Covering regular graphs with forests of small trees

H. Assiyatun and N. C. Wormald*

Department of Mathematics and Statistics
University of Melbourne
Parkville, VIC 3052, Australia
email: hilda@ms.unimelb.edu.au; nick@ms.unimelb.edu.au

Abstract

A (d,k)*-forest* is a forest consisting of trees whose diameters are at most d and whose maximum vertex degree Δ is at most k. The (d,k)*-arboricity* of a graph G is the minimum number of (d,k)-forests needed to cover $E(G)$. This concept is a common generalization of linear k-arboricity and star arboricity. Using a probabilistic approach developed recently for linear k-arboricity, we obtain an upper bound on the (d,k)-arboricity of r-regular graphs .

1 Introduction

We are concerned in this paper with decompositions of regular graphs into forests. A *linear forest* is a forest consisting entirely of paths, and the *linear arboricity* of a graph G is the minimum number of linear forests required to partition $E(G)$. An outstanding unsolved problem for decompositions of regular graphs into linear forests is the conjecture of Akiyama et al. [1] that every r-regular graph has linear arboricity exactly $\lceil (r+1)/2 \rceil$. This was shown to be asymptotically correct as $r \to \infty$ by Alon [3].

A *linear k-forest* is a forest consisting of paths of length at most k. The *linear k-arboricity* of G, introduced by Bermond et al. [7], is the minimum number of linear k-forests required to partition $E(G)$, and is denoted by $la_k(G)$. The linear k-arboricity of an r-regular graph must be at least $(k+1)r/(2k)$, simply by counting edges. For large k, this is close to the following upper bound shown in [6]: there is an absolute constant $c > 0$ such that for every r-regular graph G and every k, $\sqrt{d} > k \geq 2$, $la_k(G) \leq (k+1)d/(2k) + c\sqrt{kd\log d}$. Moreover, for suitable k this upper bound gives the upper bound in [3] for linear arboricity. In this paper we extend this upper bound so as to apply to forests in which the trees have upper bounds on their diameter and vertex degrees. This concept is a common generalization of the

*Supported by the Australian Research Council

problems of linear arboricity and star arboricity (in which the trees are all stars, as in the papers by Truszczynski [9] and Alon et al. [4]).

Define a *(d,k)-forest* to be a forest consisting of trees whose diameters are at most d and whose maximum vertex degree Δ is at most k. The *(d,k)-arboricity* of a graph G, denoted by $a_{d,k}(G)$, is the minimum number of (d,k)-forests needed to cover $E(G)$.

For any r-regular graph G and $k \geq 2$, it was shown in [6, Lemma 1] that,

$$la_k(G) \leq r.$$

Immediately we have

$$a_{d,k}(G) \leq la_d(G) \leq r. \tag{1}$$

In this paper we give an upper bound on the (d,k)-arboricity of regular graphs, which is better than (1) when r is fairly large. Instead of working directly with (d,k)-arboricity, we will prove the following theorem. To obtain the theorem, which generalizes [6, Theorem 3], we use the same probabilistic approach as in [6].

We use DK_t to denote the complete directed graph on t vertices. We could equally well use K_t, the complete graph on t vertices in which every pair of vertices is joined by double edges, but DK_t is slightly more convenient.

Theorem 1 *There exists a positive constant c such that the following holds. Let G be an r-regular graph and let* $3 \leq t < \sqrt{r}$ *so that for a given set of trees* $\mathcal{T}$ *there is a covering of* DK_t *by t trees in* $\mathcal{T}$. *Then G can be covered by* $\dfrac{rt}{2t-2} + c\sqrt{tr\log r}$ *forests of subtrees of trees in* $\mathcal{T}$.

The proof of this theorem is given in Section 2. In Section 3, we give a decomposition of complete directed graphs into isomorphic directed trees whose diameters and degrees are bounded. By this decomposition, Theorem 1 gives us the desired result of the upper bound on the (d,k)-arboricity of regular graphs.

Theorem 2 *There exists a positive constant c such that for every r-regular graph G and for every* $d,k \geq 2$ *with* $f(d,k) < \sqrt{r}$

$$a_{d,k}(G) \leq \frac{rf(d,k)}{2f(d,k)-2} + c\sqrt{f(d,k)r\log r}$$

where

$$f(d,k) = \begin{cases} 1 + \sum\limits_{i=1}^{l} k(k-1)^{i-1} & \text{if } d = 2l \\ 2\left(1 + \sum\limits_{i=1}^{l}(k-1)^i\right) & \text{if } d = 2l+1. \end{cases}$$

As a maximum tree with diameter d and maximum vertex degree k has $f(d,k)-1$ edges, we note that by counting edges the (d,k)-arboricity of an r-regular graph must be at least $rf(d,k)/(2f(d,k)-2)$.

Since Theorem 2 is obtained using some results in Section 3, we defer the proof until the end of Section 3.

2 Proof of Theorem 1

To prove Theorem 1 we need the following lemma. The proof given here is extracted from the argument in [6], while the undirected version of the lemma can be found in [2].

Lemma 1 *There exists a positive constant c_0 such that the following holds. Let G be an r-regular graph with r is even and let $3 \le k < \sqrt{r}$. Then the vertex set of G can be coloured by k colours $1, \dots, k$ so that for each $v \in V$ and each colour i, $1 \le i \le k$, the numbers $N^+(v,i)$ and $N^-(v,i)$ of out-neighbours and in-neighbours of v in G with colour i satisfy*

$$\left|N^+(v,i) - \frac{r}{2k}\right| \le c_0\sqrt{\frac{r\log r}{k}}, \qquad \left|N^-(v,i) - \frac{r}{2k}\right| \le c_0\sqrt{\frac{r\log r}{k}}. \tag{2}$$

Proof. Orient the edges of G along an Euler cycle. Then each vertex has in-degree and out-degree $r/2$. Colour the vertices randomly by k colours. Let $A^{\pm}_{v,i}$ be the event that $N^{\pm}(v,i)$ does not satisfy inequality (2). Every event $A^-_{v,i}$ is independent of all other events except the events $A^-_{v,j}$ where $i \ne j$ (there are $k-1$ of these), $A^-_{w,j}$ if (u,v) and (u,w) are in G for some vertex u (there are $k(r/2)(r/2-1)$ of these), and event $A^+_{w,j}$ if (u,v) and (w,u) are in G for some vertex u (there are $k(r/2)^2$ of these). Therefore, every event is independent of all except at most $kr(r-1)/2 + k - 1 < r^3$ of the others.

Since $N^{\pm}(v,i)$ is distributed as the binomial random variable $Bin(r/2, 1/k)$, by [5, Theorem A.11], for every $v \in V$ and $1 \le i \le k$

$$Pr(A^+_{v,i}) < e^{-4\log r}, \qquad Pr(A^-_{v,i}) < e^{-4\log r}.$$

Since $e(r^3+1)/r^4 < 1$, by the Lovasz Local Lemma [5, Corollary 1.2] there exists a vertex colouring such that no event $A^{\pm}_{v,i}$ occurs. Hence, there is a vertex colouring which satisfies (2) for all $v \in V$ and $1 \le i \le k$. ∎

Given a set of trees $\mathcal{T}$ let $a_{\mathcal{T}}(G)$ denote the minimum number of forests of subtrees of trees in $\mathcal{T}$ needed to cover edges of G. For convenience, we define

$$a_{\mathcal{T}}(r) = \max_{G \text{ is r-regular}} a_{\mathcal{T}}(G).$$

Then

$$a_{\mathcal{T}}(G) \le a_{\mathcal{T}}(\Delta(G)) \tag{3}$$

since every graph whose maximum degree r occurs as a subgraph of an r-regular graph and the restriction of a forest of subtrees of trees in $\mathcal{T}$ to a subgraph is still such a forest.

We now prove Theorem 1. In view of Lemma 1, we can assume r is even and colour the vertices of G by t colours, such that the set of edges from a vertex of colour i to one colour j, $i \ne j$ forms a bipartite graph whose vertex degrees are at most $F(r) = r/2t + c_0\sqrt{r\log r/t}$. These edges can be covered by $F(r)$ matchings.

Now consider the complete directed graph DK_t where each vertex represents a colour. Suppose this can be covered by t trees in $\mathcal{T}$. Since for every two different colours there are at most $F(r)$ matchings and the t trees in $\mathcal{T}$ cover DK_t, all edges joining vertices of different colours can be covered by

$$tF(r) = t\left(\frac{r}{2t} + c_0\sqrt{\frac{r\log r}{t}}\right) = \frac{r}{2} + c_0\sqrt{tr\log r}$$

forests of subtrees of trees in $\mathcal{T}$. The remaining graph which joins the vertices of the same colours has in-degree (out-degree) at most $F(r)$. Therefore, by equation (3) it can be covered by $a_{\mathcal{T}}(2F(r))$ forests of subtrees of trees in $\mathcal{T}$.

Thus,

$$a_{\mathcal{T}}(G) \leq \frac{r}{2} + c_0\sqrt{tr\log r} + a_{\mathcal{T}}(2F(r)) \tag{4}$$

By applying iteration on r equation (4) gives

$$\begin{aligned} a_{\mathcal{T}}(G) \leq & \left(\frac{r}{2} + c_0\sqrt{tr\log r}\right) + \left(\frac{r_1}{2} + c_0\sqrt{tr_1\log r_1}\right) + \ldots + \left(\frac{r_{i_0}}{2} + c_0\sqrt{tr_{i_0}\log r_{i_0}}\right) \\ & + a_{\mathcal{T}}(r_{i_0+1}) \end{aligned} \tag{5}$$

where $r_i = 2F(r_{i-1})$ for $1 \leq i \leq i_0$ with $r_0 = r$ and $r_{i_0+1} \leq t^2$.

Using $r_i = 2F(r_{i-1})$ for $1 \leq i \leq i_0$ and $a_{\mathcal{T}}(r_{i_0+1}) \leq r_{i_0+1}$ equation (5) can be written as

$$\begin{aligned} a_{\mathcal{T}}(G) \leq & \left(\frac{r}{2} + c_0\sqrt{tr\log r}\right) + \left(\frac{r}{2t} + \frac{c_0}{t}\sqrt{tr\log r} + c_0\sqrt{tr_1\log r_1}\right) \\ & + \left(\frac{r}{2t^2} + \frac{c_0}{t^2}\sqrt{tr\log r} + \frac{c_0}{t}\sqrt{tr_1\log r_1} + c_0\sqrt{tr_2\log r_2}\right) + \ldots \\ & + \left(\frac{r}{2t^{i_0}} + \frac{c_0}{t^{i_0}}\sqrt{tr\log r} + \frac{c_0}{t^{i_0-1}}\sqrt{tr_1\log r_1} + \ldots \right. \\ & \left. + \frac{c_0}{t}\sqrt{tr_{i_0-1}\log r_{i_0-1}} + c_0\sqrt{tr_{i_0}\log r_{i_0}}\right) + t^2 \\ \leq & \frac{r}{2}\left(1 + \frac{1}{t} + \frac{1}{t^2} + \ldots + \frac{1}{t^{i_0}}\right) \\ & + c_0\left[\sum_{j=0}^{i_0}\frac{1}{t^j}\sqrt{tr\log r} + \sum_{j=0}^{i_0-1}\frac{1}{t^j}\sqrt{tr_1\log r_1} + \ldots \right. \\ & \left. + \sum_{j=0}^{1}\frac{1}{t^j}\sqrt{tr_{i_0-1}\log r_{i_0-1}} + \sqrt{tr_{i_0}\log r_{i_0}}\right] + t^2. \end{aligned} \tag{6}$$

We note here that if $r_i/\log r_i > 4c_0^2 t$ then $r_{i+1} = 2F(r_i) < 2r_i/t \leq 2r_i/3$. Otherwise, $t < \sqrt{r_i}$ implies r_i is bounded above, thus the number of terms inside the square brackets in equation (6) corresponding to such r_i is bounded. Hence, in any case the terms inside the square brackets in equation (6) are dominated by $O(\sqrt{tr\log r})$. The assertion follows immediately by choosing c sufficiently large. The case of odd r is done by applying to graphs with maximum degree $r+1$. ∎

3 Decomposition of complete directed graphs into isomorphic directed trees

A directed tree is a tree with oriented edges. (We introduce orientations merely as a convenient way to handle a complete graph with doubled edges.) Given a directed tree T and vertices u and v, there may or may not be a $u-v$ path in T. Therefore in this paper we define the diameter of a directed tree as the diameter of the underlying tree.

Let us define $S_{k,l} = 1 + \sum_{i=1}^{l}(k-1)^i$. Let T be a (directed) tree with diameter $diam(T) \leq d$ and $\Delta(T) \leq k$, where $d, k \geq 2$.
Then

$$|V(T)| \leq \begin{cases} 1 + kS_{k,l-1} & \text{if } d = 2l \\ 2S_{k,l} & \text{if } d = 2l+1. \end{cases}$$

For convenience, we denote the set $\{1, 2, \ldots, k\}$ by $[k]$.

Figure 1: A maximum tree with $\Delta = 4$ and diameter 3

Lemma 2 *For $k \geq 2$ and $l \geq 1$, $DK_{2S_{k,l}}$ can be decomposed into $2S_{k,l}$ isomorphic directed trees whose diameters are $2l+1$ and maximum vertex degrees are k.*

Proof. We construct the $2S_{k,l}$ isomorphic directed trees in the following way.
Put $2S_{k,l}$ points on a circle, and label them as $0, 1, \ldots, 2S_{k,l} - 1$.

A maximum (directed) tree T with $diam(T) = 2l+1$ and $\Delta(T) = k$ has two main branches at the central edge (see Figure 1). Let us consider that in each branch there are l levels of arcs and consider the single arc connecting the two branches as level 0. In level i of each branch there are $(k-1)^i$ arcs, $i = 1, 2, \ldots, l$.
In one main branch, we take arcs in each level in the following way.

1. Level 0:

   The arc $(0, S_{k,l})$.

2. Level r, where r is odd:

   Define $s_{0,1} = 0$.
   For $j \in [(k-1)^r]$, let $s_{r,j} = s_{r-1,i} + (j - (i-1)(k-1))S_{k,l-r}$, where $i = \lceil \frac{j}{k-1} \rceil$.
   Then for each $s_{r-1,j}$, $j \in [(k-1)^{r-1}]$,
   take arcs $\left(s_{r-1,j}, s_{r,(k-1)^r-(j-1)(k-1)-t+1}\right)$, where $t \in [k-1]$.

   For example, if $r = 1$ then from this step we obtain arcs $(s_{0,1}, s_{1,k-t})$ for $t \in [k-1]$.

3. Level r, where r is even:

   Define $s_{r-1,0} = s_{r-2,1}$, where $s_{1,0} = s_{0,1} = 0$.

   For $j \in [(k-1)^r]$, let $s_{r,j} = s_{r-1,i-1} + (j - 1 - (i-1)(k-1))S_{k,l-r} + 1$, where $i = \lceil \frac{j}{k-1} \rceil$.

   Then for each $s_{r-1,j}$, $j \in [(k-1)^{r-1}]$,

   take arcs $\left(s_{r-1,j}, s_{r,(k-1)^r-(j-1)(k-1)-t+1}\right)$, where $t \in [k-1]$.

So far we have constructed one of the main branches of one tree. To obtain the other branch of the tree, repeat all steps after 1 using vertex $S_{k,l}$ in place of 0 and reverse the orientation of the arcs. This gives one copy of the isomorphic directed trees. We can obtain the other copies by rotating the first one around the circle. This means the i-th copy has an arc (j,k) if and only if the first copy has an arc $(j-i+1, k-i+1) \pmod{2S_{k,l}}$. ∎

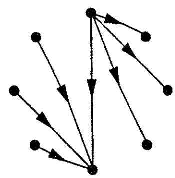

Figure 2: The first factor in DK_8-decomposition

Corollary 1 *$K_{S_{k,l}}$ can be decomposed into $S_{k,l}$ isomorphic trees whose diameters are $2l+1$ and maximum vertex degrees are k, where $k \geq 2$ and $l \geq 1$.*

Proof. We rotate the first copy of the underlying tree around half of the circle instead of the whole circle. ∎

Lemma 3 *Let $R_{k,l} = 1 + kS_{k,l-1}$, with $k \geq 2$ and $l \geq 1$. Then $DK_{R_{k,l}}$ can decomposed into $R_{k,l}$ isomorphic directed trees whose diameters are $2l$ and maximum degrees are k.*

Proof. Put $1 + kS_{k,l-1}$ points on a circle and label them as $0, 1, \ldots, kS_{k,l-1}$. We construct the $R_{k,l}$ isomorphic directed trees in the following way.

A maximum (directed) tree T with $diam(T) = 2l$ and $\Delta(T) = k$ can be viewed as a tree with l levels of arcs. In level i, there are $k(k-1)^{i-1}$ arcs, $i = 1, \ldots, l$ of distance $i-1$ from the central vertex. We define the arcs in each level of one such tree as follows.

1. Level 1:

   Define $s_{0,1} = 0$ and $s_{1,j} = jS_{k,l-1}$, for all $j \in [k]$.

   Take edges $(s_{0,1}, s_{1,j})$, where $j \in [k]$.

2. Level r, where r is even:

   Define $s_{r-1,0} = s_{r-2,1}$, where $s_{1,0} = s_{0,1} = 0$. For $j \in [k(k-1)^{r-1}]$, let $s_{r,j} = s_{r-1,i-1} + (j - 1 - (i-1)(k-1))S_{k,l-r} + 1$, where $i = \lceil \frac{j}{k-1} \rceil$.

   Then for each $s_{r-1,j}$, $j \in [k(k-1)^{r-2}]$, take edges $\left(s_{r-1,j}, s_{r,k(k-1)^{r-1}-(j-1)(k-1)-t+1}\right)$, where $t \in [k-1]$.

   For example, if $r = 2$ and $j = 1$ then from this step we obtain edges $\left(s_{1,1}, s_{2,k(k-1)-t+1}\right)$ for $t \in [k-1]$.

3. Level r, where r is odd:

   For $j \in [k(k-1)^{r-1}]$, let $s_{r,j} = s_{r-1,i} + (j-(i-1)(k-1))S_{k,l-r}$, where $i = \lceil \frac{j}{k-1} \rceil$.

   Then for each $s_{r-1,j}$, $j \in [k(k-1)^{r-2}]$, take edges $\left(s_{r-1,j}, s_{r,k(k-1)^{r-1}-(j-1)(k-1)-t+1}\right)$, where $t \in [k-1]$.

4. Direct the tree obtained from previous steps so that on every arc the direction points to the vertex with larger label.

So far we have constructed one copy of the isomorphic directed trees. To obtain the other copies, rotate the first copy around the circle. That is, the i-th copy has an arc (j,k) if and only if the first copy has an arc $(j-i+1, k-i+1) \pmod{1+kS_{k,l-1}}$. ∎

Proof of Theorem 2:
By Lemma 2 or Lemma 3 there exists a covering of $DK_{f(d,k)}$ by $f(d,k)$ (d,k)-trees. Theorem 2 now comes immediately from Theorem 1. ∎

References

[1] J. Akiyama, G. Exoo and F. Harary, Covering and packing in graphs III, cyclic and acyclic invariants, *Math. Slovaca* **30** (1980), 405-417.

[2] I. Algor and N. Alon, The star arboricity of graphs, *Discrete Mathematics* **75**, North Holland, Amsterdam, 1989, 11-22.

[3] N. Alon, The linear arboricity of graphs, *Israel Journal of Mathematics*, 62, No. 3, (1988), 311-325.

[4] N. Alon, C. McDiarmid and B. Reed, Star arboricity, *Combinatorica* **12 (4)**, 1989, 375-380.

[5] N. Alon, J.H. Spencer and P. Erdős, *The Probabilistic Method*, Wiley, New York, 1992.

[6] N. Alon, V. Teague and N.C. Wormald, Linear arboricity and linear k-arboricity of regular graphs, submitted to *Graphs and Combinatorics*.

[7] J.C. Bermond, J.L. Fouquet, M. Habib and B. Peroche, On linear k-arboricity, *Discrete Math.* **52** (1984), 123-132.

[8] J.A. Bondy and U.S.R. Murty, *Graph Theory and Applications*, Macmillan, London, 1978.

[9] M. Truszczynski, Decomposing graphs into forests of stars, *Congressus Numerantium* **54**, 1986, 73-86.

(Received 15/11/99)