor \ge 5/8$. But this is impossible since m is even. □

Theorem 8 *Every graph $S_{m,2}$ has a near α-labeling.*

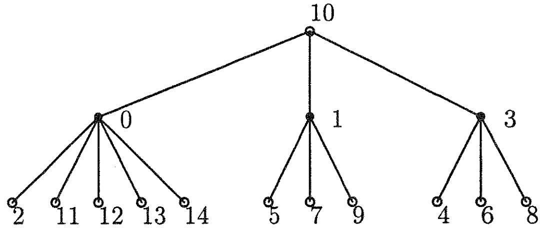

Figure 4: A near α-labeling of $B_{5,3}$

Proof. If m is congruent to 0 or 1 modulo 3, then the lemma applies. Suppose $m \equiv 2 \pmod 3$. Then $S_{m-1,2}$ has a near α-labeling γ such that $\gamma(x) = 2(m-1)$. We consider $S_{m-1,2}$ to be a subgraph of $S_{m,2}$ in the natural way. Define γ^* on the vertices of $S_{m,2}$ by $\gamma^* = \gamma + 1$ on the vertices of $S_{m-1,2}$, $\gamma^*(y_m) = 0$ and $\gamma^*(z_m) = 2m$. It can easily be verified that γ^* is a near α-labeling of $S_{m,2}$. □

Some examples of near α-labelings of $S_{m,2}$ are given in Figure 3. We give another infinite family of trees which have near α-labelings, but no α-labelings.

Let $B_{m,n}$ denote the tree with the $m + 2n + 4$ vertices x, y_i, $1 \le i \le 3$, z_{1j}, $1 \le j \le m$, and z_{ij}, $2 \le i \le 3$, $1 \le j \le n$, and the $m + 2n + 3$ edges $\{x, y_i\}$, $1 \le i \le 3$, $\{y_1, z_{1j}\}$, $1 \le j \le m$, and $\{y_i, z_{ij}\}$, $2 \le i \le 3$, $1 \le j \le n$. Note that if m and n are positive integers, then $B_{m,n}$ has diameter 4 and contains $S_{m,2}$ as a subtree and so does not have an α-labeling by the previously mentioned result of Rosa.

Theorem 9 *If m and n are positive integers, then $B_{m,n}$ has a near α-labeling.*

Proof. Define γ on the vertices of $B_{m,n}$ by $\gamma(x) = 2n + 4$, $\gamma(y_1) = 0$, $\gamma(y_2) = 1$, $\gamma(y_3) = 3$, $\gamma(z_{11}) = 2$, $\gamma(z_{1j}) = 2n + j + 3$, $2 \le j \le m$, and $\gamma(z_{ij}) = 2j + 5 - i$, $2 \le i \le 3$, $1 \le j \le n$ (see Figure 4). We will show that γ is a near α labeling with $V_1 = \{y_1, y_2, y_3\}$. Note that for $1 \le j \le m$ the function γ takes the values $2, 2n + 5, 2n + 6, 2n + 7, \ldots, m + 2n + 3$ on the vertices z_{1j}, while for $1 \le j \le n$ the function γ takes the values $5, 7, 9, \ldots, 2n + 3$ on the vertices z_{2j}, and the values $4, 6, 8, \ldots, 2n + 2$ on the vertices z_{3j}. From this it is easy to see that γ is one-to-one.

Now the differences $\gamma(x) - \gamma(y_i)$ are $\{2n + 1, 2n + 3, 2n + 4\} = A$, the differences $\gamma(z_{1j}) - \gamma(y_1)$ are $\{2\} \cup [2n + 5, m + 2n + 3] = B$, the differences $\gamma(z_{2j}) - \gamma(y_2)$ are $\{4, 6, 8, \ldots, 2n + 2\} = C$, and the differences $\gamma(z_{3j}) - \gamma(y_3)$ are $\{1, 3, 5, \ldots, 2n - 1\} = D$. Note that these are all positive. Moreover $1 \in D$, $2 \in B$, $[3, 2n] \subseteq C \cup D$, $2n + 1 \in A$, $2n + 2 \in C$, $\{2n + 3, 2n + 4\} \subseteq A$, and $[2n + 5, m + 2n + 3] \subseteq B$. Thus γ is a near α-labeling. □

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(Received 24/8/99)