Vertex-disjoint cycles containing specified edges in a bipartite graph

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Abstract

Dirac and Ore-type degree conditions are given for a bipartite graph to contain vertex disjoint cycles each of which contains a previously specified edge. This solves a conjecture of Wang in [6].

1 Introduction

In this paper, we only consider finite undirected graphs without loops and multiple edges. For a vertex x of a graph G, the neighborhood of x in G is denoted by $N_G(x)$, and $d_G(x) = |N_G(x)|$ is the degree of x in G. For a subgraph H of G and a vertex $x \in V(G) - V(H)$, we also denote $N_H(x) = N_G(x) \cap V(H)$ and $d_H(x) = |N_H(x)|$. For a subgraph H and a subset S of $V(G)$, define $d_H(S) = \sum_{x \in S} d_H(x)$. The subgraph induced by S is denoted by $\langle S \rangle$, and define $G - S = \langle V(G) - S \rangle$ and $G - H = \langle G - V(H) \rangle$. For a graph G, $|G| = |V(G)|$ is the order of G, $\delta(G)$ is the minimum degree of G, and

$$\sigma_2(G) = \min\{d_G(x) + d_G(y) \mid x, y \in V(G),\ x \neq y,\ xy \notin E(G)\}$$

is the minimum degree sum of nonadjacent vertices. (When G is a complete graph, we define $\sigma_2(G) = \infty$.) For a bipartite graph G with partite sets V_1 and V_2,

$$\delta_{1,1}(G) = \min\{d_G(x) + d_G(y) \mid x \in V_1,\ y \in V_2\}$$

and

$$\sigma_{1,1}(G) = \min\{d_G(x) + d_G(y) \mid x \in V_1,\ y \in V_2,\ xy \notin E(G)\}.$$

(When G is a complete bipartite graph, we define $\sigma_{1,1}(G) = \infty$.) Two edges e and f are adjacent if they have a common endvertex, and they are independent if they are nonadjacent. A set F of independent edges in G is a perfect matching when $|F| = |G|/2$.

In this paper, "disjoint" means "vertex-disjoint," since we only deal with partitions of the vertex set.

Suppose $H_1, \cdots, H_k$ are disjoint cycles of G such that $V(G) = \bigcup_{i=1}^{k} V(H_i)$. Then the union of these H_i is a 2-factor of G with k components. A sufficient condition for the existence of a 2-factor with a specified number of components was given by Brandt et al. [1].

Theorem A *Suppose $|G| = n \geq 4k$ and $\sigma_2(G) \geq n$. Then G can be partitioned into k cycles, that is, G contains k disjoint cycles $H_1, \cdots, H_k$ satisfying $V(G) = \bigcup_{i=1}^{k} V(H_i)$.*

Wang [4] considered partitioning a graph into cycles passing through specified edges, and conjectured that if $k \geq 2$, n is sufficiently large compared with k, and $\sigma_2(G) \geq n + 2k - 2$, then for any independent edges $e_1, \cdots, e_k$, G can be partitioned into cycles $H_1, \cdots, H_k$ such that $e_i \in E(H_i)$. This conjecture was completely solved by Egawa et al. [3].

Theorem B *Suppose $k \geq 2$, $|G| = n \geq 3k$ and either*

$$\sigma_2(G) \geq \max\left\{n + 2k - 2, \left\lfloor \frac{n}{2} \right\rfloor + 4k - 2\right\}$$

or

$$\delta(G) \geq \max\left\{\left\lceil \frac{n}{2} \right\rceil + k - 1, \left\lceil \frac{n + 5k}{3} \right\rceil - 1\right\}.$$

Then for any independent edges $e_1, \cdots, e_k$, G can be partitioned into cycles $H_1, \cdots, H_k$ such that $e_i \in E(H_i)$.

In this paper, we consider analogous results for a bipartite graph, and in the rest of this paper, G denotes a bipartite graph with partite sets V_1 and V_2 satisfying $|V_1| = |V_2| = n$.

Wang [5] proved the following analogue of Theorem A for bipartite graphs.

Theorem C *Suppose $n \geq 2k+1$ and $\delta(G) \geq n/2+1$. Then G can be partitioned into k cycles.*

The assumption $\delta(G) \geq n/2+1$ is sharp when $n = 2k+1$. However, a weaker condition is sufficient when n is large.

Theorem D (Chen et al. [2]) *Suppose $n \geq \max\{51, k^2/2+1\}$ and $\delta_{1,1}(G) \geq n+1$. Then G can be partitioned into k cycles.*

Wang [6] conjectured that if $k \geq 2$, n is sufficiently large compared with k, and $\sigma_{1,1}(G) \geq n+k$, then for any independent edges $e_1, \cdots, e_k$, G can be partitioned into cycles $H_1, \cdots, H_k$ such that $e_i \in E(H_i)$, and verified it when $k \leq 3$.

In this paper, we solve this conjecture affirmatively.

Theorem 1 *Suppose $k \geq 2$, $n \geq 2k$, and either*

$$\sigma_{1,1}(G) \geq \max\left\{n+k, \left\lceil\frac{2n-1}{3}\right\rceil + 2k\right\}$$

or

$$\delta(G) \geq \max\left\{\left\lceil\frac{n+k}{2}\right\rceil, \left\lceil\frac{2n+4k}{5}\right\rceil\right\}.$$

Then for any independent edges $e_1, \cdots, e_k$, G can be partitioned into cycles $H_1, \cdots, H_k$ such that $e_i \in E(H_i)$.

Note that $n+k \geq \lceil\frac{2n-1}{3}\rceil + 2k$ if and only if $n \geq 3k-1$, and $\lceil\frac{n+k}{2}\rceil \geq \lceil\frac{2n+4k}{5}\rceil$ if and only if $n = 3k-5, n = 3k-3$ or $n \geq 3k-1$.

Theorem 1 is an immediate corollary of the following two theorems: One solves the packing problem, and the other one extends a packing to a partition.

Theorem 2 *Suppose $n \geq 2k$, and either*

$$\sigma_{1,1}(G) \geq \max\left\{n+k, \left\lceil\frac{2n-1}{3}\right\rceil + 2k\right\}$$

or

$$\delta(G) \geq \max\left\{\left\lceil\frac{n+k}{2}\right\rceil, \left\lceil\frac{2n+4k}{5}\right\rceil\right\}.$$

Then for any independent edges $e_1, \cdots, e_k$, G contains k disjoint cycles $C_1, \cdots, C_k$ such that $e_i \in E(C_i)$ and $|C_i| \leq 6$.

Theorem 3 *Suppose $k \geq 2$, $\sigma_{1,1}(G) \geq n+k$, $C_1, \cdots, C_k$ are disjoint cycles and $e_i \in E(C_i)$. Then there exist disjoint cycles $H_1, \cdots, H_k$ satisfying $V(G) = \bigcup_{i=1}^{k} V(H_i)$ and $e_i \in E(H_i)$.*

The sharpness of the assumptions will be discussed in the final section.

We will use the notation $C[u,v]$ to denote the segment of the cycle C from u to v (including u and v) under some orientation of C, and $C[u,v) = C[u,v] - \{v\}$ and $C(u,v) = C[u,v] - \{u,v\}$. Given a cycle C with an orientation, we let v^+ (resp. v^-) denote the successor (resp. the predecessor) of v along C according to this orientation, and $v^{++} = (v^+)^+$ (resp. $v^{--} = (v^-)^-$).

Let $F = \{e_1, \cdots, e_k\}$ be a set of independent edges, where $e_i = x_i y_i$, $x_i \in V_1$, $y_i \in V_2$, and set $T = \{x_1, y_1, \cdots, x_k, y_k\}$. A set of disjoint cycles $\{C_1, \cdots, C_r\}$ is called *admissible* for F if $|E(C_i) \cap F| = 1$ and $|V(C_i) \cap T| = 2$ for $1 \le i \le r$.

2 Proof of Theorem 2

The following lemma will be used several times in the proof of Theorem 2.

Lemma 4 *Suppose C is a cycle in G, $e \in E(C)$, $u \in V(G-C) \cap V_1$, $v \in V(G-C) \cap V_2$ and $d_C(u) + d_C(v) \ge |C|/2 + 2$. Then, either $\langle V(C) \cup \{v\} \rangle$ contains a shorter cycle than C passing through e, or there exists $w \in N_C(u)$ such that $\langle V(C) \cup \{v\} - \{w\} \rangle$ contains a cycle passing through e.*

Proof. If $d_C(v) \ge 3$, $\langle V(C) \cup \{v\} \rangle$ contains a shorter cycle than C passing through e. Hence we may assume that $d_C(v) \le 2$. Then $d_C(v) = 2$ and $d_C(u) = |C|/2$, that is, $N_C(u) = V(C) \cap V_2$. We may assume that $N_C(v) = \{a, b\}$ with $e \in E(C[b,a])$. If $|C(a,b)| > 1$, $\langle V(C) \cup \{v\} \rangle$ contains a shorter cycle than C passing through e. Hence we may assume that $C(a,b) = \{w\}$. Then $w \in N_C(u)$ and $\langle V(C) \cup \{v\} - \{w\} \rangle$ contains a (spanning) cycle passing through e. ∎

Let G be an edge-maximal counterexample of Theorem 2, and set $F = \{e_1, \cdots, e_k\}$. In the rest of the proof, 'admissible' means 'admissible for F,' and a cycle is called *short* if its length is equal to 4 or 6. If G is a complete bipartite graph, G contains k admissible cycles of length 4. Hence G is not complete bipartite. Let $x \in V_1$ and $y \in V_2$ be nonadjacent vertices of G, and define $G' = G + xy$, the graph obtained from G by adding the edge xy. Then G' is not a counterexample by the maximality of G, and so G' contains admissible short cycles $C_1, \cdots, C_k$. Without loss of generality, we may assume that $xy \notin \bigcup_{i=1}^{k-1} E(C_i)$. This means that G contains $k-1$ admissible short cycles $C_1, \cdots, C_{k-1}$ such that $\sum_{i=1}^{k-1} |C_i| \le 2n - 4$. We choose those admissible short cycles $C_1, \cdots, C_{k-1}$ so that $\sum_{i=1}^{k-1} |C_i|$ is as small as possible. Let L be the subgraph of G induced by $\bigcup_{i=1}^{k-1} V(C_i)$.

We may assume that $e_i \in E(C_i)$, $1 \le i \le k-1$. Let $e_i = x_i y_i$ with $x_i \in V_1$ and $y_i \in V_2$ for $1 \le i \le k$, $M = G - L$, $|M| = 2m$, and $D = M - \{x_k, y_k\}$. Note that $|D| \ge 2$ and $|V(D) \cap V_1| = |V(D) \cap V_2|$. In most parts of the proof, we only use the assumption that $\sigma_{1,1}(G) \ge n + k$.

Claim 2.1 *We may assume that $d_D(x_k) > 0$ and $d_D(y_k) > 0$.*

Proof. Suppose $d_D(x_k) = 0$ and take any $z \in V(D) \cap V_2$. Then

$$d_M(x_k) + d_M(z) \leq 1 + (m - 1) = m.$$

This implies that

$$d_L(x_k) + d_L(z) \geq n + k - m = k + \sum_{i=1}^{k-1} \frac{|C_i|}{2} > \sum_{i=1}^{k-1} (\frac{|C_i|}{2} + 1).$$

This means that for some i, $1 \leq i \leq k - 1$,

$$d_{C_i}(x_k) + d_{C_i}(z) \geq \frac{|C_i|}{2} + 2.$$

By Lemma 4, there exists $w \in N_{C_i}(x_k)$ such that $\langle V(C_i) \cup \{z\} - \{w\} \rangle$ contains a cycle passing through e_i.

Similarly, by replacing cycles if necessary, we may assume that $N_D(y_k) \neq \emptyset$. ∎

Take any $z \in N_D(x_k)$ and $z' \in N_D(y_k)$. Since M does not contain an admissible short cycle, z and z' are nonadjacent.

We distinguish two cases according to the value $|D|$.

Case 1. $|D| \geq 4$.

Claim 2.2 *We may assume that $d_D(z) > 0$ and $d_D(z') > 0$.*

Proof. Suppose $N_D(z) = \emptyset$ and take any $w \in V(D) \cap V_1 - \{z'\}$. Then

$$d_M(z) + d_M(w) \leq 1 + (m - 1) = m.$$

The rest of the proof is similar to that of Claim 2.1. ∎

Take any $w \in N_D(z)$ and $w' \in N_D(z')$. Let

$$D_1 = N_D(y_k) \cap N_D(w') - \{z'\},$$

and

$$D_2 = N_D(x_k) \cap N_D(w) - \{z\}.$$

Claim 2.3 *We may assume that $|D_1| + |D_2| \leq m - 3$.*

Proof. Suppose $|D_1| + |D_2| \geq m - 2$. Then $D_1 \neq \emptyset$ and $D_2 \neq \emptyset$. Take any $u \in D_2$ and $u' \in D_1$. Since $N_{D_1}(u) = \emptyset$ and $N_{D_2}(u') = \emptyset$,

$$d_M(u) + d_M(u') \leq (m - |D_1| - 1) + (m - |D_2| - 1) = 2m - (|D_1| + |D_2|) - 2 \leq m.$$

By Lemma 4, we can replace the cycles to decrease $|D_1| + |D_2|$. ∎

Let $S = \{w, z, x_k, y_k, z', w'\}$. Since

$$d_M(S) = 10 + |E(S, M - S)| \leq 10 + |M - S| + |D_1| + |D_2| \leq 3m + 1,$$

we get

$$d_L(S) \geq 3(n+k) - 3m - 1 = \sum_{i=1}^{k-1} \frac{3}{2}|C_i| + 3k - 1 > \sum_{i=1}^{k-1}(\frac{3}{2}|C_i| + 3).$$

This means that for some i,

$$d_{C_i}(S) \geq \frac{3}{2}|C_i| + 4.$$

First, suppose $C_i = x_i y_i a a' x_i$ and $d_{C_i}(S) \geq 10$. If $wa', y_k a, z' y_i, w' x_i$ are edges in G, $\langle S \cup V(C_i) \rangle$ contains two admissible cycles $x_k y_k a a' w z x_k$ and $x_i y_i z' w' x_i$. So $|E(G) \cap \{wa', y_k a, z' y_i, w' x_i\}| \leq 3$. Similarly, $|E(G) \cap \{w'a, x_k a', z x_i, w y_i\}| \leq 3$. This means that za and $z'a'$ are edges. If zx_i and $x_k a'$ are edges, $\langle S \cup V(C_i) \rangle$ contains two admissible cycles $x_k y_k z' a' x_k$ and $x_i y_i a z x_i$. So $|E(G) \cap \{z x_i, x_k a'\}| \leq 1$. Similarly, $|E(G) \cap \{z' y_i, y_k a\}| \leq 1$. This means that $wa', w y_i, w' x_i, w'a$ are edges. Then $\langle S \cup V(C_i) \rangle$ contains two admissible cycles $x_k y_k z' a' w z x_k$ and $x_i y_i a w' x_i$.

Next, suppose $C_i = x_i y_i a b b' a' x_i$ and $d_{C_i}(S) \geq 13$. Note that $d_{C_i}(s) \leq 2$ for every $s \in S - \{x_k, y_k\}$ by the minimality of L. Hence $d_{C_i}(\{x_k, y_k, z, z'\}) \geq 9$. By symmetry, we may assume that $d_{C_i}(x_k) = 3$ and $d_{C_i}(z') = 2$. Then $x_k b$ and $z'b$ are edges, and $x_k y_k z' b x_k$ is an admissible cycle shorter than C_i. ∎

Case 2. $|D| = 2$.

Claim 2.4 *For some i, $|C_i| = 4$ and $d_{C_i}(z) = d_{C_i}(z') = 2$.*

Proof. Since $d_M(z) = d_M(z') = 1$,

$$\sum_{i=1}^{k-1} d_{C_i}(\{z, z'\}) \geq n + k - 2 = \sum_{i=1}^{k-1} |C_i|/2 + k > \sum_{i=1}^{k-1}(|C_i|/2 + 1).$$

This means that $d_{C_i}(\{z, z'\}) \geq |C_i|/2 + 2$ for some i. On the other hand, $d_{C_i}(\{z, z'\}) \leq 4$. Hence $|C_i| = 4$ and $d_{C_i}(z) = d_{C_i}(z') = 2$. ∎

We may assume that $d_{C_{k-1}}(z) = d_{C_{k-1}}(z') = 2$ and $C_{k-1} = x_{k-1} y_{k-1} w w' x_{k-1}$. Let $L' = L - C_{k-1}$, $M' = G - L'$ and $S = \{w, z, x_k, y_k, z', w'\}$.

Now we use the assumption that $\sigma_{1,1}(G) \geq \frac{2n-1}{3} + 2k$ or $\delta(G) \geq \frac{2n+4k}{5}$. First, suppose $\sigma_{1,1}(G) \geq \frac{2n-1}{3} + 2k$. Since $w y_k, z z', x_k w' \notin E(G)$,

$$d_G(S) \geq 3\sigma_{1,1}(G) \geq 2n + 6k - 1.$$

Since $d_{M'}(S) \leq 18$,

$$d_{L'}(S) \geq 2n + 6k - 19 = \sum_{i=1}^{k-2} |C_i| + 6k - 11 > \sum_{i=1}^{k-2}(|C_i| + 6).$$

This means that $d_{C_i}(S) \geq |C_i| + 7$ for some i, $1 \leq i \leq k-2$.

Suppose $C_i = x_i y_i a a' x_i$ and $d_{C_i}(S) \geq 11$. By symmetry, we may assume that $d_{C_i}(x_k) = d_{C_i}(z') = d_{C_i}(w') = 2$. If $y_k a$ is an edge, $\langle V(M') \cup V(C_i) \rangle$ contains three admissible cycles $x_k y_k a a' x_k$, $x_{k-1} y_{k-1} w z x_{k-1}$ and $x_i y_i z' w' x_i$. On the other hand, if $z x_i$ and $z a$ are edges, $\langle V(M') \cup V(C_i) \rangle$ contains three admissible cycles $x_k y_k z' a' x_k$, $x_{k-1} y_{k-1} w w' x_{k-1}$ and $x_i y_i a z x_i$.

Suppose $C_i = x_i y_i a b b' a' x_i$ and $d_{C_i}(S) \geq 13$. By symmetry, we may assume that $d_{C_i}(x_k) = 3$ and $d_{C_i}(z') = 2$. Then $x_k b$ and $z' b$ are edges, and $x_k y_k z' b x_k$ is an admissible cycle shorter than C_i.

Next, suppose $\delta(G) \geq \frac{2n+4k}{5}$, and let $S' = \{x_k, y_k, z, z'\}$. Then

$$\begin{aligned} d_{L'}(\{w, w'\}) + 2d_{L'}(S') &\geq 10\delta(G) - 30 \geq 4n + 8k - 30 \\ &= 2\sum_{i=1}^{k-2} |C_i| + 8k - 14 > \sum_{i=1}^{k-2} (2|C_i| + 8). \end{aligned}$$

This means that

$$d_{C_i}(\{w, w'\}) + 2d_{C_i}(S') \geq 2|C_i| + 9$$

for some i, $1 \leq i \leq k-2$. Suppose $C_i = x_i y_i a a' x_i$ and $d_{C_i}(\{w, w'\}) + 2d_{C_i}(S') \geq 17$. In particular, $d_{C_i}(S') \geq 7$. By symmetry, we may assume that $d_{C_i}(x_k) = d_{C_i}(z') = 2$. If $z x_i$ and $z a$ are edges, $\langle V(M') \cup V(C_i) \rangle$ contains three admissible cycles. Similarly, if $w' x_i$ and $w' a$ are edges, $\langle V(M') \cup V(C_i) \rangle$ contains three admissible cycles. Hence $|E(G) \cap \{z x_i, z a\}| \leq 1$ and $|E(G) \cap \{w' x_i, w' a\}| \leq 1$. This means $w a'$, $w y_i$, $y_k a$ are edges. Furthermore, either $z x_i$ or $z a$ is an edge, but in either case $\langle V(M') \cup V(C_i) \rangle$ contains three admissible cycles. Suppose $C_i = x_i y_i a b b' a' x_i$ and $d_{C_i}(\{w, w'\}) + 2d_{C_i}(S') \geq 21$. By symmetry, we may assume that $d_{C_i}(x_k) = 3$ and $d_{C_i}(z') = 2$. Then $x_k b$ and $z' b$ are edges, and $x_k y_k z' b x_k$ is an admissible cycle shorter than C_i.

This completes the proof of Theorem 2.

3 Proof of Theorem 3

We prepare several lemmas before proving Theorem 3.

Lemma 5 *Suppose $k \geq 2$, G is not complete bipartite, and $\sigma_{1,1}(G) \geq n + k$. Then G is $(k+1)$-connected.*

Proof. Suppose G is not $(k+1)$-connected. Then $G - S$ is disconnected for some S with $|S| \leq k$. Let A be a component of $G - S$, and $B = V(G) - (S \cup A)$. We may assume that $|A \cap V_1| + |B \cap V_2| \geq |A \cap V_2| + |B \cap V_1|$. First, suppose $A \cap V_1 \neq \emptyset$ and $B \cap V_2 \neq \emptyset$, and take $u \in A \cap V_1$ and $v \in B \cap V_2$. Then

$$\begin{aligned} d_G(u) + d_G(v) &\leq |A \cap V_2| + |B \cap V_1| + |S| \\ &\leq |G - S|/2 + |S| \\ &\leq n + k/2, \end{aligned}$$

but this contradicts the assumption. Next, suppose $A \cap V_1 = \emptyset$ or $B \cap V_2 = \emptyset$. By symmetry, we may assume that $A \cap V_1 = \emptyset$. If $B \cap V_1 = \emptyset$, $n = |V_1| \leq |S| \leq k$. On the other hand, $k \leq n-2$, since $\sigma_{1,1}(G) \leq 2n-2$ when G is not complete bipartite. This is a contradiction. Hence $B \cap V_1 \neq \emptyset$. Take $u \in B \cap V_1$ and $v \in A \cap V_2$. Then $d_G(u) \leq n-1$ and $d_G(v) \leq |S| \leq k$. This contradicts the assumption that $\sigma_{1,1}(G) \geq n+k$. ∎

Lemma 6 *Suppose C is a cycle in G, $e \in E(C)$, $u \in V(G-C) \cap V_1$, $v \in V(G-C) \cap V_2$, and G contains no cycle D satisfying $e \in E(D)$ and $V(D)$ properly contains $V(C)$. Then*
(1) $d_C(u) + d_C(v) \leq |C|/2 + 1$.
(2) *If $d_C(u) + d_C(v) = |C|/2 + 1$, u and v belong to different components of $G - C$.*

Proof. We may assume that $C = w_1 w_2 \cdots w_r w_1$ with $e = w_1 w_r$ and $w_1 \in V_1$.
(1) If $d_C(u) + d_C(v) \geq |C|/2 + 2$, there exist i and j $(1 \leq i < j \leq r-1)$ with $vw_i, uw_{i+1}, vw_j, uw_{j+1} \in E(G)$. Then the cycle

$$w_1 \cdots w_i v w_j \cdots w_{i+1} u w_{j+1} \cdots w_r w_1$$

passes through e and properly contains $V(C)$.
(2) Suppose $d_C(u) + d_C(v) = |C|/2 + 1$ and u and v belong to the same component of $G - C$. Then there exists i $(1 \leq i \leq r-1)$ with $vw_i, uw_{i+1} \in E(G)$, and a path P connecting u and v in $G - C$. By joining P and $uw_{i+1} \cdots w_r w_1 \cdots w_i v$, we get a cycle that passes through e and properly contains $V(C)$. ∎

A set of admissible cycles $\{C_1, \cdots, C_r\}$ is called *maximal* if there are no admissible cycles $D_1, \cdots, D_r$ such that $\bigcup_{i=1}^r V(D_i)$ properly contains $\bigcup_{i=1}^r V(C_i)$.

Lemma 7 *Suppose $\{C_1, \cdots, C_k\}$ is a maximal set of admissible cycles, and $\sigma_{1,1}(G) \geq n+k$. Then $G - \bigcup_{i=1}^k V(C_i)$ is connected.*

Proof. Suppose $M = G - \bigcup_{i=1}^k V(C_i)$ is not connected. Let M_0 be a component of M and set $M_1 = M - M_0$. We may assume that $|V(M_0) \cap V_1| \geq |V(M_0) \cap V_2|$. Then $|V(M_1) \cap V_1| \leq |V(M_1) \cap V_2|$. Take $u \in V(M_0) \cap V_1$ and $v \in V(M_1) \cap V_2$. Then

$$d_M(u) + d_M(v) \leq |V(M_0) \cap V_2| + |V(M_1) \cap V_1| \leq |M|/2.$$

Hence

$$\sum_{i=1}^k (d_{C_i}(u) + d_{C_i}(v)) \geq n + k - |M|/2 = \sum_{i=1}^k (|C_i|/2 + 1).$$

If $d_{C_i}(u) + d_{C_i}(v) \geq |C_i|/2 + 2$ for some i, there exists a cycle D in $\langle V(C_i) \cup V(M) \rangle$ that passes through e_i and properly contains $V(C_i)$ by Lemma 6. This contradicts the maximality of $\{C_1, \cdots, C_k\}$. Hence $d_{C_i}(u) + d_{C_i}(v) = |C_i|/2 + 1$ for all i and $d_M(u) + d_M(v) = |M|/2$. This means that $|V(M_0) \cap V_1| = |V(M_0) \cap V_2|$, $|V(M_1) \cap V_1| = |V(M_1) \cap V_2|$, and $d_M(u) = V(M_0) \cap V_2$ and $d_M(v) = V(M_1) \cap V_1$. This holds for any $u \in V(M_0) \cap V_1$ and $v \in V(M_1) \cap V_2$. Hence M_0 and M_1 are complete bipartite. Take any $u' \in V(M_0) \cap V_2$ and $v' \in V(M_1) \cap V_1$. By the same arguments as above, $d_{C_i}(u') + d_{C_i}(v') = |C_i|/2 + 1$ for all i. Then $d_{C_1}(\{u, u', v, v'\}) = |C_1| + 2$. By symmetry, we may assume that $d_{C_1}(u) + d_{C_1}(u') \geq |C_1|/2 + 1$. Since u and u' belong to the same component of M, there exists a cycle D in $\langle V(C_1) \cup V(M) \rangle$ that passes through e_1 and properly contains $V(C_1)$ by Lemma 6. This contradicts the maximality of $\{C_1, \cdots, C_k\}$. ∎

Proof of Theorem 3. Let $F = \{e_1, \cdots, e_k\}$, $e_i = x_i y_i$, $x_i \in V_1$, $y_i \in V_2$, and in the rest of the proof, 'admissible' means 'admissible for F.'

Choose admissible cycles $C_1, \cdots, C_k$ such that $\sum_{i=1}^{k} |C_i|$ takes the maximum value, and set $\mathcal{C} = \{C_1, \cdots, C_k\}$. Let $L = \langle \bigcup_{i=1}^{k} V(C_i) \rangle$ and $M = G - L$. Since $\mathcal{C}$ is maximal, M is connected by Lemma 7.

Claim 3.1 *Either $N_{C_i}(M) \cap V_1 = \emptyset$ or $N_{C_i}(M) \cap V_2 = \emptyset$ for every i, $1 \leq i \leq k$.*

Proof. Suppose $N_{C_i}(M) \cap V_1 \neq \emptyset$ and $N_{C_i}(M) \cap V_2 \neq \emptyset$. We may assume $i = 1$, and choose uw and $vz \in E(G)$ with $u \in V(M) \cap V_1$, $v \in V(M) \cap V_2$, and $w, z \in V(C_1)$ satisfying $e_1 \in E(C_1[z, w])$ and $N(M) \cap C_1(w, z) = \emptyset$. If $z = w^+$, there exists a longer admissible cycle than C_1 in $\langle V(C_1) \cup V(M) \rangle$, which contradicts the choice of $\mathcal{C}$. Hence $|C_1(w, z)| \geq 2$. Let D be the cycle obtained by joining $C_1[z, w]$, a path P connecting u and v in M, and the two edges uw and vz. If

$$d_{C_1[z,w]}(\{w^+, z^-\}) \geq |C_1[z, w]|/2 + 2,$$

$C_1[w^+, z^-]$ can be inserted into D, and $\langle V(D) \cup C_1[w^+, z^-] \rangle$ contains a spanning cycle passing through e_1. This contradicts the choice of $\mathcal{C}$. Hence

$$d_{C_1[z,w]}(\{w^+, z^-\}) \leq |C_1[z, w]|/2 + 1.$$

Similarly, if

$$d_{C_i}(\{w^+, z^-\}) \geq |C_i|/2 + 1$$

for some i $(2 \leq i \leq k)$, $\langle V(C_i) \cup C_1[w^+, z^-] \rangle$ contains a spanning cycle passing through e_i, and this contradicts the choice of $\mathcal{C}$. Hence

$$d_{C_i}(\{w^+, z^-\}) \leq |C_i|/2.$$

On the other hand, if

$$d_{C_1[z,w]}(\{u, v\}) \geq |C_1[z, w]|/2 + 2,$$

P can be inserted into C_1, and $\langle V(C_1) \cup V(P) \rangle$ contains a spanning cycle passing through e_1, a contradiction. Also,

$$d_{C_i}(\{u, v\}) \leq |C_i|/2$$

by Lemma 6. Since $d_{C_1(w,z)}(\{u,v\}) = 0$, $d_{C_1(w,z)}(\{w^+, z^-\}) \leq |C_1(w,z)|$, $d_M(\{u,v\}) \leq |M|$ and $d_M(\{w^+, z^-\}) = 0$,

$$d_G(\{u, v, w^+, z^-\}) \leq |M| + \sum_{i=1}^{k} |C_i| + 2 = 2n + 2.$$

This is not possible when $k \geq 2$, since $d_G(u) + d_G(z^-) \geq n+k$ and $d_G(v) + d_G(w^+) \geq n + k$. ∎

By Lemma 5, $|N_L(M)| \geq k+1$. This means $|N_{C_i}(M)| \geq 2$ for some i, and we may assume that $i = 1$. Choose two vertices w and z in $N_{C_1}(M)$ such that $e_1 \in E(C_1[z,w])$ and $N(M) \cap C_1(w,z) = \emptyset$. By Claim 3.1, we may assume that $w, z \in V_2$.

Claim 3.2 $|C_1(w,z)| \geq 3$.

Proof. Suppose $C_1(w,z) = \{a\}$. Then $\langle C_1[z,w] \cup V(M) \rangle$ contains an admissible cycle D such that $V(D)$ properly contains $C_1[z,w]$. Since $N_M(a) = \emptyset$, $G - (V(D) \cup \bigcup_{i=2}^{k} V(C_i))$ is disconnected, and $\{D, C_2, \cdots, C_k\}$ is not maximal by Lemma 7. This contradicts the choice of $\mathcal{C}$. ∎

Take any $u \in N_M(w)$, $u' \in N_M(z)$ and $v \in V(M) \cap V_2$, and set $S = \{w^+, z^{--}, u, v\}$. Note that $z^{--} \in C_1(w,z) \cap V_2$ by Claim 3.2. If $e_1 \neq aa^+$ and $\{a, a^+\} \subset N(\{w^+, z^{--}\})$ for some $a \in C_1[z,w)$, then there is an admissible cycle that contains $(V(C_1) - \{z^-\}) \cup \{u, u'\}$. If $u \neq u'$, this contradicts the maximality of the choice of $\mathcal{C}$. Even if $u = u'$, let D be the admissible cycle such that $V(D) = (V(C_1) - \{z^-\}) \cup \{u\}$. Since $N_M(z^-) = \emptyset$, $G - V(D) - \bigcup_{i=2}^{k} V(C_i)$ is disconnected. By Lemma 7, $\{D, C_2, \cdots, C_k\}$ is not maximal, but this contradicts the choice of $\mathcal{C}$. Hence $d_{C_1[z,w]}(\{w^+, z^{--}\}) \leq (|C_1[z,w]| + 1)/2$. Also, $d_{C_1[z,w]}(v) = 0$ by Claim 3.1, $d_{C_1[z,w]}(u) \leq (|C_1[z,w]| + 1)/2$, $d_{C_1(w,z)}(S) = d_{C_1(w,z)}(\{w^+, z^{--}\}) \leq |C_1(w,z)|$, $d_M(S) = d_M(\{u,v\}) \leq |M|$, and $d_{C_i}(\{w^+, z^{--}\}) \leq |C_i|/2$ and $d_{C_i}(\{u,v\}) \leq |C_i|/2$ for $2 \leq i \leq k$. Summing up these inequalities,

$$d_G(S) \leq |M| + |C_1(w,z)| + |C_1[z,w]| + 1 + \sum_{i=2}^{k} |C_i| = 2n + 1.$$

On the other hand, $d_G(S) \geq 2(n+k)$ since $w^+v, z^{--}u \notin E(G)$. This is a contradiction.

This completes the proof of Theorem 3.

4 Examples

The degree conditions of Theorem 2 are sharp in the following sense. (In the following examples, $E_{i,j} = \{xy \mid x \in V_i, y \in V_j\}$.)

Example 1. Suppose $n \geq 2k$, and let $V(G) = V_1 \cup V_2 \cup V_3 \cup V_4$, where $|V_1| = k$, $|V_2| = 2k-1$, $|V_3| = n-k$ and $|V_4| = n-2k+1$, and $E(G) = E_{1,2} \cup E_{2,3} \cup E_{3,4}$. Then any k independent edges in $\langle V_1 \cup V_2 \rangle$ cannot be contained in k disjoint cycles, while $\sigma_{1,1}(G) = (2k-1)+(n-k) = n+k-1$.

Example 2. Suppose $2k \leq n \leq 3k-2$, and let $V(G) = \bigcup_{i=1}^{8} V_i$, where $|V_1| = |V_2| = n-2k+1$, $|V_3| = |V_6| = \lceil (2n-1)/3 \rceil - k$, $|V_4| = |V_5| = n - \lceil (2n-1)/3 \rceil$ and $|V_7| = |V_8| = 3k-1-n$, and $E(G) = \bigcup_{i=1}^{5} E_{i,i+1} \cup E_{6,1} \cup E_{7,8} \cup \bigcup_{i=1}^{3} (E_{7,2i} \cup E_{8,2i-1})$. Let F_1 be any perfect matching in $\langle V_1 \cup V_2 \rangle$ and F_2 be any perfect matching in $\langle V_7 \cup V_8 \rangle$. Then $|F_1 \cup F_2| = k$, but $F_1 \cup F_2$ cannot be contained in k disjoint cycles. (In fact, if such cycles exist, only edges in F_2 can be contained in cycles of length 4. Hence $n \geq 3k-(3k-1-n) = n+1$, which is impossible.) On the other hand,

$$\begin{aligned}\sigma_{1,1}(G) &\geq 2n - \max\left\{n-2k+1+n-\left\lceil\frac{2n-1}{3}\right\rceil, 2\left(\left\lceil\frac{2n-1}{3}\right\rceil - k\right)\right\} \\ &= \min\left\{2k-1+\left\lceil\frac{2n-1}{3}\right\rceil, 2k+2n-2\left\lceil\frac{2n-1}{3}\right\rceil\right\} \\ &= 2k-1+\left\lceil\frac{2n-1}{3}\right\rceil.\end{aligned}$$

Example 3. Suppose $n \geq 2k$, and let $V(G) = \bigcup_{i=1}^{6} V_i$, where $|V_1| = |V_2| = \lceil (n-k+1)/2 \rceil$, $|V_3| = |V_4| = k-1$ and $|V_5| = |V_6| = \lfloor (n-k+1)/2 \rfloor$, and $E(G) = E_{1,2} \cup E_{1,4} \cup E_{2,3} \cup E_{3,4} \cup E_{3,6} \cup E_{4,5} \cup E_{5,6} \cup \{uv\}$, where $u \in V_1$ and $v \in V_6$. Let F be any perfect matching in $\langle V_3 \cup V_4 \rangle$. Then $F \cup \{uv\}$ cannot be contained in k disjoint cycles, while $\delta(G) = \lfloor (n-k+1)/2 \rfloor + k - 1 = \lceil (n+k)/2 \rceil - 1$.

Example 4. Suppose $2k \leq n \leq 3k-2$, $n \neq 3k-3$ and let $s = \lceil (3n-4k+1)/5 \rceil$, and $V(G) = \bigcup_{i=1}^{8} V_i$, where $|V_1| = |V_2| = \cdots = |V_6| = s$ and $|V_7| = |V_8| = n-3s$, and $E(G) = \bigcup_{i=1}^{5} E_{i,i+1} \cup E_{6,1} \cup E_{7,8} \cup \bigcup_{i=1}^{3} (E_{7,2i} \cup E_{8,2i-1})$. Let F_1 be any perfect matching in $\langle V_1 \cup V_2 \rangle$ and F_2 be any matching of size $k-s$ in $\langle V_7 \cup V_8 \rangle$. (Note that $n-3s \geq k-s$.) Then $F_1 \cup F_2$ cannot be contained in k disjoint cycles. (In fact, if such cycles exist, the number of cycles of length 4 is at most $2(n-k-2s)+(k-s) = 2n-k-5s$. Hence $n \geq 3k-(2n-k-5s)$, which is impossible.) On the other hand, $\delta(G) = n-s = \lceil (2n+4k)/5 \rceil - 1$.

The degree condition of Theorem 3 is sharp in the following sense.

Example 5. Suppose $n \geq 2k+1$, and let $V(G) = V_1 \cup V_2 \cup V_3 \cup V_4$, where $|V_1| = 1$, $|V_2| = k$, $|V_3| = n-1$ and $|V_4| = n-k$, and $E(G) = E_{1,2} \cup E_{2,3} \cup E_{3,4}$. Then k independent edges in $\langle V_2 \cup V_3 \rangle$ are contained in k disjoint cycles, but they cannot be extended to a partition. On the other hand, $\sigma_{1,1}(G) = k+(n-1) = n+k-1$.

Furthermore, the assumption $k \geq 2$ in Theorem 1 and in Theorem 3 is necessary.

Example 6. Let $V(G) = \bigcup_{i=1}^{6} V_i$, where $|V_1| = |V_2| = m$, $|V_3| = |V_4| = 1$, $|V_5| = |V_6| = n-m-1$, and $E(G) = E_{1,2} \cup E_{1,4} \cup E_{2,3} \cup E_{3,4} \cup E_{3,6} \cup E_{4,5} \cup E_{5,6}$. Then

$\sigma_{1,1}(G) = n+1$ and the edge e in $E_{3,4}$ is contained in a cycle, but there is no hamiltonian cycle containing e when $1 \le m \le n-2$.

From this example, we also see that the assumption $k \ge 2$ is necessary in Lemma 5.

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(Received 20/10/99)