Partitioning Segre varieties and projective spaces*

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Abstract

The recent interest both in partitions of finite geometries into other geometric objects and in the classical Segre varieties over finite fields are the background motivation for this paper. More precisely, partitions of Segre varieties into Segre varieties are investigated and the idea of nested partitions is introduced. Other partitions, namely of projective spaces and hyperbolic quadrics, are also studied.

* The authors are partially supported by GNSAGA and by the Italian Ministry for University Research and Technology (project: *Strutture geometriche, combinatoria e loro applicazioni*).

1 Introduction

Corrado Segre [12] introduced the varieties which are named after him a bit over one century ago and the projective spaces he considered are real or complex. However, the definition, as well as many of their properties, carry over to any field, as B. Segre noted in [11], where the finite field case was first studied.

Quite recently, new interest in the field arose, as it was shown in [1] that Segre varieties can be partitioned into Veronese surfaces. On the other hand, projective spaces can be partitioned into Segre varieties [2] which shows that such varieties, in a certain sense, behave like projective spaces. Also, in a recent paper [3], Leedham-Green and O'Brien study tensor products of vector spaces as projective geometries and the geometric object naturally associated with a tensor product is a Segre variety. These facts inspired the present paper whose aim is to point out further links between Segre varieties and projective spaces, mainly over finite fields, and to provide new partitions of geometric objects.

First of all, we show that projective spaces are naturally embedded in Segre varieties using incidence properties only.

Next, we provide partitions of Segre varieties into Segre varieties. Thus, we construct such partitions and by iterating the construction we achieve what we call a nested partition. Several methods of constructing partitions of Segre varieties into Segre varieties are shown.

It is well known that the smallest non-trivial Segre variety is a hyperbolic quadric in a 3–space; for such a quadric, under the assumption the ground field has square order q^2, we construct a partition into elliptic quadrics over the subfield of order q.

Only Segre varieties with two indices are considered, i.e. varieties which are products of two projective spaces. Most of the results extend to more general cases, even if notation becomes a bit messy.

2 Notation

Let $PG(h,F)$ and $PG(k,F)$ be projective spaces over a field F, with $h,k \geq 0$. Set $N := (h+1)(k+1)-1$.

For all points $x = (x_0,\ldots,x_h) \in PG(h,F)$ and $y = (y_0,\ldots,y_k) \in PG(k,F)$, define

$$x \otimes y = (x_0y_0,\ldots,x_0y_k,x_1y_0,\ldots,x_1y_k,\ldots,x_hy_0,\ldots,x_hy_k).$$

The *Segre variety*, product of $PG(h,F)$ and $PG(k,F)$, is the variety $S_{h,k}$ of $PG(N,F)$, consisting of all points represented by the vectors $x \otimes y$, as x and y range over all points of $PG(h,F)$ and $PG(k,F)$, respectively.

A more classical, and equivalent, definition of a Segre variety is via its parametric equations. Let x_i, with $i = 0,1,\ldots,h$, be homogeneous projective coordinates in $PG(h,F)$ and let y_j, with $j = 0,1,\ldots,k$, be homogeneous projective coordinates in $PG(k,F)$. The Segre variety $S_{h,k}$ has parametric equations:

$$\xi_{ij} = x_iy_j \ , \ i = 0,1,\ldots,h \ , \ j = 0,1,\ldots,k,$$

with ξ_{ij} homogeneous projective coordinates in $PG(N, F)$.

$S_{h,k}$ contains two systems (reguli) of spaces, say $\mathcal{M}_1$ and $\mathcal{M}_2$, such that $\mathcal{M}_1$ consists of k–dimensional spaces, each of which is an $S_{0,k}$, and $\mathcal{M}_2$ consists of h–dimensional spaces, each of which is an $S_{h,0}$. Spaces of the same system are pairwise skew and any two spaces of different systems meet in exactly one point. The spaces of each system partition $S_{h,k}$.

When F is $GF(q)$, then $S_{h,k}$ contains $\theta_h\theta_k = (q^h + q^{h-1} + \ldots + q + 1)(q^k + q^{k-1} + \ldots + q + 1)$ points, where, for any non-negative integer n, $\theta_n = q^n + q^{n-1} + \ldots + q + 1$. Moreover, $\mathcal{M}_1$ consists of θ_h k–dimensional spaces and $\mathcal{M}_2$ consists of θ_k h–dimensional spaces.

Furthermore, a Segre variety $S_{h,k}$ of $PG(N, F)$ has an automorphism group either isomorphic to $PGL(h+1, F) \times PGL(k+1, F)$ if $h \neq k$ or isomorphic to $PGL(h+1, F) \times PGL(k+1, F) \times C_2$ if $h = k$ ([10], Thm. 25.5.13).

Finally, we denote by $\mathcal{G}_{1,n}$ the Grassmannian of the lines of $PG(n, F)$, i.e. the variety of $PG(M, F)$, $M = \binom{n+1}{2} - 1$, representing, under the Plücker map, the 1-dimensional subspaces of $PG(n, F)$.

For background and more details, see [7], [8], [9], [10].

A flock of the Segre variety $S_{h,h}$ over $GF(q)$ is a partition of $S_{h,h}$ into θ_h Veronese surfaces. A flock is linear if all the spaces of the Veronese varieties of the flock share an h–dimensional space. For more details, see [1].

3 Segre varieties and projective geometries

In this section we show that in any Segre variety $S_{k,n}$, over $GF(q)$, $k < n$, there is a natural structure of projective space of dimension n.

In $PG(2n+1, q)$ fix a line A and an $n-$dimensional subspace B skew with A, and consider the lines joining all the points on A with the points on B. Under the Plücker map, these lines are mapped onto the points of $\mathcal{G}_{1,n}$ contained in a suitable $(2n+1)-$dimensional space which intersects $\mathcal{G}_{1,n}$ in $S_{1,n}$.

Define the following geometry $G^{1,n}$:
the points are the lines of $S_{1,n}$ which form the regulus of 1–spaces of the Segre variety; the $m-$dimensional subspaces $G^{1,n}_m$'s are the $S_{1,m}$'s canonically embedded in $S_{1,n}$; hence, the lines are the hyperbolic quadrics $S_{1,1}$ canonically embedded in $S_{1,n}$, and similarly for all dimensions.

Note that the number of points of $G^{1,n}$ is $\theta_n = q^n + q^{n-1} + ... + q + 1$. A subspace $G^{1,n}_m = S_{1,m}$ of positive dimension m consists of the images of the points on all the lines joining the points on the line A with the points of a suitable subspace $U \subset B$ of dimension m, and a subspace $G^{1,n}_l = S_{1,l}$ of positive dimension l consists of the images of the points on all the lines joining the points on the line A with the points of a suitable subspace $V \subset B$ of dimension l. Hence, the subspace $G^{1,n}_m \cap G^{1,n}_l$ is the image of the set of points on the lines joining the points on the line A with the points of the subspace $U \cap V$. Therefore, the incidence structure $G^{1,n}$ is a projective geometry of dimension n, as all the incidence properties which characterise a projective geometry follow from the corresponding properties of $PG(2n+1, q)$.

The argument above easily generalises by taking the Segre variety $S_{k,n}$, $k < n$, giving the geometry $G^{k,n}$. Points are the $S_{k,0}$'s (which form the regulus of k–spaces), lines are the $S_{k,1}$'s canonically embedded in $S_{k,n}$, and so on.

This proves the following result.

Theorem 3.1 *The geometry $G^{k,n}$, embedded in the Segre variety $S_{k,n}$, is a projective space isomorphic to $PG(n,q)$.* □

Example. Let $n = 3$. The projective geometry $G^{1,3} \cong PG(3,q)$ is the following.

Points are the lines of $S_{1,3}$ of one regulus, and they number $\theta_3 = q^3 + q^2 + q + 1$;

Lines are the hyperbolic quadrics $Q^+(3,q) = S_{1,1}$ canonically embedded in $S_{1,3}$, and they number $(q^2+1)(q^2+q+1)$;

Planes are the Segre varieties $S_{1,2}$'s canonically embedded in $S_{1,3}$, and they number θ_3.

Observe that Theorem 3.1 is true over any field but the proof is not by counting arguments but follows from the incidence properties.

4 Nested partitions of Segre varieties

Can a Segre variety $S_{3,3} \subset PG(15,q)$ be partitioned into hyperbolic quadrics $Q^+(3,q)$? More generally, can a Segre variety $S_{h,k}$ be partitioned into Segre varieties?

Fix an $n-$spread, say $\mathcal{S}_1$, in $PG(2n+1,q)$, and an $m-$spread, say $\mathcal{S}_2$, in $PG(2m+1,q)$.

Construct the Segre variety $S_{(2n+1),(2m+1)}$ as $PG(2n+1,q) \otimes PG(2m+1,q)$. For any $A \in \mathcal{S}_1$ and $B \in \mathcal{S}_2$, the Segre variety $S_{n,m} \cong A \otimes B$ is naturally embedded in $S_{(2n+1),(2m+1)}$. Moreover, if $(A,B) \neq (A',B')$, with $A, A' \in \mathcal{S}_1$ and $B, B' \in \mathcal{S}_2$, then $S_{n,m} \cong A \otimes B$ is disjoint from $S'_{n,m} \cong A' \otimes B'$. The set $\mathcal{F} = \{S_{n,m} \cong A \otimes B | A \in \mathcal{S}_1,\ B \in \mathcal{S}_2\}$ has cardinality $(q^{n+1}+1)(q^{m+1}+1)$ and is a partition of $S_{(2n+1),(2m+1)}$ because each $S_{n,m} \in \mathcal{F}$ consists of $\theta_n \theta_m$ points, and the number of points of $S_{(2n+1),(2m+1)}$ is $\theta_{2n+1}\theta_{2m+1} = (q^n + q^{n-1} + \ldots + q + 1)(q^m + q^{m-1} + \ldots + q + 1)(q^{n+1}+1)(q^{m+1}+1)$. Hence, we have constructed a partition of the Segre variety $S_{(2n+1),(2m+1)}$ into Segre varieties $S_{n,m}$'s. Repeating the above construction yields a partition of $S_{(4n+3),(4m+3)}$ into $S_{(2n+1),(2m+1)}$'s, each partitioned into $S_{n,m}$'s; therefore, an ascending chain of partitions into Segre varieties of $S_{(2^s(n+1)-1),(2^t(m+1)-1)}$ is defined, which we call a *nested partition* of Segre varieties with *basis* $S_{n,m}$.

For any choice of the dimensions n, m such that 2 does not divide $\gcd(n-1, m-1)$, the Segre variety $S_{n,m}$ is the "bottom" of the ascending chain of the Segre varieties in the nested partition. On the other hand, if 2 divides $\gcd(n-1, m-1) = 2d$, then write $n = 2dh + 1$, $m = 2dk + 1$, and $S_{n,m}$ is in the nested partition with basis $S_{dh,dk}$. Therefore, it suffices to consider only nested partitions with a minimal basis, which is an $S_{n,m}$ such that 2 does not divide $\gcd(n-1, m-1)$. Note that nested partitions with different (minimal) bases share no elements.

Observe that the same result holds in the infinite case even if the proof is not by counting arguments (coordinates may be used).

The construction of a nested partition, for $n = m$, can be given also via projectivities, and provides a partition of the Segre variety $S_{(2n+1),(2n+1)}$. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be the two systems of pairwise disjoint $(2n+1)$–dimensional spaces. For $i, j \in \{1, 2\}$ and $i \neq j$, the spaces of $\mathcal{M}_i$ define a projectivity ϕ between any two given spaces π_1 and π_2 of $\mathcal{M}_j$ by taking as corresponding points the intersections with the same space of $\mathcal{M}_i$. Take π_1 and π_2 in $\mathcal{M}_j$, and take any spread $\mathcal{S}_1$ of n–dimensional subspaces in π_1.

Then, $\mathcal{S}_1^{\phi}$ is a spread of n–spaces in π_2, and $\mathcal{S}_1 \wedge \mathcal{S}_1^{\phi} = \{A \wedge A^{\phi} : A \in \mathcal{S}_1\}$ is a block of the partition of $S_{(2n+1),(2n+1)}$ into $S_{n,n}$'s we want to construct. By taking a cyclic permutation σ acting on $\mathcal{S}_1^{\phi}$ and considering $\mathcal{S}_1 \wedge \mathcal{S}_1^{\phi\sigma^j}$, $j = 1, 2, \ldots, q^n$, we get the other blocks of the partition.

Further, observe that the construction in this section does not require ϕ to be a projectivity. Indeed, ϕ was used only to get the spread $\mathcal{S}_1^{\phi}$. We could also take any two spreads, $\mathcal{S}_i \in \pi_i$, $i = 1, 2$, and any bijection ϕ between them, and obtain a partition by the same argument.

We observe that this construction works also for infinite fields, provided the required infinite permutation can be found. On the other hand, since the members of the spread can always be viewed as the points of a suitable projective space, the shifting can be achieved by using the relevant projective group.

Remark 4.1 Let ϕ be the projectivity between the two spreads $\mathcal{S}_1$ and $\mathcal{S}_2$, $\mathcal{S}_j \in \pi_j$, with $j = 1, 2$, π_1 and π_2 both belonging to say $\mathcal{M}_1$, in which points correspond if and only if they lie in some $\tilde{\pi} \in \mathcal{M}_2$.

Let $\tau \in \mathrm{Stab}(\mathcal{S}_1)$; then $\phi\tau$ is in $\mathrm{Stab}(\mathcal{S}_2)$. Hence, $\mathrm{Stab}(\mathcal{S}_1) \otimes \mathrm{Stab}(\mathcal{S}_2)$ is a subgroup of $PGL(2n+1, q) \otimes PGL(2n+1, q)$ which, in the general case, is a proper subgroup.

Remark 4.2 For any $S_{h,k}$ (over any field) such a projectivity ϕ exists. It is defined between any two h–spaces by the k–spaces and between any two k–spaces by the h–spaces: two points correspond under ϕ iff they belong to the same space of the other system.

Remark 4.3 Obviously, any Segre variety which is the product of two spaces admitting a partition into subgeometries admits a partition into Segre varieties belonging to these subgeometries. E.g., $PG(2, q^2) \otimes PG(2, q^2)$ admits a partition into $PG(2, q) \otimes PG(2, q)$, [8].

Remark 4.4 By the argument involving spreads, whenever $(d+1)|(h+1)$ and $(r+1)|(k+1)$, a decomposition of $S_{h,k}$ into $S_{d,r}$'s is obtained.

Clearly, the construction above can also be presented via coordinates using the parametric equations of a Segre variety, i.e.

$$\xi_{ij} = x_i y_j \ , \ i, j = 0, 1, 2, \ldots, n-1,$$

with ξ_{ij} homogeneous projective coordinates in $PG(n^2 + 2n, F)$, and the relevant canonical forms of the involved projectivities and the considered spaces of one regulus can always be chosen so that they have easy to handle equations.

5 Nested partitions and linear flocks

We can also construct nested partitions via the action of a suitable group, naturally partitioning each Segre variety into Veronese surfaces which form linear flocks.

Theorem 5.1 *The Segre variety $S_{(2n+1),(2n+1)}$ can be partitioned into Segre varieties $S_{n,n}$'s. Furthermore, such a partition can be obtained by a group which naturally partitions each Segre variety into linear flocks.*

Proof. The proof is given for $n = 1$, the general case following by a similar argument.

Let S be a Singer cycle of $GL(4,q)$ and V its natural module. The matrix S is similar in $GL(4,q^4)$ to the diagonal matrix $D = \text{diag}\ (\omega, \omega^q, \omega^{q^2}, \omega^{q^3})$, where ω is a primitive element of $GF(q^4)$ over $GF(q)$.

It is known that $\langle S \rangle$ has projective order $(q+1)(q^2+1)$ and admits a subgroup, say A, of order q^2+1 which acts irreducibly and semiregularly on $PG(3,q)$ and whose orbits are elliptic quadrics, and a subgroup, say $B = \langle b \rangle$, which is reducible and its orbits are lines forming a regular spread of $PG(3,q)$. We are interested in the latter subgroup.

The Kronecker product $b \otimes b$ has a rational canonical form which is a block diagonal matrix. In particular, $b \otimes b$ is the direct sum of four 4×4 blocks, each of which is the lifting of a Singer cycle of $GL(2,q)$ to a collineation of $PG(3,q)$ fixing a hyperbolic quadric as described in [4]. Of course, the collineation group $\overline{\langle b \otimes b \rangle}$ of $PG(3,q)$, induced by $\langle b \otimes b \rangle$, fixes the Segre variety $S_{3,3} \subset PG(15,q)$.

We can rearrange entries in the canonical form of $b \otimes b$ so that the following matrix is obtained: diag (E, T^2, T^2, T^2, T^2), where E is the scalar matrix aI_8, where $a \in GF(q)^*$ and I_8 is the identity 8×8 matrix. Looking at the canonical form of $b \otimes b$ in $GL(16,q^4)$ shows that the element $\omega^{q^3+q^2+q+1} \in GF(q)$ appears exactly eight times. Each of such elements can be rewritten in the form η^{q+1}, with $\eta \in GF(q^2)$. In particular, the matrix E can be viewed as the direct sum of four scalar matrices of the form diag (a,a).

It follows that $\overline{\langle b \otimes b \rangle}$ fixes two 7–dimensional subspaces, say Σ_1 and Σ_2, and each of them is the direct sum of four lines.

In particular, Σ_2 is the direct sum of two copies of $PG(3,q)$, and the group $\overline{\langle b \otimes b \rangle}$ induces on each copy a regular line spread, say $\mathcal{S}_1$ and $\mathcal{S}_2$. Also, each 3–subspace Σ generated by a line in $\mathcal{S}_1$ and a line in $\mathcal{S}_2$ is fixed by $\overline{\langle b \otimes b \rangle}$ and the orbits of the group induced on Σ are lines forming again a line spread.

Consider a 3–dimensional subspace, say Ψ, generated by a line in Σ_1 and a line in Σ_2. Clearly, Ψ is fixed by $\overline{\langle b \otimes b \rangle}$ and $\Psi \cap S_{3,3}$ is either empty or a hyperbolic quadric, and each hyperbolic quadric so obtained (recall that any $PG(3,q)$ meets $S_{3,3}$ in a hyperbolic quadric) is partitioned into conics inducing a linear flock. The theorem is completely proved. □

6 Partitioning the hyperbolic quadric in a 3–space

Let $Q^+(3,q)$ be a hyperbolic quadric in $PG(3,q)$. The number of points of $Q^+(3,q)$ is $(q+1)^2$.

Which are the possible "uniform" partitions of $Q^+(3,q)$ into interesting objects? By a uniform partition, we mean a partition into objects of the same type; for instance:

1) partition into lines (the lines of a regulus of $Q^+(3,q)$ cover the pointset of $Q^+(3,q)$);
2) partition into conics (any flock of $Q^+(3,q)$);
3) partition into twisted cubics (see [4] for a possible construction).

Here, we construct a new partition of the hyperbolic quadric, assuming that the ground field has square order.

Let x_0, x_1, x_2, x_3 be homogeneous projective coordinates in $PG(3,q^2)$.

Let l_1 be the line with equations $x_0 = x_1 = 0$ and l_2 the line with equations $x_2 = x_3 = 0$. Thus, l_1 and l_2 are skew.

Consider the set $\mathcal{L}$ of all lines of $PG(3,q^2)$ which join a point on l_1 and a point on l_2. Using the Plücker correspondence between lines of $PG(3,q^2)$ and points of $PG(5,q^2)$ on the Klein quadric $\mathcal{K}$, the set $\mathcal{L}$ is represented on $\mathcal{K}$ by a hyperbolic quadric $Q^+(3,q^2)$ obtained by intersecting $\mathcal{K}$ with the 3–subspace $\Sigma : p_{01} = p_{23} = 0$, where the p_{ij}'s are Plücker coordinates in $PG(5,q^2)$ (or Plücker line coordinates in $PG(3,q^2)$).

Next, consider the lines of $\mathcal{L}$ joining any point $P = (0,0,a,b)$ on l_1 with the point $P' = (a^q, b^q, 0, 0)$ on l_2. Call E this set of lines. Then the locus, on $\mathcal{K}$, described by E is an elliptic quadric $\mathcal{E}$ embedded in a subgeometry $\Sigma_1 \cong PG(3,q)$ of Σ.

The Plücker coordinates of the lines of E are $(0, a^{q+1}, a^q b, ab^q, b^{q+1}, 0)$, where $a, b \in GF(q^2)$, a, b not both zero.

The point with these coordinates describes a rational curve in Σ, i.e. the curve $\mathcal{C} = \{(1, t, t^q, t^{q+1}) | t \in GF(q^2)\} \cup \{(0,0,0,1)\}$. There exists a collineation of Σ mapping this curve onto an elliptic quadric in Σ_1 [5].

Next, let S be a Singer cycle in $GL(2,q^2)$. Then S is similar in $GL(2,q^4)$ to the diagonal matrix diag (ω, ω^{q^2}), where ω is a primitive element of $GF(q^4)$ over $GF(q^2)$. Consider the Kronecker product $D \otimes D^q$. This is the diagonal matrix diag $(\omega^{q+1}, \omega^{q^2+q}, \omega^{q^3+1}, \omega^{q^3+q^2})$. If we view ω as a primitive element of $GF(q^4)$ over $GF(q)$, then $D \otimes D^q$ coincides with the canonical form of the unique subgroup of order q^2+1 of a Singer cycle of $GL(4,q)$ whose orbits are elliptic quadrics [6]. It follows that the linear collineation group, induced by $\langle S \otimes S^q \rangle$, has order q^2+1, fixes a hyperbolic quadric $Q^+(3,q^2)$ (by definition) and its orbits (all of size q^2+1) are elliptic quadrics (defined over the field $GF(q)$) partitioning the pointset of $Q^+(3,q^2)$.

We can summarise the above in the following result.

Theorem 6.1 *The hyperbolic quadric $Q^+(3,q^2)$ in $PG(3,q^2)$ admits a partition into elliptic quadrics $Q^-(3,q)$'s defined over $GF(q)$ (i.e., belonging to subgeometries $PG(3,q)$).* □

Alternatively, Theorem 6.1 can be proved by observing that a partition of $Q^+(3,q^2)$ into $Q^-(3,q)$'s can also be obtained in a purely geometric way. More precisely, let l_1, l_2, E and S be as above. By construction, E is a linear elliptic congruence and its image under the Plücker map is an elliptic quadric $Q^-(3,q)$ over $GF(q)$. Put

$$T = \begin{pmatrix} I_2 & 0_2 \\ 0_2 & S \end{pmatrix},$$

where I_2 is the 2×2 identity matrix and 0_2 is the 2×2 zero matrix.

The orbit of E under $\langle T\rangle$ consists of q^2+1 pairwise disjoint linear elliptic congruences whose images under the Plücker map are the elliptic quadrics $Q^-(3,q)$ partitioning $Q^+(3,q^2)$.

Note that $PG(3,q^2)$ does not admit a partition into subgeometries $PG(3,q)$ (see [8]).

7 Partitioning projective spaces into Segre varieties

In [2], using suitable subgroups of Singer cyclic groups, some properties of regular n–spreads are proven and, in particular, that, for $n = mk$ and $\gcd(m,k) = 1$, the space $PG(n-1,q)$ can be partitioned into Segre varieties $S_{m-1,k-1}$'s. Here, we give a shorter proof of this result.

Let ω be a primitive element of $GF(q^6)$ over $GF(q)$. Then $\omega^{q^4+q^2+1}$ is a primitive element of $GF(q^2)$ and ω^{q^3+1} is a primitive element of $GF(q^3)$. Thus, $D =$ diag $(w^{q^4+q^2+1}, \omega^{q^5+q^3+q})$ is the canonical form of a Singer cycle S of $GL(2,q)$ and diag $D' =$ diag $(\omega^{q^3+1}, \omega^{q^4+q}, \omega^{q^5+q^2})$ is the canonical form of a Singer cycle T of $GL(3,q)$.

Consider the Kronecker product $D'' = D\otimes D'$. Therefore, $D'' =$ diag $(\omega^{q^4+q^3+q^2+2}$, $\omega^{2q^4+q^2+q+1}, \omega^{q^5+q^4+2q^2+1}, \omega^{q^5+2q^3+q+1}, \omega^{q^5+q^4+q^3+2q}, \omega^{2q^5+q^3+q^2+q})$.

We observe that the entries of D'' are distinct elements of $GF(q^6)$ conjugate over $GF(q)$. It follows that the cyclic group $\langle S\otimes T\rangle$ induces a collineation group, say $\langle\Phi\rangle$, of the projective space $PG(5,q)$ which fixes no subspace, i.e. its action on the pointset of $PG(5,q)$ is irreducible. Also, each orbit of $\langle S\otimes T\rangle$ generates $PG(5,q)$ (since the linear transformation $S\otimes T$ has six distinct eigenvalues).

Lemma 7.1 *The order of* $\langle\Phi\rangle$ *is* $(q+1)(q^2+q+1)$.

Proof. Set $\alpha = (q+1)(q^2+q+1)$. It is easily seen that $\omega^{(q^4+q^3+q^2+2)\alpha} = \omega^{5(q^5+q^4+q^3+q^2+q+1)} = \beta^5$, with $\beta \in GF(q)$. Hence, the order of $S\otimes T$ is at least α. On the other hand, the α–th power is the least power of $\omega^{q^4+q^3+q^2+2}$ such that this element lies in $GF(q)$. □

Lemma 7.2 *All the point–orbits of* $\langle\Phi\rangle$ *have size* $(q+1)(q^2+q+1)$.

Proof. The order of Φ is $(q+1)(q^2+q+1)$ and its action is semiregular. □

Theorem 7.3 *([2], Prop. 3) The projective space $PG(n-1,q)$, with $n = mk$, $gcd(m,k) = 1$, can be partitioned into Segre varieties $S_{m-1,k-1}$'s.*

Proof. We restrict ourselves to the case $m = 2$, $k = 3$, the general case immediately follows by a similar argument using the corresponding appropriate statements of Lemmas 7.1 and 7.2. Actually, the required primitive element of $GF(q^m)$ over $GF(q)$ is $\omega^{q^{m(k-1)}+q^{m(k-2)}+\ldots+q^m+1}$, and the required primitive element of $GF(q^k)$ over $GF(q)$ is $\omega^{q^{k(m-1)}+q^{k(m-2)}+\ldots+q^k+1}$.
By definition, $\langle\Phi\rangle$ fixes a Segre variety $S_{1,2}$. Since $S_{1,2}$ has $(q+1)(q^2+q+1)$ points, it is a point–orbit of $\langle\Phi\rangle$, and the projective space $PG(5,q)$ can be partitioned into Segre varieties $S_{1,2}$'s. $\square$

References

[1] L. Bader, A. Cossidente, G. Lunardon, Generalizing flocks of $Q^+(3,q)$, Adv. Geom., to appear.

[2] A. Bonisoli, A. Cossidente, D. Saeli, Partitioning projective geometries into Segre varieties, J. Geom., 57 (1996), 58–62.

[3] C.R. Leedham-Green, E.A. O'Brien, Tensor Products are Projective Geometries, J. Algebra, 189 (1997), 514–528.

[4] A. Cossidente, Mixed partitions of $PG(3,q)$, J. Geom., 68 (2000), 48–57.

[5] A. Cossidente, A. Siciliano, On the geometry of Hermitian matrices of order three over finite fields, European J. Combin., to appear.

[6] G.L. Ebert, Partitioning projective geometries into caps, Canad. J. Math., 37 (1985), 1163–1175.

[7] J. Harris, *Algebraic Geometry. A First Course*, GTM 133, Springer-Verlag, New York, 1992.

[8] J.W.P. Hirschfeld, *Projective Geometry over Finite Fields*, Oxford University Press, Oxford, 1998.

[9] J.W.P. Hirschfeld, *Finite Projective Spaces of Three Dimensions*, Oxford University Press, Oxford, 1985.

[10] J.W.P. Hirschfeld, J.A. Thas, *General Galois Geometries*, Oxford University Press, Oxford, 1991.

[11] B. Segre, Teoria di Galois, fibrazioni proiettive e geometrie non desarguesiane, Ann. Mat. Pura ed Appl., Serie IV, 64 (1964), 1–76.

[12] C. Segre, Sulle varietà che rappresentano le coppie di punti di due piani o spazi, Rend. Circ. Mat. Palermo, 5 (1891), 192–204.

(Received 19 Sep 2000)