On a conjecture of Hilton

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Abstract

We show that if $\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_k$ are collections of distinct subsets from an n-element set such that these collections are incomparable and uncomplemented, then $\sum_{i=1}^{k} |\mathcal{A}_i| \leq 2^{n-1}$ under certain conditions. Upper bounds are also given for $\sum_{i=1}^{k} |\mathcal{A}_i|$ with or without the "uncomplemented" condition.

1 Introduction

Let $\mathcal{A}_1, \cdots, \mathcal{A}_k$ be k collections of distinct subsets of $S = \{1, 2, \ldots, n\}$. These k collections of distinct subsets are called *incomparable* if $A_i \in \mathcal{A}_i$ and $A_j \in \mathcal{A}_j$, $(i \neq j)$, then $A_i \not\subset A_j$. A collection of subsets $\mathcal{C}$ is called *uncomplemented* if $A \in \mathcal{C}$, then $\bar{A} \notin \mathcal{C}$, where $\bar{A} = S \backslash A$.

It is well known that if $\mathcal{C}$ is a collection of distinct subsets of $\{1, 2, \ldots, n\}$ which are uncomplemented, then $|\mathcal{C}| \leq 2^{n-1}$. Hilton extended this result to two incomparable, uncomplemented collections

Theorem 1 [2] *If $\mathcal{A}_1$ and $\mathcal{A}_2$ are collections of distinct subsets of S such that these collections are incomparable and uncomplemented, then*

$$|\mathcal{A}_1| + |\mathcal{A}_2| \leq 2^{n-1}.$$

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He also posed the following conjecture.

Conjecture 1 [4] *If $\mathcal{A}_1$, $\mathcal{A}_2$, ..., $\mathcal{A}_k$ are collections of distinct subsets of n-element set S such that these collections are incomparable and uncomplemented, then*

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq 2^{n-1}.$$

In this paper, we will investigate this conjecture. We give an upper bound and show that this conjecture is true under certain conditions. We also discuss the case when $k = 3$.

The following lemma from Kleitman will be used in our proof.

Lemma 2 [3] *Let $\mathcal{U}$ and $\mathcal{V}$ be collections of subsets of an n element set S, such that*
(i) if $X \in \mathcal{U}$ and $X \subset Y \subset S$, then $Y \in \mathcal{U}$,
(ii) if $X \in \mathcal{V}$ and $Y \subset X \subset S$, then $Y \in \mathcal{V}$. Then

$$|\mathcal{U} \cap \mathcal{V}| \cdot 2^n \leq |\mathcal{U}||\mathcal{V}|.$$

2 Main results

Theorem 3 *Let $\mathcal{A}_1$, $\mathcal{A}_2$, ..., $\mathcal{A}_k$ be incomparable collections of distinct subsets of n-element set S. Then for any $1 \leq j \leq k$,*

$$\sum_{i=1}^{k} |\mathcal{A}_i| + 2[|\mathcal{A}_j| \sum_{i \neq j} |\mathcal{A}_i|]^{\frac{1}{2}} \leq 2^n.$$

Proof. Without loss of generality, we will show

$$\sum_{i=1}^{k} |\mathcal{A}_i| + 2[|\mathcal{A}_1| \sum_{i \neq 1} |\mathcal{A}_i|]^{\frac{1}{2}} \leq 2^n.$$

Let

$$\mathcal{H} = \{Z : \exists A_1 \in \mathcal{A}_1, A_1 \subseteq Z, \exists D \in \cup_{i=2}^{k} \mathcal{A}_i, D \subseteq Z\},$$
$$\mathcal{I}_i = \{Z : \exists A_i \in \mathcal{A}_i, A_i \subseteq Z, \nexists D \in \cup_{j \neq i} \mathcal{A}_j, D \subseteq Z\},$$
$$\mathcal{L} = \{Z : \nexists A_i \in \mathcal{A}_i, A_i \subseteq Z, 1 \leq i \leq k\}.$$

Then clearly, $\mathcal{H} \cap \mathcal{L} = \emptyset$, $\mathcal{H} \cap \mathcal{I}_i = \emptyset$, $\mathcal{L} \cap \mathcal{I}_i = \emptyset$ for $1 \leq i \leq k$ and $\mathcal{I}_i \cap \mathcal{I}_j = \emptyset$ for any $i \neq j$. Therefore,

$$|\mathcal{H}| + \sum_{i=1}^{k} |\mathcal{I}_i| + |\mathcal{L}| \leq 2^n.$$

Let $\mathcal{U} = \mathcal{H} \cup \mathcal{I}_1$ and $\mathcal{V} = \mathcal{L} \cup \mathcal{I}_1$. We claim that both $\mathcal{U}$ and $\mathcal{V}$ satisfy the conditions in Lemma 2. Let $X \in \mathcal{U}$ and $X \subset Y \subset S$. Then there exists an $A_1 \in \mathcal{A}_1$ such that $A_1 \subset X \subset Y$ by the definitions of $\mathcal{H}$ and $\mathcal{I}_1$. If there is a $D \in \cup_{i=2}^{k} \mathcal{A}_i$ such that $D \subseteq X \subset Y$, then $Y \in \mathcal{H} \subset \mathcal{U}$. Otherwise, $Y \in \mathcal{I}_1 \subset \mathcal{U}$.

Now let $X \in \mathcal{V}$ and $Y \subset X$. If there is no $A_1 \in \mathcal{A}_1$ such that $A_1 \subset Y$, then $Y \in \mathcal{L}$ and hence $Y \in \mathcal{V}$. Otherwise, $X \in \mathcal{I}_1$. This implies that there is no $D \in \cup_{j\neq 1}\mathcal{A}_j$ with $D \subseteq Y$. Therefore, $Y \in \mathcal{I}_1 \subset \mathcal{V}$.

By Lemma 2, we have

$$|\mathcal{U} \cap \mathcal{V}|2^n \leq |\mathcal{U}||\mathcal{V}|.$$

That is,

$$|\mathcal{I}_1| \cdot 2^n \leq (|\mathcal{H}| + |\mathcal{I}_1|)(|\mathcal{L}| + |\mathcal{I}_1|).$$

Then

$$|\mathcal{I}_1|(|\mathcal{H}| + \sum_{i=1}^{k} |\mathcal{I}_i| + |\mathcal{L}|) \leq (|\mathcal{H}| + |\mathcal{I}_1|)(|\mathcal{L}| + |\mathcal{I}_1|).$$

Simplify,

$$|\mathcal{I}_1|(\sum_{i=2}^{k} |\mathcal{I}_i|) \leq |\mathcal{H}||\mathcal{L}| \leq [\frac{|\mathcal{H}| + |\mathcal{L}|}{2}]^2 \leq [\frac{2^n - \sum_{i=1}^{k} |\mathcal{I}_i|}{2}]^2.$$

Therefore,

$$\sum_{i=1}^{k} |\mathcal{I}_i| + 2[|\mathcal{I}_1| \sum_{i\neq 1} |\mathcal{I}_i|]^{\frac{1}{2}} \leq 2^n.$$

We note that $\mathcal{A}_i \subseteq \mathcal{I}_i$ for any $i = 1, \ldots, k$ as $\mathcal{A}_1, \ldots \mathcal{A}_k$ are incomparable collections. Hence $|\mathcal{A}_i| \leq |\mathcal{I}_i|$ for $1 \leq i \leq k$. Therefore,

$$\sum_{i=1}^{k} |\mathcal{A}_i| + 2[|\mathcal{A}_1| \sum_{i\neq 1} |\mathcal{A}_i|]^{\frac{1}{2}} \leq 2^n.$$

This completes the proof. ∎

Corollary 4 *Let $\mathcal{A}_1$, $\mathcal{A}_2$, ..., $\mathcal{A}_k$ be incomparable collections of distinct subsets of n-element set S. Let I and J be any partition of $\{1, \ldots, k\}$. Then*

$$\sum_{i=1}^{k} |\mathcal{A}_i| + 2[\textstyle\sum_{j\in J} |\mathcal{A}_j| \displaystyle\sum_{i\in I} |\mathcal{A}_i|]^{\frac{1}{2}} \leq 2^n.$$

Proof. The corollary follows from the fact that $\cup_{i\in I}\mathcal{A}_i$ and $\cup_{j\in J}\mathcal{A}_j$ are incomparable. ∎

The following theorem gives an upper bound if there is no $\mathcal{A}_i$ having its cardinality too large.

Theorem 5 *Let $\mathcal{A}_1$, $\mathcal{A}_2$, ..., $\mathcal{A}_k$ be incomparable collections of distinct subsets of n-element set S. If there is an $I \subset \{1, \ldots, k\}$ such that $\frac{\sum_{i=1}^{k} |\mathcal{A}_i|}{k} \leq \sum_{i\in I} |\mathcal{A}_i| \leq \frac{\sum_{i=1}^{k} |\mathcal{A}_i|}{2}$, then*

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq \frac{k}{2\sqrt{k-1} + k} 2^n.$$

Proof. From Corollary 4,

$$\sum_{i=1}^{k} |\mathcal{A}_i| + 2[\sum\nolimits_{j\in J} |\mathcal{A}_j| \sum_{i\in I} |\mathcal{A}_i|]^{\frac{1}{2}} \le 2^n,$$

where $J = \{1,\ldots,k\} - I$. That is,

$$\sum_{i=1}^{k} |\mathcal{A}_i| + 2[\sum\nolimits_{i\in I} |\mathcal{A}_i| (\sum_{i=1}^{k} |\mathcal{A}_i| - \sum\nolimits_{i\in I} |\mathcal{A}_i|)]^{\frac{1}{2}} \le 2^n.$$

The function $f(x) = \sqrt{x(a-x)}$, where $a = \sum_{i=1}^{k} |\mathcal{A}_i|$ is a constant, is an increasing function for $0 \le x \le \frac{a}{2}$. Therefore, we can replace $\sum_{i\in I}|\mathcal{A}_i|$ by the average $\frac{\sum_{i=1}^{k}|\mathcal{A}_i|}{k}$ in the above inequality. We have

$$\sum_{i=1}^{k} |\mathcal{A}_i| + 2[\frac{\sum_{i=1}^{k} |\mathcal{A}_i|}{k} (\sum_{i=1}^{k} |\mathcal{A}_i| - \frac{\sum_{i=1}^{k} |\mathcal{A}_i|}{k})]^{\frac{1}{2}} \le 2^n.$$

Solving for $\sum_{i=1}^{k} |\mathcal{A}_i|$ yields

$$\sum_{i=1}^{k} |\mathcal{A}_i| \le \frac{k}{2\sqrt{k-1}+k} 2^n.$$

This completes the proof. ∎

Corollary 6 *If $\mathcal{A}_1$ and $\mathcal{A}_2$ are incomparable collections of distinct subsets of n-element set S with $|\mathcal{A}_1| = |\mathcal{A}_2|$, then*

$$|\mathcal{A}_1| + |\mathcal{A}_2| \le 2^{n-1}.$$

Corollary 7 *If $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k$ are incomparable and uncomplemented collections of distinct subsets of n-element set S, then either*

$$\sum_{i=1}^{k} |\mathcal{A}_i| \le \frac{k}{2\sqrt{k-1}+k} 2^n$$

or

$$\sum_{i=1}^{k} |\mathcal{A}_i| < \frac{k}{k-1} 2^{n-1}.$$

Proof. Without loss of generality, we assume $|\mathcal{A}_1| \le |\mathcal{A}_2| \le \cdots \le |\mathcal{A}_k|$. If $|\mathcal{A}_k| \le \sum_{i=1}^{k-1} |\mathcal{A}_i|$, then we take $I = \{k\}$ in Theorem 5, we have $\frac{\sum_{i=1}^{k}|\mathcal{A}_i|}{k} \le \sum_{i\in I} |\mathcal{A}_i| \le \frac{\sum_{i=1}^{k}|\mathcal{A}_i|}{2}$. Therefore,

$$\sum_{i=1}^{k} |\mathcal{A}_i| \le \frac{k}{2\sqrt{k-1}+k} 2^n.$$

Thus, we may assume $|\mathcal{A}_k| > \sum_{i=1}^{k-1} |\mathcal{A}_i|$. If $\frac{\sum_{i=1}^{k} |\mathcal{A}_i|}{k} \leq \sum_{i=1}^{k-1} |\mathcal{A}_i|$, then we take $I = \{1, \ldots, k-1\}$ and have $\frac{\sum_{i=1}^{k} |\mathcal{A}_i|}{k} \leq \sum_{i \in I} |\mathcal{A}_i| \leq \frac{\sum_{i=1}^{k} |\mathcal{A}_i|}{2}$. Hence

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq \frac{k}{2\sqrt{k-1}+k} 2^n$$

by Theorem 5 again.

Therefore, $\frac{\sum_{i=1}^{k} |\mathcal{A}_i|}{k} > \sum_{i=1}^{k-1} |\mathcal{A}_i|$. That is $\sum_{i=1}^{k} |\mathcal{A}_i| > k \sum_{i=1}^{k-1} |\mathcal{A}_i|$. Thus, $|\mathcal{A}_k| > (k-1) \sum_{i=1}^{k-1} |\mathcal{A}_i|$. This is equivalent to $k|\mathcal{A}_k| > (k-1) \sum_{i=1}^{k} |\mathcal{A}_i|$. But $|\mathcal{A}_k| \leq 2^{n-1}$ as $\mathcal{A}_k$ is uncomplemented. Therefore,

$$\sum_{i=1}^{k} |\mathcal{A}_i| < \frac{k}{k-1} 2^{n-1}.$$

This completes the proof. ∎

In [5], Seymour proved the following result.

Theorem 8 *If $\mathcal{A}$ is a collection of subsets of n-set S such that for all $A, B \in \mathcal{A}$, $A \cap B \neq \emptyset$ and $A \cup B \neq S$, then $|\mathcal{A}| \leq 2^{n-2}$.*

Combining Theorems 5 and 8, we have the following result.

Theorem 9 *Let $\mathcal{A}_1$, $\mathcal{A}_2$, ..., $\mathcal{A}_k$ be incomparable collections of distinct subsets of n-element set S. If for each $\mathcal{A}_i$, $A, B \in \mathcal{A}_i$, $A \cap B \neq \emptyset$ and $A \cup B \neq S$, then*

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq \frac{k}{2\sqrt{k-1}+k} 2^n.$$

Proof. We have that for each i, $|\mathcal{A}_i| \leq 2^{n-2}$ by Seymour's result. Let $a = \sum_{i=1}^{k} |\mathcal{A}_i|$. If $a \leq 2^{n-1}$, then we are done. Otherwise, we have that for any i, $|\mathcal{A}_i| \leq \frac{a}{2}$ from Theorem 8. Therefore,

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq \frac{k}{2\sqrt{k-1}+k} 2^n,$$

by Theorem 5. ∎

Lemma 10 *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be collections of distinct subsets of n-element set S such that $\mathcal{A}_1$ and $\mathcal{A}_2$ are incomparable and $\mathcal{A}_1$ is uncomplemented. Then*

(a) $|\mathcal{A}_1| \leq 2^{n-1} - 2^{\lfloor \frac{n}{2} \rfloor} - 2^{\lceil \frac{n}{2} \rceil} + 2$ if $\mathcal{A}_2$ contains a pair of complemented sets.

(b) $|\mathcal{A}_1| \leq 2^{n-1} - 2^{\lfloor \frac{n}{2} \rfloor} - 2^{\lceil \frac{n}{2} \rceil} - 2^{\lfloor \frac{n}{2} \rfloor - 1} + 2$ if $\mathcal{A}_2$ contains more than one pair of complemented sets.

Proof. (a) Let $\mathcal{A}_{1i}$ $(1 \leq i \leq 2)$ and $\mathcal{A}_{2j}$ $(1 \leq j \leq 3)$ be such that

$$\begin{cases} \mathcal{A}_1 = \mathcal{A}_{11} \cup \mathcal{A}_{12}, \\ \mathcal{A}_2 = \mathcal{A}_{21} \cup \mathcal{A}_{22} \cup \mathcal{A}_{23} \cup \bar{\mathcal{A}}_{23}, \end{cases}$$

where $\mathcal{A}_{12} = \bar{\mathcal{A}}_{21}$, $\mathcal{A}_{11} \cap \mathcal{A}_{12} = \emptyset$, $\mathcal{A}_{2i} \cap \mathcal{A}_{2j} = \emptyset$ for $i \neq j$ and $1 \leq i,j \leq 3$, $\mathcal{A}_{2i} \cap \bar{\mathcal{A}}_{23} = \emptyset$ for $1 \leq i \leq 3$, and $\mathcal{A}_2 \cap \bar{\mathcal{A}}_{22} = \emptyset$.

Since $|\mathcal{A}_{23}| \neq 0$, we can choose $A_{23} \in \mathcal{A}_{23}$. Clearly, $S = A_{23} \cup \bar{A}_{23}$. We let $A_{23} = \{a_1, \cdots, a_k\}$ and $\bar{A}_{23} = \{a_{k+1}, \cdots, a_n\}$. Without loss of generality, we assume that $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

We have that, for any $A_1 \in \mathcal{A}_1$,

$$\begin{cases} A_1 \cap A_{23} \neq \emptyset, \\ A_1 \cap \bar{A}_{23} \neq \emptyset, \\ \bar{A}_1 \cap A_{23} \neq \emptyset, \\ \bar{A}_1 \cap \bar{A}_{23} \neq \emptyset. \end{cases} \tag{*}$$

This claim is true since otherwise $\mathcal{A}_1$ and $\mathcal{A}_2$ are not incomparable, which contradicts our assumption.

It follows from (*) that any element A_1 of $\mathcal{A}_1$ can be written as $A_1 = A_{11} \cup A_{12}$, where A_{11} and A_{12} are proper subsets of A_{23} and $\bar{A}_{23}$, respectively. Obviously, $1 \leq |A_{11}| \leq k-1$ and $1 \leq |A_{21}| \leq n-k-1$. It is easy to see that there are at most

$$\sum_{j=1}^{n-k-1} \sum_{i=1}^{k-1} \binom{k}{i} \binom{n-k}{j} = (2^k - 2)(2^{n-k} - 2)$$

such subsets satisfying the property (*).

Note that if $A_1 = A_{11} \cup A_{12}$, where $A_{11} \subset A_{23}$ and $A_{12} \subset \bar{A}_{23}$, then $(A_{23} \backslash A_{11}) \cup (\bar{A}_{23} \backslash A_{12})$ is also a subset satisfying the property (*). Since $\mathcal{A}_1$ is uncomplemented, we have that

$$\begin{aligned} |\mathcal{A}_1| &\leq \frac{1}{2}(2^k - 2)(2^{n-k} - 2) \\ &= 2^{n-1} - 2^k - 2^{n-k} + 2. \end{aligned} \tag{**}$$

It is easy to verify that the function $2^x + 2^{n-x}$ is a decreasing function if $1 \leq x \leq \lfloor \frac{n}{2} \rfloor$. Therefore, taking $x = \lfloor \frac{n}{2} \rfloor$, we have

$$\begin{aligned} |\mathcal{A}_1| &\leq 2^{n-1} - 2^{\lfloor \frac{n}{2} \rfloor} - 2^{n - \lfloor \frac{n}{2} \rfloor} + 2 \\ &= 2^{n-1} - 2^{\lfloor \frac{n}{2} \rfloor} - 2^{\lceil \frac{n}{2} \rceil} + 2. \end{aligned}$$

This completes the proof of (a).

(b) We divide the proof of (b) into two cases.

Case 1. $1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$.

Taking $x = \lfloor \frac{n}{2} \rfloor - 1$ in (**), we have

$$\begin{aligned} |\mathcal{A}_1| &\leq 2^{n-1} - 2^{\lfloor \frac{n}{2} \rfloor - 1} - 2^{n - \lfloor \frac{n}{2} \rfloor + 1} + 2 \\ &= 2^{n-1} - 2^{\lfloor \frac{n}{2} \rfloor - 1} - 2^{\lceil \frac{n}{2} \rceil + 1} + 2 \\ &\leq 2^{n-1} - 2^{\lfloor n/2 \rfloor} - 2^{\lceil n/2 \rceil} - 2^{\lfloor n/2 \rfloor - 1} + 2. \end{aligned}$$

Case 2. $k = \lfloor \frac{n}{2} \rfloor$.

In this case, we have, that for any $A_{23} \in \mathcal{A}_{23}$, $|A_{23}| = \lfloor \frac{n}{2} \rfloor$. We pick a $B_1 \in \mathcal{A}_{23}$ and a $B_2 \in \mathcal{A}_{23}$ where $B_1 \neq \overline{B_2}$, $B_1 \neq B_2$, and $|B_1| = |B_2| = \lfloor \frac{n}{2} \rfloor$.

Case 2.1. $n \equiv 1 \pmod 2$.

Observe that $B_1 \cap B_2 < \lfloor \frac{n}{2} \rfloor$. Otherwise we would have a contradiction. First we assume that $1 \leq |B_1 \cap B_2| = x < \lfloor \frac{n}{2} \rfloor$. Then $|B_1 \cap \overline{B_2}| = \lfloor \frac{n}{2} \rfloor - x$. By repeating the argument in the proof of (a) we deduce that the number of A's which intersect B_1 and $\overline{B_1}$ properly, and do not contain all, is $(2^{\lfloor \frac{n}{2} \rfloor} - 2)(2^{n-\lfloor \frac{n}{2} \rfloor} - 2)$. The number of these A's contained in B_2 is $(2^x - 1)(2^{\lfloor \frac{n}{2} \rfloor - x} - 1)$. The number of these A's containing B_2 is $(2^{x+1} - 1)(2^{\lfloor \frac{n}{2} \rfloor - x} - 1)$. The number of these A's contained in $\overline{B_2}$ is $(2^{x+1} - 1)(2^{\lfloor \frac{n}{2} \rfloor - x} - 1)$. The number of these A's containing $\overline{B_2}$ is $(2^x - 1)(2^{\lfloor \frac{n}{2} \rfloor - x} - 1)$. Therefore,

$$\begin{aligned}|\mathcal{A}_1| &\leq \frac{1}{2}\{(2^{\lfloor \frac{n}{2} \rfloor} - 2)(2^{n-\lfloor \frac{n}{2} \rfloor} - 2) - 2(2^x - 1)(2^{\lfloor \frac{n}{2} \rfloor - x} - 1) \\ &\quad -2(2^{x+1} - 1)(2^{\lfloor \frac{n}{2} \rfloor - x} - 1)\} \\ &\leq 2^{n-1} - 2^{\lfloor n/2 \rfloor} - 2^{n-\lfloor n/2 \rfloor} - 2^{\lfloor n/2 \rfloor - 1} + 2.\end{aligned}$$

Next we assume $|B_1 \cap B_2| = 0$. Then $B_2 \subset \overline{B_1}$ and $|\overline{B_1} \cap \overline{B_2}| = 1$. It follows that $|\overline{B_1}| = |\overline{B_2}| = \lfloor \frac{n}{2} \rfloor + 1$. Repeating the proof in the above, we deduce that the number of A's which intersect B_1 and $\overline{B_1}$ properly, and do not contain all, is $(2^{\lfloor \frac{n}{2} \rfloor} - 2)(2^{n-\lfloor \frac{n}{2} \rfloor} - 2)$. The number of these A's contained in $\overline{B_2}$ is $2^{\lfloor n/2 \rfloor} - 2$. The number of these A's containing B_2 is $2^{\lfloor n/2 \rfloor} - 2$. Therefore,

$$\begin{aligned}|\mathcal{A}_1| &\leq \frac{1}{2}\{(2^{\lfloor \frac{n}{2} \rfloor} - 2)(2^{n-\lfloor \frac{n}{2} \rfloor} - 2) - 2(2^{\lfloor n/2 \rfloor} - 2)\} \\ &\leq 2^{n-1} - 2^{\lfloor n/2 \rfloor} - 2^{n-\lfloor n/2 \rfloor} - 2^{\lfloor n/2 \rfloor - 1} + 2.\end{aligned}$$

Case 2.2. $n \equiv 0 \pmod 2$.

In this case we only have that $1 \leq |B_1 \cap B_2| \leq \lfloor n/2 \rfloor - 1$. By repeating the argument of Case 2.1, we conclude that

$$\begin{aligned}|\mathcal{A}_1| &\leq \frac{1}{2}\{(2^{\lfloor \frac{n}{2} \rfloor} - 2)(2^{n-\lfloor \frac{n}{2} \rfloor} - 2) - 2(2^{x+1} - 1)(2^{\lfloor \frac{n}{2} \rfloor - x} - 1) \\ &\quad -2(2^x - 1)(2^{\lfloor \frac{n}{2} \rfloor - x} - 1)\} \\ &\leq 2^{n-1} - 2^{\lfloor n/2 \rfloor} - 2^{n-\lfloor n/2 \rfloor} - 2^{\lfloor n/2 \rfloor - 1} + 2.\end{aligned}$$

This completes the proof. ∎

Theorem 11 *If $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k$ are collections of distinct subsets of n-element set S such that these collections are incomparable and uncomplemented, then*

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq 2^{n-1},$$

if $max_{1 \leq i \leq k}\{|\mathcal{A}_i|\} > 2^{n-1} - 2^{\lfloor \frac{n}{2} \rfloor} - 2^{\lceil \frac{n}{2} \rceil} + 2$.

Proof. Without loss of generality, we assume that $|\mathcal{A}_1| = max_{1\leq i\leq k}\{|\mathcal{A}_i|\}$. Let $\mathcal{B} = \cup_{i=2}^{n}\mathcal{A}_i$. Then $\mathcal{A}_1$ and $\mathcal{B}$ are incomparable. If $\mathcal{B}$ is not uncomplemented, then $|\mathcal{A}_1| \leq 2^{n-1} - 2^{\lfloor\frac{n}{2}\rfloor} - 2^{\lceil\frac{n}{2}\rceil} + 2$ by Lemma 10 (a), which is a contradiction. Therefore, both $\mathcal{A}_1$ and $\mathcal{B}$ are uncomplemented and hence $|\mathcal{A}_1| + |\mathcal{B}| \leq 2^{n-1}$ by Theorem 1. That is, $\sum_{i=1}^{k} |\mathcal{A}_i| \leq 2^{n-1}$. ∎

Theorem 12 *If $\mathcal{A}_1$, $\mathcal{A}_2$, …, $\mathcal{A}_k$ are collections of distinct subsets of n-elements set S such that these collections are incomparable and uncomplemented, then*

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq 2^{n-1} + 1,$$

if $max_{1\leq i\leq k}\{|\mathcal{A}_i|\} > 2^{n-1} - 2^{\lfloor\frac{n}{2}\rfloor} - 2^{\lceil\frac{n}{2}\rceil} - 2^{\lfloor\frac{n}{2}\rfloor-1} + 2$.

Proof. Without loss of generality, we assume that $|\mathcal{A}_1| = max_{1\leq i\leq k}\{|\mathcal{A}_i|\}$. Let $\mathcal{B} = \cup_{i=2}^{n}\mathcal{A}_i$. Then $\mathcal{A}_1$ and $\mathcal{B}$ are incomparable. If $\mathcal{B}$ contains more than one pair of complemented sets, then $|\mathcal{A}_1| \leq 2^{n-1} - 2^{\lfloor\frac{n}{2}\rfloor} - 2^{\lceil\frac{n}{2}\rceil} - 2^{\lfloor\frac{n}{2}\rfloor-1} + 2$ by Lemma 10 (b), which is a contradiction. Therefore, $\mathcal{B}$ contains at most one pair of complemented sets. Let U be one of the set in the pair. Then $\mathcal{B}$ - U is uncomplemented, therefore, $|\mathcal{A}_1| + |\mathcal{B} - U| \leq 2^{n-1}$. That is, $\sum_{i=1}^{k} |\mathcal{A}_i| \leq 2^{n-1} + 1$. ∎

3 The case $k = 3$

Let $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ be collections of distinct subsets of n-element set S such that these collections are incomparable and uncomplemented. Then we can partition $\mathcal{A}_1$ into $\mathcal{A}_{11}$, $\mathcal{A}_{12}$, $\mathcal{A}_{13}$ such that $\overline{\mathcal{A}}_{12}$ is contained in $\mathcal{A}_2$ and $\overline{\mathcal{A}}_{13}$ is contained in $\mathcal{A}_3$. The similar partition applies to $\mathcal{A}_2$ and $\mathcal{A}_3$. Therefore, we have the following partitions:
$\mathcal{A}_1 = \mathcal{A}_{11} \cup \mathcal{A}_{12} \cup \mathcal{A}_{13}$, $\mathcal{A}_2 = \mathcal{A}_{21} \cup \mathcal{A}_{22} \cup \mathcal{A}_{23}$, $\mathcal{A}_3 = \mathcal{A}_{31} \cup \mathcal{A}_{32} \cup \mathcal{A}_{33}$,
such that $\overline{\mathcal{A}}_{i,j} = \mathcal{A}_{j,i}$ for $i \neq j$.

We have the following result.

Theorem 13 *Let $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ be collections of distinct subsets of n-element set S such that these collections are incomparable and uncomplemented. Then for any $1 \leq i, j \leq 3$,*

$$|\mathcal{A}_1| + |\mathcal{A}_2| + |\mathcal{A}_3| \leq 2^{n-1} + |\mathcal{A}_{i,j}|.$$

Proof. Without loss of generality, we need only to show that

$$|\mathcal{A}_1| + |\mathcal{A}_2| + |\mathcal{A}_3| \leq 2^{n-1} + |\mathcal{A}_{3,j}|,$$

for $j = 1, 2, 3$. There are three cases.

Case 1. $j = 1$.

Let $\mathcal{B}_1 = \mathcal{A}_1 \cup \mathcal{A}_{32} \cup \mathcal{A}_{33}$ and $\mathcal{B}_2 = \mathcal{A}_2$. Then $\mathcal{B}_1$ and $\mathcal{B}_2$ are collections of uncomplemented and incomparable. By Theorem 1,

$$|\mathcal{B}_1| + |\mathcal{B}_2| \leq 2^{n-1}.$$

Therefore,

$$|\mathcal{A}_1| + |\mathcal{A}_2| + |\mathcal{A}_3| \leq 2^{n-1} + |\mathcal{A}_{3,1}|.$$

Case 2. $j = 2$.

The proof is similar to Case 1.

Case 3. $j = 3$.

Let $\mathcal{C}_1 = \mathcal{A}_1 \cup \mathcal{A}_{32}$ and $\mathcal{C}_2 = \mathcal{A}_2 \cup \mathcal{A}_{31}$. Then $\mathcal{C}_1$ and $\mathcal{C}_2$ are collections of uncomplemented subsets from S. To show that they are incomparable, we need to show that if $A \in \mathcal{C}_1, A = A_{32} \in \mathcal{A}_{32}$, and $B \in \mathcal{C}_2, B = A_{31} \in \mathcal{A}_{31}$, then $A \not\subset B$ and $B \not\subset A$. We observe that $A_{32} \not\subset A_{31}$. Otherwise, $\overline{A_{31}} \subset \overline{A_{32}}$. But $\overline{A_{31}}$ is in $\mathcal{A}_{13}$ and $\overline{A_{32}}$ is in $\mathcal{A}_{23}$, which contradicts the fact that $\mathcal{A}_1$ and $\mathcal{A}_2$ are incomparable. Similarly, $A_{31} \not\subset A_{32}$. Therefore, $\mathcal{C}_1$ and $\mathcal{C}_2$ are incomparable.

By Theorem 1 again,

$$|\mathcal{C}_1| + |\mathcal{C}_2| \leq 2^{n-1}.$$

Therefore,

$$|\mathcal{A}_1| + |\mathcal{A}_2| + |\mathcal{A}_3| \leq 2^{n-1} + |\mathcal{A}_{3,3}|.$$

This completes the proof. ∎

Remark *We note that in many cases,* $\min\{|\mathcal{A}_{ij}| : 1 \leq i, j \leq 3\}$ *is zero.*

Corollary 14 *Let $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ be collections of distinct subsets of n-element set S such that these collections are incomparable and uncomplemented. Then*

$$|\mathcal{A}_1| + |\mathcal{A}_2| + |\mathcal{A}_3| \leq \frac{9}{8} \cdot 2^{n-1}.$$

Proof. By Theorem 13, we have that for any $1 \leq i, j \leq 3$,

$$|\mathcal{A}_1| + |\mathcal{A}_2| + |\mathcal{A}_3| \leq 2^{n-1} + |\mathcal{A}_{i,j}|.$$

Summing up over all $1 \leq i, j \leq 3$, we have

$$9(|\mathcal{A}_1| + |\mathcal{A}_2| + |\mathcal{A}_3|) = 9 \times 2^{n-1} + (|\mathcal{A}_1| + |\mathcal{A}_2| + |\mathcal{A}_3|).$$

Therefore,

$$|\mathcal{A}_1| + |\mathcal{A}_2| + |\mathcal{A}_3| \leq \frac{9}{8} \cdot 2^{n-1}.$$

This completes the proof. ∎

Acknowledgments

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