# Two results concerning distance-regular directed graphs

D. Combe
School of Mathematics
The University of New South Wales. NSW 2052 Australia
diana@maths.unsw.edu.au

D. E. Taylor
School of Mathematics and Statistics
The University of Sydney. NSW 2006 Australia
don@maths.usyd.edu.au

**Abstract**

The study of distance-regular directed graphs can be reduced to that of short distance-regular directed graphs. We consider the eigenspaces of the intersection matrix of a short distance-regular directed graph and show that nearly all the eigenvalues are nonreal. Next we show that a nontrivial short distance-regular directed graph is primitive.

## 1. Introduction

Distance-regular and distance-transitive directed graphs are the directed versions of distance-regular and distance-transitive (undirected) graphs. However, in the directed case, very few examples are known. Distance-transitive directed graphs were introduced by Lam [8] and have also been considered by others, for example in [1], [3], [4], [5], [9], [10], [11] and [12].

A *directed graph* or *digraph* $\Gamma$ is a pair $\Gamma = (V, E)$ consisting of a finite set $V$ of *vertices* and a set $E$ of *edges*. The elements of $E$ are ordered pairs of distinct elements of $V$. A (directed) *path* of *length* $h$ from $x$ to $y$ is a sequence of vertices $x = x_0, x_1, \ldots, x_h = y$, such that $h > 0$ and $(x_i, x_{i+1}) \in E$ for $i = 0, 1 \ldots, h-1$. If $x = y$ then the path is a *circuit*. A digraph $\Gamma$ is *strongly connected* if, for every $x, y \in V$, there is a path from $x$ to $y$. The length of a shortest path from a vertex $x$ to a vertex $y$ is the *distance* from $x$ to $y$, and is denoted $d(x, y)$. The *diameter* $d$ of a strongly connected digraph is the maximum value taken by this distance function over all $x, y \in V$. The *girth* $g$ is the minimum length of a circuit. Clearly $g \leq d + 1$.

For every vertex $x$ we define the $i$th *directed shell* $\Gamma_i(x)$ to be

$$\Gamma_i(x) = \{\, w \in V : d(x, w) = i \,\}.$$

A digraph $\Gamma$ is *distance-regular* if it is strongly connected and for all vertices $x$ and $y$, and for $0 \le i, j \le d$, $|\Gamma_i(x) \cap \Gamma_j(y)|$ depends only on $i$, $j$ and $d(x,y)$.

An *automorphism* of a digraph is a permutation $\rho$ of the vertices which preserves the edges; that is $(x, y) \in E$ if and only if $(x^\rho, y^\rho) \in E$. A strongly connected digraph is *distance-transitive* if, for all vertices $x, y, z, w$ with $d(x,y) = d(z,w)$, there is an automorphism of the digraph such that $x^\rho = z$ and $y^\rho = w$. Clearly a distance-transitive digraph is distance-regular.

A directed circuit with $n$ vertices is an example of a distance-transitive digraph with girth $g = d + 1 = n$. For each prime power $q \equiv 3 \pmod 4$, the Paley tournament with $q$ vertices is the digraph whose vertices are the elements of the finite field with $q$ elements. There is an edge from $x$ to $y$ if and only if $y - x$ is a nonzero square in the field. A Paley tournament is a distance-transitive digraph with girth $g = d + 1 = 3$.

Let $\Gamma$ be a distance-regular digraph. A distance-regular digraph of girth 2 is essentially the same as the underlying undirected graph and so from now on we always assume $\Gamma$ has girth at least 3. We define an involutory permutation on the set $\{0, 1, \ldots, g\}$, by setting $0^* = 0$, $g^* = g$ and $i^* = g - i$ for $0 < i < g$. Damerell [4] showed that for all vertices $x, y$ of $\Gamma$, the distance function satisfies $d(y,x) = d(x,y)^*$ and hence the girth and diameter differ by at most 1. A distance-regular digraph is said to be *short* if $g = d + 1$, otherwise it is said to be *long*. Damerell showed that every long graph can be constructed from an associated short graph thus reducing the classification of distance-regular directed graphs to those which are short.

Leonard and Nomura [9] showed that a short distance-regular directed graph which is not simply a directed circuit has girth at most 8, and, furthermore, that there always exist edges within the first directed shell of any vertex. There are examples of short distance-regular directed graphs of girth 3 or 4, see for example [8], [5], or [10]. Bannai, Cameron and Kahn [1] showed that if the girth of a short distance-transitive directed graph is odd, then $g = 3$. It is known [7] that in this case the directed graph is a Paley tournament.

In §2 we recall definitions and results concerning the adjacency and intersection algebras of a short distance-regular digraph and prove the first result of this paper. This result was known (Cameron [2]) in the case where the digraph has a distance transitive group of automorphisms.

**Theorem A.** *Let $C$ be the intersection matrix of a short distance-regular directed graph with girth $g$ and valency $k$. Then $C$ has $g$ distinct eigenvalues, and*

- (*i*) *if $g$ is odd, then $C$ has exactly one real eigenvalue $k$, and*
- (*ii*) *if $g$ is even, then $C$ has exactly two real eigenvalues, one of which is $k$, the other of which is a negative real number.*

In §3 we recall the definition of primitivity for a directed graph. Damerell [4] showed that a long distance-regular digraph is imprimitive. It follows immediately that its automorphism group acts imprimitively on the vertices. For short distance-regular digraphs we prove the following theorem:

**Theorem B.** *Let $\Gamma$ be a short distance-regular directed graph with valency $k$ and $n$ vertices.*

(*i*) *If $k = 1$, then $\Gamma$ is a directed circuit with $n$ vertices and is primitive if and only if $n$ is prime.*

(*ii*) *If $k > 1$, then $\Gamma$ is primitive.*

**Corollary.** *Let $\Gamma$ be a short distance-regular directed graph with valency $k > 1$. Then any distance-transitive group of automorphisms acts primitively on the vertices.*

## 2. Adjacency and intersection matrices.

In this section we will always assume that $\Gamma$ is a short distance-regular digraph with $n$ vertices, diameter $d$, girth $g = d + 1$, and labelled vertex set $V = \{x_1, x_2, \ldots, x_n\}$. For $i, j, k = 0, 1, \ldots, d$, the *intersection numbers* $p_{i,j,k}$ are

$$p_{i,j,k} = |\Gamma_i(x) \cap \Gamma_{j^*}(y)|, \quad \text{for } x, y \text{ any vertices of } \Gamma \text{ with } d(x, y) = k.$$

For any matrix $M$ of complex numbers, $M^{\mathrm{T}}$ denotes the transposed matrix and $\overline{M}$ denotes the complex conjugate.

For any digraph with vertex set $\{y_1, y_2, \ldots, y_m\}$ the *adjacency matrix* $A$ is the matrix of 0s and 1s whose $(r, s)$-th entry is $(A)_{r,s}$, where

$$(A)_{r,s} = 1 \quad \text{if and only if } (y_r, y_s) \text{ is an edge.}$$

For $i = 0, 1, \ldots, d$, the *distance matrix* $A_i$ of $\Gamma$ is the matrix of 0s and 1s with

$$(A_i)_{r,s} = 1 \quad \text{if and only if } d(x_r, x_s) = i.$$

Thus $A_0$ is the identity matrix. The matrix $A = A_1$ is the *adjacency matrix* of the digraph. It is clear that in general $A_{i^*} = A_i^{\mathrm{T}}$, and that $A_i$ is a symmetric matrix if and only if $i = 0$ or $i = g/2$.

By counting paths it is easy to see that, for $0 \le i, h \le d$, the distance matrices satisfy $A_i A_h = \sum_{j=0}^{d} p_{i,h,j} A_j$. Hence the linear span of $A_1, A_2, \ldots, A_d$, is closed under multiplication and is thus an algebra. This is the *adjacency algebra* $\mathcal{A}$ of the digraph. It is well known (see [8]) that the adjacency algebra is commutative with dimension $g = d + 1$. Each of the sets $\{A_0, A_1, \ldots, A_d\}$ and $\{I, A^1, A^2, \ldots, A^d\}$ forms a basis of $\mathcal{A}$, where the $A_i$ are the distance matrices and the $A^i$ are the powers of $A$.

For each $i = 0, 1, \ldots, d-1$, we have $p_{i,1,i+1} > 0$ and $A_i A = \sum_{j=0}^{i+1} p_{i,1,j} A_j$. Thus for $i = 1, 2, \ldots d$, this equation recursively defines real polynomials $V_i(x)$ of degree $i$ such that $V_i(A) = A_i$, where $V_0(x) = 1$.

For $h = 0, 1, \ldots, d$, the $h$-th *intersection matrix* $C_h$ is defined to be the $(d+1) \times (d+1)$ matrix whose $(i, j)$-th entry is

$$(C_h)_{i,j} = p_{i,h,j} = p_{h,i,j}.$$

The matrix $C_0$ is the identity matrix. The matrix $C = C_1$ is called the *intersection matrix*. The algebra $\mathcal{A}$ acts on itself by multiplication on the right. Right multiplication by $A_h$, regarded as a linear mapping of the adjacency algebra to itself with respect to the basis $\{A_0, A_1, \ldots, A_d\}$ can be faithfully represented by the transposed $h$-th intersection matrix $C_h^{\mathrm{T}}$.

The two matrices $A$ and $C$ have the same minimum polynomial and so have the same eigenvalues.

The matrix $A$ is a real nonsymmetric matrix which commutes with its transpose. Therefore $A$ is a normal matrix and hence is diagonalizable; that is, there is a basis of $\mathbb{C}^n$ which consists of eigenvectors of $A$. Since the minimum polynomial of $A$ has degree $d+1$, $A$ has $d+1$ distinct eigenvalues, which we denote by $\lambda_0 = 1$, $\lambda_1$, $\ldots$, $\lambda_d$.

We call an eigenvector of a matrix *standard* if its first coordinate is 1.

**Lemma 2.1.** *Let $\lambda$ be an eigenvalue of $C$. Then*

(*i*) *we can construct a unique standard left eigenvector* $\mathbf{w}(\lambda)$, *and a unique standard right eigenvector* $\mathbf{v}(\lambda)$ *corresponding to* $\lambda$,

(*ii*) *for* $i = 0, 1, \ldots, d$, *the eigenvectors* $\mathbf{w}(\lambda)$ *and* $\mathbf{v}(\lambda)$ *are standard left and right eigenvectors of* $C_i$ *with corresponding eigenvalue* $V_i(\lambda)$,

(*iii*) *the eigenvalue* $\lambda$ *is real if and only if* $\mathbf{v}(\lambda)$ *is a real vector if and only if* $\mathbf{w}(\lambda)$ *is a real vector, and*

(*iv*) $\mathbf{v}(\overline{\lambda}) = \overline{\mathbf{v}(\lambda)}$ *and* $\mathbf{w}(\overline{\lambda}) = \overline{\mathbf{w}(\lambda)}$.

*Proof.* Let $\lambda$ be any eigenvalue of $C$. The corresponding left (or right) eigenspace has dimension 1 and so any standard eigenvector must be unique. Right and left eigenvectors in standard form can be constructed in the following manner:

If $\mathbf{v} = [v_0, v_1, \ldots, v_d]^{\mathrm{T}}$ then the equation $C\mathbf{v} = \lambda\mathbf{v}$ becomes $\sum_{j=0}^{i+1} p_{i,1,j} v_j = \lambda v_i$. Setting $v_0 = 1$ we get the same system of equations as for the distance matrices. Therefore $v_i = V_i(\lambda)$. The vector $\mathbf{v}(\lambda) = [1, \ldots, V_i(\lambda), \ldots, V_d(\lambda)]^T$ is the unique standard right eigenvector corresponding to $\lambda$.

Similarly a left eigenvector $\mathbf{w}(\lambda)$ corresponding to $\lambda$ can be constructed. If $\mathbf{w} = [w_0, w_1, \ldots, w_d]$ then the system $\mathbf{w}\lambda = \mathbf{w}C$ becomes $\sum_{i=0}^{d} w_i p_{1,i,j} = \lambda w_j$. This time each $w_j$ is $w_d$ times a polynomial in $\lambda$. Setting $w_d = \frac{\lambda}{k} \neq 0$, we get $\lambda w_0 = w_d p_{d,1,0} = w_d p_{1,d,0} = \frac{\lambda}{k} k = \lambda$. Therefore $w_0 = 1$, and $\mathbf{w}(\lambda) = [1, \ldots, \frac{\lambda}{k}]$, the unique standard left eigenvector corresponding to $\lambda$.

For each $i$, $C_i = V_i(C)$, and so an eigenvector of $C$ corresponding to eigenvalue $\lambda$ is an eigenvector of $C_i$ with corresponding eigenvalue $V_i(\lambda)$.

Finally, since $C$ is a real matrix, $\mathbf{v}(\overline{\lambda}) = \overline{\mathbf{v}(\lambda)}$ and $\mathbf{w}(\overline{\lambda}) = \overline{\mathbf{w}(\lambda)}$. Furthermore the eigenvalues corresponding to real eigenvectors must be real. Conversely, if $\lambda$ is a real eigenvalue then for each $i$, the $i$th entry of $\mathbf{v}(\lambda)$ is $V_i(\lambda)$ which is real. Similarly the entries of $\mathbf{w}(\lambda)$ are the values of real polynomials evaluated at $\lambda$, and hence are real. Therefore the eigenvalue $\lambda$ is real if and only if $\mathbf{v}(\lambda)$ is a real vector if and only if $\mathbf{w}(\lambda)$ is a real vector. $\square$

The next lemma is a standard result.

**Lemma 2.2.** *If* $\lambda_i \neq \lambda_j$ *then* $\mathbf{w}(\lambda_i)\mathbf{v}(\lambda_j) = 0$. □

With respect to the eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_d$ of $C$, we define the *eigenmatrix* $\Lambda$ of the digraph $\Gamma$ to be $(d+1) \times (d+1)$ matrix whose $j$th column is the standard eigenvector $\mathbf{v}(\lambda_j)$. We denote the $(i,j)$th entry of $\Lambda$ by $\lambda_{ij}$. By the previous construction, $\lambda_{ij} = V_i(\lambda_j)$, and the $i$th row of $\Lambda$ consists of the eigenvalues of $C_i$. Row 0 of $\Lambda$ consists of all 1s, and column 0 is $\mathbf{v}(k) = [k_0, k_1, \ldots, k_d]^{\mathrm{T}}$. We prove first that $\Lambda$ has at most two real columns.

Denote by $\mathbf{K}$ the diagonal matrix $\mathbf{K} = \mathrm{diag}(k_0, k_1, \ldots, k_d)$. For each $i$, the distance matrices $A_i$ and $A_{i^*}$ are related by $A_{i^*} = A_i^{\mathrm{T}}$. The next lemma links $C_{i^*}$, $C_i^{\mathrm{T}}$ and $\mathbf{K}$.

**Lemma 2.3.** *For* $i = 0, 1, \ldots, d$, $C_{i^*}^{\mathrm{T}} = \mathbf{K}^{-1} C_i \mathbf{K}$.

*Proof.* Let $h, j \in \{0, 1, \ldots, d\}$ and $x$ be a vertex of $\Gamma$. We count in two ways the elements of the set $\mathbf{P} = \{\, (u,v) : u \in \Gamma_i(x),\ v \in \Gamma_j(x) \text{ and } d(u,v) = h \,\}$. For each $u \in \Gamma_i(x)$ there are $p_{j,h^*,i} = p_{h^*,j,i}$ corresponding vertices $v$, and for each $v \in \Gamma_j(x)$ there are $p_{i,h,j} = p_{h,i,j}$ corresponding vertices $u$.

Therefore $|\mathbf{P}| = k_i p_{h^*,j,i} = k_j p_{h,i,j}$, and thus we have

$$(C_{h^*}^{\mathrm{T}})_{i,j} = p_{h^*,j,i} = k_j p_{h,i,j} k_i^{-1} = k_j (C_h)_{i,j} k_i^{-1}.$$

Hence $C_{i^*}^{\mathrm{T}} = \mathbf{K}^{-1} C_i \mathbf{K}$. □

For each $i = 0, 1, \ldots, d$, the matrices $C_i$ and $C_{i^*}$ have the same set of eigenvalues and the same set of standard eigenvectors. The eigenvalues of $C$ are distinct and we define $\sigma$ to be the permutation of $\{0, 1, \ldots, d\}$ such that $\lambda_{dj} = \lambda_{1j^\sigma}$ $(= \lambda_{j^\sigma})$. This gives rise to a permutation of the eigenvalues of $C$ which is in fact complex conjugation.

**Lemma 2.4.** *For* $j = 0, 1, \ldots, d$,

(*i*) *for* $i = 0, 1, \ldots, d$ *we have* $\lambda_{ij^\sigma} = \overline{\lambda_{ij}} = \lambda_{i^*j}$, *and*

(*ii*) *the standard right eigenvector is* $\mathbf{v}(\overline{\lambda_j}) = \mathbf{K}\mathbf{w}(\lambda_j)^{\mathrm{T}}$.

*Proof.* For $j = 0, 1, \ldots, d$ we have

$$\begin{aligned} C\mathbf{K}\mathbf{w}(\lambda_j)^{\mathrm{T}} &= (\mathbf{K}C_d^{\mathrm{T}})\mathbf{w}(\lambda_j)^{\mathrm{T}} \\ &= \mathbf{K}(\mathbf{w}(\lambda_j)C_d)^{\mathrm{T}} \\ &= \mathbf{K}(\lambda_{dj}\mathbf{w}(\lambda_j))^{\mathrm{T}} \\ &= \lambda_{1j^\sigma}(\mathbf{K}\mathbf{w}(\lambda_j)^{\mathrm{T}}). \end{aligned}$$

Therefore $\mathbf{K}\mathbf{w}(\lambda_j)^{\mathrm{T}} = \mathbf{v}(\lambda_{j^\sigma})$ because the first entry in $\mathbf{K}\mathbf{w}(\lambda_j)^{\mathrm{T}}$ is 1.

Since $\mathbf{K}$ is a diagonal matrix with positive diagonal entries,

$$\mathbf{w}(\overline{\lambda_j})\mathbf{v}(\lambda_{j^\sigma}) = \overline{\mathbf{w}(\lambda_j)}\mathbf{K}\mathbf{w}(\lambda_j)^{\mathrm{T}} > 0.$$

Hence $\overline{\lambda_j} = \lambda_{j^\sigma}$, and thus $\mathbf{v}(\overline{\lambda_j}) = \mathbf{v}(\lambda_{j^\sigma}) = \mathbf{K}\mathbf{w}(\lambda_j)^{\mathrm{T}}$. Furthermore $\lambda_{ij^\sigma} = V_i(\lambda_{j^\sigma}) = V_i(\overline{\lambda_j}) = \overline{V_i(\lambda_j)} = \overline{\lambda_{ij}}$.

Finally, for $i, j \in \{0, 1, \ldots, d\}$ we have

$$\begin{aligned}
\lambda_{i^*j}\mathbf{w}(\lambda_j) &= \mathbf{w}(\lambda_j)C_{i^*} \\
&= \mathbf{v}(\overline{\lambda}_j)^{\mathrm{T}}\mathbf{K}^{-1}C_{i^*} \\
&= \mathbf{v}(\overline{\lambda}_j)^{\mathrm{T}}C_i^{\mathrm{T}}\mathbf{K}^{-1} \\
&= (C_i\mathbf{v}(\overline{\lambda}_j))^{\mathrm{T}}\mathbf{K}^{-1} \\
&= \lambda_{ij^\sigma}\mathbf{v}(\overline{\lambda}_j)^{\mathrm{T}}\mathbf{K}^{-1} \\
&= \overline{\lambda_{ij}}\mathbf{w}(\lambda_j).
\end{aligned}$$

Therefore $\lambda_{i^*j} = \overline{\lambda}_{ij}$. □

As a corollary to this lemma we have

**Corollary 2.5.** *The eigenmatrix $\Lambda$ has at most two real rows. Furthermore,*

(*i*) *if the girth $g$ is odd, $\Lambda$ has 1 real row: the 0th row;*
(*ii*) *if the girth $g$ is even, $\Lambda$ has 2 real rows: the 0th row and row $g/2$.*

*Proof.* The eigenmatrix is nonsingular since its columns are the $d+1$ standard eigenvectors corresponding to the $d+1$ distinct eigenvalues of $C$. Therefore the rows are certainly distinct. However, the previous proposition shows that row $i^*$ and row $i$ are conjugate. Therefore the $i$th row of $\Lambda$ is real if and only if $i^* = i$. If $g$ is odd, then $i^* = i$ if and only if $i = 0$. If $g$ is even, then $i^* = i$ if and only if $i = 0$ or $g/2$. □

We can now prove Theorem A.

**Theorem A.** *Let $C$ be the intersection matrix of a short distance-regular directed graph with girth $g$ and valency $k$. Then $C$ has $g$ distinct eigenvalues, and*

(*i*) *if $g$ is odd, then $C$ has exactly one real eigenvalue $k$, and*
(*ii*) *if $g$ is even, then $C$ has exactly two real eigenvalues, one of which is $k$, the other of which is a negative real number.*

*Proof.* The eigenmatrix $\Lambda$ is nonsingular. The permutation of the entries which takes $\lambda_{ij} \mapsto \lambda_{i^*j}$ is an involution and comes from an involutory permutation of the rows of $\Lambda$. Let $L$ be the permutation matrix such that $\Lambda \mapsto L\Lambda$ effects this permutation of the rows. Then $(L\Lambda)_{ij} = \lambda_{i^*j} = \overline{\lambda_{ij}}$.

The action on the columns of $\Lambda$, given by $\mathbf{v}(\lambda_j) \mapsto \mathbf{v}(\overline{\lambda_j})$, is also an involution. Let $R$ be the permutation matrix such that $\Lambda \mapsto \Lambda R$ effects this permutation. Then the matrix entry $(\Lambda R)_{ij} = (\mathbf{v}(\overline{\lambda_j}))_i = \overline{\mathbf{v}(\lambda_j)_i} = \overline{\lambda_{ij}} = (L\Lambda)_{ij}$.

Therefore $L\Lambda = \Lambda R$, and so $L = \Lambda^{-1}R\Lambda$. The number of fixed points of a permutation is equal to the trace of the corresponding permutation matrix, and the traces of similar matrices are equal. Therefore $L$ and $R$ have the same number of fixed points in their actions.

By the previous corollary, there are at most two fixed rows, and so $C$ has at most two real standard right eigenvectors and thus at most two real eigenvalues.

The eigenvalue $\lambda_0 = k$ is real and the nonreal eigenvalues occur in conjugate pairs. The number of eigenvalues is $g$. Therefore, if $g$ is odd, then $C$ has exactly one real eigenvalue. If $g$ is even, then $C$ has exactly two real eigenvalues, one of which is $\lambda_0 = k$, and the rest are pairs of complex conjugate eigenvalues. The product of the eigenvalues is the determinant of $C$ which can be conveniently calculated by expanding down the first column. Thus $\det(C) = -k\prod_{j=0}^{d-1} C_{j,j+1}$, which is $-k \times$ a product of positive integers, and so the second real eigenvalue must be negative. □

Note that the argument used in the proof that $L$ and $R$ have the same number of fixed points in their actions is a special case of a combinatorial lemma due to Brauer. (See Feit [6; 12.1, page 66].)

## 3. Primitivity

Throughout this section $\Gamma = (V, E)$ is a distance-regular digraph, not necessarily short. If the valency $k$ is 1 the digraph is said to be *trivial*, and in this case it is clear that it is simply a directed circuit.

Let $\mathbb{V} = \{I, E_1, \ldots, E_d\}$ be the partition of $V^2$ defined by $(x, y) \in E_i$ if and only if $y \in \Gamma_i(x)$. For each $i$, we define $\Gamma_i$ to be the digraph with vertex set $V$ and edge set $E_i$ and so $\Gamma_i$ is the directed graph with adjacency matrix $A_i$. If $i = g/2$ then $\Gamma_i$ has girth 2. The digraphs $\Gamma_i$ are not necessarily connected.

The digraph $\Gamma$ is said to be *primitive* if the two trivial relations, $I$ and $V^2$, are the only equivalence relations which are unions of members of $\mathbb{V}$, otherwise $\Gamma$ is called *imprimitive.*

**Lemma 3.1.** *The digraph $\Gamma$ is primitive if and only if each $\Gamma_i$ is connected.*

*Proof.* Clear. □

**Lemma 3.2.** (Damerell [4]) *A long distance-regular digraph is imprimitive.*

*Proof.* If $\Gamma$ is a long distance-regular digraph, then $E = I \cup \Gamma_g$ is a nontrivial equivalence relation and so $\Gamma$ is imprimitive. □

Before we complete the proof of Theorem B, we need several results concerning edges and circuits within directed shells when $\Gamma$ is nontrivial.

**Lemma 3.3.** (Leonard and Nomura [9]). *If $\Gamma$ is short and nontrivial then $p_{1,1,1} > 0$. That is, there are edges in the first directed shell.* □

**Corollary 3.4.** *If $\Gamma$ is short and nontrivial, and $x$ is any vertex, then there is a closed path entirely contained in $\Gamma_1(x)$.*

*Proof.* Let $u_0$ be any vertex in $\Gamma_1(x)$. Choose $u_1$ to be any out-neighbour of $u_0$ in $\Gamma_1(x)$. Choose $u_2$ to be any out-neighbour of $u_1$ in $\Gamma_1(x)$. Continue in this way constructing a path $u_0, u_1, \ldots u_j$ in $\Gamma_1(x)$. Since $\Gamma_1(x)$ is finite, for some smallest $m$ we have $u_m = u_h$ for some $h < m$. Then $u_h \longrightarrow u_{h+1} \longrightarrow \cdots \longrightarrow u_m = u_h$ is a closed path with all $u_j$ in $\Gamma_1(x)$. □

As a further corollary we have that in the first directed shell of any vertex there are vertices $y$ and $z$ with $d(y,z) = j$ for every $j \leq d$; that is, the entries in column 1 of $C$ are all nonzero.

**Corollary 3.5.** *If $\Gamma$ is short and nontrivial then $p_{1,i,1} > 0$ for $i = 0, 1, \ldots, d$.* □

**Corollary 3.6.** *If $\Gamma$ is short and nontrivial then $p_{d,1,i} > 0$ for $i = 0, 1, \ldots, d$.*

*Proof.* Let $x, y, z$ be vertices with $d(x,y) = 1$, $d(y,z) = i^*$ and $d(x,z) = 1$. These exist since $p_{1,i^*,1} > 0$. Then $d(z,x) = 1^* = d$, $d(x,y) = 1$ and $d(z,y) = (i^*)^* = i$. Therefore $p_{d,1,i} > 0$. □

**Lemma 3.7.** *If $\Gamma$ is a short nontrivial distance-regular digraph, and $0 < i \leq g/2$, $x \in V$ and $y \in \Gamma_i(x)$, then there is a vertex $z \in \Gamma_i(x)$ such that $0 < d(y,z) < i$.*

*Proof.* Suppose that $0 < i \leq g/2$. We use the commutativity of the intersection matrices and, in particular, the equality $(C_1C_d)_{1i} = (C_dC_1)_{1i}$.

On the one hand we have

$$(C_1C_d)_{1i} = \sum_{j=0}^{d} p_{1,1,j}p_{d,j,i} = p_{1,1,1}p_{d,1,i} + p_{1,1,2}p_{d,2,i}.$$

and on the other

$$(C_dC_1)_{1i} = \sum_{j=0}^{d} p_{d,1,j}p_{1,j,i} = p_{d,1,i-1}p_{1,i-1,i} + p_{d,1,i}p_{1,i+1,i} \cdots + p_{d,1,d}p_{1,d,i}.$$

Now $p_{1,1,1} = p_{d,1,d}$ and $p_{d,1,i} = p_{1,d,i}$. Therefore $p_{1,1,2}p_{d,2,i} \geq p_{d,1,i-1}p_{1,i-1,i} > 0$ and so $p_{d,2,i} > 0$. Therefore there are edges from $\Gamma_i(x)$ to $\Gamma_2(x)$.

Now suppose $y \in \Gamma_i(x)$. Since $p_{2,d,i} = p_{d,2,i} > 0$, there exists $w \in \Gamma_2(x) \cap \Gamma_1(y)$. Since $d(x,w) = 2$ we have $d(w,x) = 2^* = d - 1$ and there is a (minimal) closed path of length $g$ of the form

$$x_0 = x \longrightarrow x_1 \longrightarrow x_2 = w \longrightarrow x_3 \longrightarrow \cdots \longrightarrow x_i \longrightarrow x_{i+1} \longrightarrow \cdots \longrightarrow x_g = x.$$

Setting $z = x_i$ we have $z \in \Gamma_i(x)$, with $d(w,z) = i - 2$. Since $d(w,y) = d \neq i - 2$, the vertices $z$ and $y$ are distinct elements of $\Gamma_i(x)$.

Therefore $0 < d(y,z) \leq d(y,w) + d(w,z) = i - 1 < i$. □

Setting $i = 2$ in this lemma we have that there are edges in the second directed shell.

**Corollary 3.8.** *If $\Gamma$ is short and nontrivial, then $p_{1,2,2} > 0$.* □

**Corollary 3.9.** *If $\Gamma$ is short and nontrivial, and $x$ is any vertex, then there is a closed path entirely contained in $\Gamma_2(x)$.* □

We can now prove Theorem B.

**Theorem B.** *Let $\Gamma$ be a short distance-regular directed graph with valency $k$ and $n$ vertices.*

(*i*) *If $k = 1$ then $\Gamma$ is a directed circuit with $n$ vertices and is primitive if and only if $n$ is prime.*

(*ii*) *If $k > 1$ then $\Gamma$ is primitive.*

*Proof.* (*i*) If $k = 1$ then $\Gamma$ is a directed circuit with $n$ vertices. Clearly in this case $\Gamma$ is primitive if and only if $n$ is prime.

(*ii*) Suppose $k > 1$ and that $E$ is an equivalence relation on $V$ which is a union of members of $\mathbb{V}$. Then for some $\ell$, a divisor of $g$, $E = I \cup E_\ell \cup E_{2\ell} \cup \cdots \cup E_{g-\ell}$. Any two vertices $y$ and $z$ are in the same equivalence class if and only if $\ell$ divides $d(y, z)$.

Suppose $1 < \ell \leq g/2$, and $x \in V$. The equivalence class containing $x$ also contains $\Gamma_\ell(x)$. By the previous lemma there exist $y, z \in \Gamma_\ell(x)$ such that $0 < d(y, z) < \ell$. However, since $y$ and $z$ are in the same equivalence class $\ell$ divides $d(y, z)$. This is a contradiction, and so $\ell = 1$ and the equivalence relation is trivial.

Therefore any short nontrivial distance-regular digraph is primitive. □

A group $G$ which acts transitively on a set acts *primitively* if the only partitions of the set which it preserves are the trivial ones, and so the corollary follows immediately.

**Corollary.** *Let $\Gamma$ be a short distance-regular directed graph with valency $k > 1$. Then any distance-transitive group of automorphisms acts primitively on the vertices.*

## References


1. E. Bannai and P.J. Cameron and J. Kahn, *Nonexistence of certain distance-transitive digraphs*, J. Combinatorial Theory, Ser. B **31** (1981), 105-110.
2. P.J. Cameron, *Suborbits in Transitive Permutation Groups*, Combinatorics (M. Hall and J. H. Van Lint, eds.), 1974, pp. 98-129.
3. D. Combe, *Distance-transitive directed graphs and multiplicity free primitive permutation representations with almost no real irreducible constituents*, PhD thesis, University of Sydney, 1998.
4. R.M. Damerell, *Distance-transitive digraphs*, J. Combinatorial Theory, Ser. B **31** (1981), 46-53.
5. H. Enomoto and R.A. Mena, *Distance-regular digraphs of girth 4*, J. Combinatorial Theory, Ser. B **43** (1987), 293-302.
6. W. Feit, *Characters of Finite Groups*, W.A.Benjamin, Amsterdam, 1967.
7. W.M. Kantor, *Automorphism groups of designs*, Math. Z. **109** (1967), 246-252.
8. C.W.H. Lam, *Distance-transitive digraphs*, Discrete Math. **9** (1980), 265–274.
9. D.A. Leonard and K. Nomura, *The girth of a directed distance-regular graph*, J. Combinatorial Theory, Ser. B **58** (1993), 34-39.
10. R.A. Liebler and R.A. Mena, *Certain distance-regular digraphs and related rings of characteristic* 4, J. Combinatorial Theory, Ser. A **47** (1988), 111-123.

11. T. Takashi, *Distance-regular digraphs of girth 6*, Mem. Fac. Sci., Kyushu Univ., Ser. A **45** (1991), 155-166.
12. M. Yamada, *Distance regular digraphs of girth* 4 *over an extension ring of* $\mathbb{Z}/4\mathbb{Z}$, Graphs and Combinatorics **6** (1990), 381-394.

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