Growth in products of graphs

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Abstract

We present some results on the growth in various products of graphs. In particular we study the Cartesian, strong, lexicographic, tensor and free product of graphs. We show that with respect to distances the tensor product behaves differently from other products. In general the results are valid for rooted graphs but have especially nice structure in the case of vertex-transitive factors.

1 Introduction

Some basic facts on growth of graphs are presented. Although the concept is most useful for vertex-transitive graphs and comes primarily from groups [9], we consider finite or special classes of locally finite graphs. The emphasis is on the actual computation of generating functions for various operations on graphs, where we try to express the growth series of various products in terms of growth series of their factors. Many of these operations can be carried our completely automatically, as shown by Flajolet and Salvy [6]. All graphs in this paper are simplicial and connected, unless specified otherwise.

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In Section 2 we define two different growth functions: the sphere growth and the ball growth. Although they are related, we find the spherical growth function easier to compute for most products. The only exception is the strong product which is more compatible with the ball growth.

In Section 3 we consider growth in the Cartesian product of graphs.

In Section 4 we introduce the notion of the free product of labeled graphs and study its growth. The motivation for the free product comes from groups via Cayley graphs. The same way as the Cartesian product of rooted graphs is related to the direct product of groups, so is the free product related to the free product of groups. Note that the free product is not considered in the monograph of Imrich and Klavžar [12] since it can only be defined for rooted graphs.

In Section 5 we turn to the growth in strong products of graphs.

In Section 6 we consider the lexicographic product of graphs, we also mention tensor product and some other operations on graphs. One such operation is truncation and was considered quite recently; see [18]. It may come as a surprise but the growth function of the tensor product is not determined by the growth functions of its factors.

2 Growth in rooted graphs

Let G be connected, finite or locally finite graph. Select a vertex v and call it a *root*. By G_v we denote the *graph G rooted at v*. Define the *spherical growth sequence* $\{\delta(G, v, n)|n = 0, 1, 2, ...\}$, where $\delta(G, v, n)$ denotes the number of vertices at distance n from v. Similarly define the *ball growth sequence* $\{\gamma(G, v, n)|n = 0, 1, 2, ...\}$ where $\gamma(G, v, n)$ is the number of vertices at distance at most n from v. Furthermore, let $\Delta(G, v; x)$ be the *(spherical) growth series or function* , i.e. the generating function for $\delta(G, v, n)$ of G at v.

$$\Delta(G, v; x) = \sum_{n=0}^{\infty} \delta(G, v, n)x^n.$$

In the same way let $\Gamma(G, v; x)$ denote the *ball growth series or function*, i.e. the generating function for $\gamma(G, v, n)$ of G at v.

$$\Gamma(G, v; x) = \sum_{n=0}^{\infty} \gamma(G, v, n)x^n.$$

Note that the spherical growth sequence $\{\delta(G, v, n)|n = 0, 1, 2, ...\}$ is sometimes called the *distance degree sequence*, [2, 3, 11].

Let

$$\mathcal{D}(G) = \{\Delta(G, v; x)|v \text{ from } V(G)\} \text{ and } \mathcal{G}(G) = \{\Gamma(G, v; x)|v \text{ from } V(G)\}$$

In finite graphs define the averages

$$\bar{\Delta}(G;x) = \frac{1}{|V|}\sum_v \Delta(G,v;x) \text{ and } \bar{\Gamma}(G;x) = \frac{1}{|V|}\sum_v \Gamma(G,v;x)$$

Instead of $\delta(G,v,n), \Delta(G,v;x), \gamma(G,v,n), \Gamma(G,v;x)$, etc. we sometimes write $\delta(G_v,n), \Delta(G_v;x), \gamma(G_v,n), \Gamma(G_v;x)$ We will drop G, v, when the context is clear. In particular, for finite and infinite vertex transitive graphs $\Delta(G_v;x) = \bar{\Delta}(G;x)$ and $\Gamma(G_v;x) = \bar{\Gamma}(G;x)$, and we write $\Delta(G;x)$ and $\Gamma(G;x)$ or just $\Delta(x)$ and $\Gamma(x)$. Note that $\delta(G_v,n) = \gamma(G_v,n) - \gamma(G_v,n-1), n > 0$.

Recall that for a vertex u in a finite graph G the *eccentricity* of u is the maximum distance from u to any other vertex of G; see [10]. Let d denote the eccentricity of u and let n be the number of vertices of G. Since the series for $\Gamma(v;x)$ has only a finite number of non-constant terms, we can introduce a polynomial $h(v;x)$ that contains only the variable coefficients of $\Gamma(v;x)$. Then we have:

$$n = \gamma(v;d), \Delta(v;x) = (1-x)h(v;x) + nx^{d+1}, h(v;x) = \frac{\Delta(v;x) - nx^{d+1}}{1-x}$$

and

$$\Gamma(v;x) = h(v;x) + \frac{nx^{d+1}}{1-x}.$$

For all graphs we have:

$$\Delta(v;x) = (1-x)\Gamma(v;x)$$

$$\Gamma(v;x) = \frac{\Delta(v;x)}{1-x}$$

Let A be the automorphism group of G that partitions the vertex set $V(G)$ into orbits:

$$V(G) = [v_1] \cup [v_2] \cup \cdots \cup [v_s].$$

Observe that if the automorphism γ maps u to v then $\Delta(v;x) = \Delta(u;x)$. In other words, if u and v belong to the same orbit then $\Delta(u;x) = \Delta(v;x)$.

$$\bar{\Delta}(x) = \frac{1}{|V|}\sum_{i=1}^{s} \Delta(v_i;x)|[v_i]|$$

There are graphs G where $\Delta(u;x)$ is independent of u but G is not transitive. We call such graphs *growth regular*. In the literature [2, 3, 11] they are also known as distance degree regular.

Example 2.1 *The graph*

$$G = K_n - C_{n-3} - C_3.$$

is not transitive but growth regular with $\Delta(G;x) = 1 + (n-3)x + 2x^2$ *independent of vertex* u.

Let $K_{n(r)}$ denote the complete multipartite graph on nr vertices with n parts of size r and let M_n denote the Möbius ladder on $2n$ vertices. In the following example only the graphs P_n and $K_{m,n}, m \neq n$ are not vertex transitive.

Example 2.2 *Some examples of growth functions:*

$$
\begin{aligned}
\Delta(C_n;x) &= 1+2x+\cdots+2x^{(n-1)/2} = \frac{2x^{(n+1)/2}-x-1}{x-1}, \ldots n \ \mathit{odd}\\
\Delta(C_n;x) &= 1+2x+\cdots+2x^{(n-2)/2}+x^{n/2}\\
&= \frac{x^{n/2}+x^{(n+2)/2}-x-1}{x-1}, \ldots n \ \mathit{even}\\
\bar{\Delta}(P_n;x) &= 1+2((1-1/n)x+(1-2/n)x^2+\cdots+(1-(n-1)/n)x^{(n-1)})\\
\Delta(K_n;x) &= 1+(n-1)x\\
\Delta(K_{m,n},u;x) &= 1+nx+(m-1)x^2\\
\Delta(K_{m,n},v;x) &= 1+mx+(n-1)x^2\\
\bar{\Delta}(K_{m,n};x) &= 1+\frac{2mn}{m+n}x+\frac{(m(m-1)+n(n-1)}{m+n}x^2\\
\Delta(K_{n,n};x) &= 1+nx+(n-1)x^2\\
\Delta(K_{n(r)};x) &= 1+r(n-1)x+(r-1)x^2\\
\Delta(K_{2,2,2};x) &= 1+4x+x^2\\
\Delta(M_n;x) &= 1+3x+4x^2+\cdots+4x^{(n-1)/2}+2x^{(n+1)/2}\\
&= 4\frac{x^{(n+1)/2}-1}{x-1}+2x^{(n+1)/2}-x-3, \ldots n \ \mathit{odd}\\
\Delta(M_n;x) &= 1+3x+4x^2+\cdots+4x^{n/2}\\
&= 4\frac{x^{n/2+1}-1}{x-1}-x-3, \ldots n \ \mathit{even}
\end{aligned}
$$

Here are some interpretations of the numbers introduced so far. Clearly, $\Delta(1)$ represents the number of vertices in G. The expression $\frac{n}{2}\bar{\Delta}'(1)$ is equal to the sum of all distances in a graph which is sometimes called *Wiener index* of a graph; see for instance [12], p.60. $\bar{\Delta}'(1)$ is therefore the *average distance* in a finite graph. Here $\bar{\Delta}'$ denotes the first derivative of $\bar{\Delta}'$.

Before turning to products, let us introduce an operation on graphs. Let G_r and H_p be disjoint graphs rooted at vertices r and p and let u be a vertex of G_r. Let $G_r \bullet_u H_p$ denote the graph obtained by attaching a copy of H_p to the vertex u in such a way that the root p of H_p is identified with u. We may easily generalize this operation. Let U be an arbitrary set of vertices of G_r, called the *set of active vertices.* Then $G_r \bullet_U H_p$ denotes the graph obtained by attaching a different copy of H_p to each active vertex of U. This operation is called a *partial corona of graph G_r with rays H_p with respect to $U \subseteq V(G)$* or simply a *U-corona.* When all vertices are active, i.e. $U = V(G)$, we omit the subscript and denote the resulting graph by

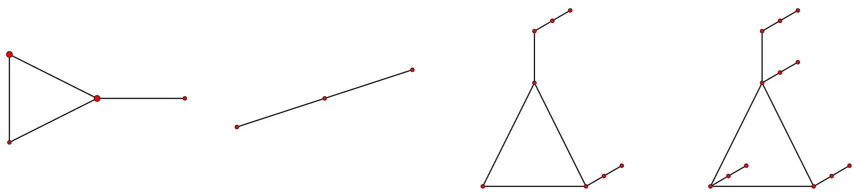

Figure 1: Factors, partial corona and corona.

$G_r \bullet H_p$. This operation is called the *corona of graph G_r with rays H_p*. We assume that r remains the root in the corona.

The vertices of $G_r \bullet_U H_p$ may be regarded as *old* if they belong to $V(G)$ and *new* if they do not belong to $V(G)$. Furthermore, the old vertices are partitioned to *active* and *passive*. Finally, the old vertices retain the label from G_r while each new vertex is labeled by a pair $(u, v), u \in U, v \in V(H)$. Sometimes we refer to v as the *rightmost label.*

Here we need a slightly more general definition of the growth sequence and the growth function.

Let $\delta(G_v, U, n)$ denote the number of vertices from U at distance n from v. Define the *spherical growth sequence with respect to U* to be $\{\delta(G_v, U, n)|n = 0, 1, 2, ...\}$. Furthermore, let $\Delta(G_v, U; x)$ be the *(spherical) growth series or function with respect to U*, i.e. the generating function for $\delta(G_v, U, n)$ of G at v.

$$\Delta(G_v, U; x) = \sum_{n=0}^{\infty} \delta(G_v, U, n)x^n.$$

The following proposition gives an alternative definition of the familiar notion of the breath-first search tree (BFS-tree).

Proposition 2.3 *For any connected graph G_r, a spanning tree T_r rooted at r is a BFS-tree if and only if the growth functions $\Delta(G_r; x)$and $\Delta(T_r; x)$ are equal.*

Proof. The result follows from the fact that a BFS-tree T_r is characterized by the fact that the distance from r to any other vertex v is the same in G_r as in T_r. ■

Proposition 2.4 *The spherical growth function of the partial corona $G_r \bullet_U H_p$ is given by:*

$$\Delta(G_r \bullet_U H_p, r; x) = \Delta(G_r, U; x)\Delta(H_p; x) + \Delta(G_r, V(G) - U; x)$$

Proof. Consider an arbitrary vertex at distance d from r in $G_r \bullet_U H_p$. There are two disjoint cases. Either it is passive, i.e. belongs to $V(G) - U$, or it belongs to some copy of H_p. There may be d_1 steps in G needed to reach the root of H_p and then d_2 steps in the appropriate copy of H_p with $d_1 + d_2 = d$. By combining the two cases we obtain the result. ■

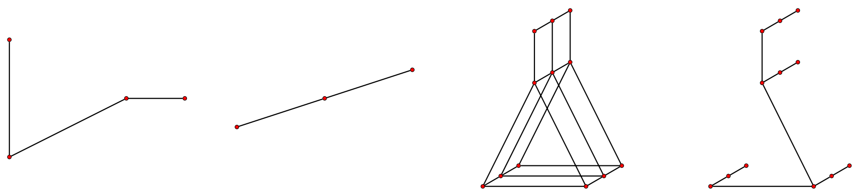

Figure 2: BFS spanning trees of the two factors and in the Cartesian product.

3 Cartesian products

Let $\square$ denote the Cartesian product of graphs. For rooted graphs we have:

Theorem 3.1 *Let G be rooted at r and H be rooted at p. The Cartesian product $G\square H$ (rooted at (r,p)) satisfies:*

$$\Delta(G\square H,(r,p);x) = \Delta(G_r;x)\Delta(H_p;x)$$

PROOF. In the proof we use Propositions 2.3 and 2.4. Namely, If T_r is a BFS spanning tree in G_r and S_p is a BFS spanning tree if H_p then the corona $T_r \bullet S_p$ is a BFS spanning tree in $G\square H_{(r,p)}$; see Figure 2. ■

In the examples we exploit the following fact.

Corollary 3.2 *If G and H are growth regular then the Cartesian product $G\square H$ is growth regular.*

PROOF. Follows directly from Theorem 3.1. If the righthand side of the equation

$$\Delta(G\square H,(r,p);x) = \Delta(G_r;x)\Delta(H_p;x)$$

is independent of the root, then so is the lefthand side. Also, any root of $G\square H$ is of form (r,p) and projects to the corresponding roots of G and H. ■

The following fact is well-known and may simplify computation of the growth function in case of vertex-transitive factors.

Proposition 3.3 *If G and H are vertex transitive, then $G\square H$ is also vertex transitive.*

PROOF. Folklore. ■

Example 3.4 *Since*

$$\Delta(K_2;x) = 1+x$$

and the $n-$cube Q_n is an n-fold Cartesian product of K_2's, $Q_n = K_2\square K_2 ... \square K_2$

$$\Delta(Q_n;x) = (1+x)^n$$

Example 3.5 *The line graph of $K_{m,n}$ is the Cartesian product of complete graphs. Hence:*

$$\Delta(L(K_{m,n});x) = (1+(n-1)x)(1+(m-1)x)$$

Example 3.6 *Let C_∞ denote the 2-way infinite path. Then $C_\infty^2 = C_\infty \Box C_\infty$ represents a square grid in the plane.*

As $\Delta(C_\infty;x) = 1+2x+2x^2+2x^3+... = \frac{1+x}{1-x}$ we have:

$$\Delta(C_\infty^k;x) = (\frac{1+x}{1-x})^k$$

for a grid in k-dimensional space.

Example 3.7 *Let P_∞ denote the one-way infinite path rooted at the vertex of valence 1. Clearly $\Delta(P_\infty;x) = 1+x+x^2+x^3+... = \frac{1}{1-x}$. Obviously $\Delta(P_\infty^2;x) = \frac{1}{(1-x)^2}$ is the generating function for the spherical growth of the grid of the quarter-plane.*

Note that Cartesian products of graphs correspond to direct products of groups. If $\langle X|R\rangle$ is a presentation for group $\mathfrak{G}$ and $\langle Y|S\rangle$ is a presentation for the group $\mathfrak{H}$ then the direct product $\mathfrak{G}\Box\mathfrak{H}$ has a presentation $\langle X,Y|R,S,[X,Y]\rangle$. This means that we take all the generators X and Y , all relators R and S and then let each generator $x \in X$ commute with each generator $y \in Y$: $xy = yx$. Let $C(\mathfrak{G},X)$ denote the Cayley graph for $\mathfrak{G}$ and generating set X. Then

$$C(\mathfrak{G}\Box\mathfrak{H}, X\cup Y) = C(\mathfrak{G},X)\Box C(\mathfrak{H},Y).$$

4 Free Product

The idea of the free product of groups can be generalized to labeled graphs via Cayley graphs. If $\langle X|R\rangle$ is a presentation for group $\mathfrak{G}$ and $\langle Y|S\rangle$ is a presentation for the group $\mathfrak{H}$ then their free product $\mathfrak{G}*\mathfrak{H}$ has a presentation $\langle X,Y|R,S\rangle$. We would like to define the free product of their Cayley graphs as follows:

$$C(\mathfrak{G},X)*C(\mathfrak{H},Y) = C(\mathfrak{G}*\mathfrak{H}, X\cup Y).$$

We can define the free product $G_r * H_p$ of two rooted graphs G_r and H_p as follows. Let the vertex set of $G_r * H_p$ be composed of all finite sequences of vertices from $(V(H)-p)\cup(V(G)-r)$ where the elements alternate from $V(H)-p$ and $V(G)-r$. Two sequences α and β are adjacent in $G_r * H_p$ if and only if either $\alpha = \gamma$ and $\beta = \gamma x$ and x is adjacent to the corresponding root vertex, or $\alpha = \gamma x$ and $\beta = \gamma y$ and x and y are adjacent in the corresponding factor. The root in $G_r * H_p$ corresponds to the empty sequence ϵ.

The free product can be defined in several equivalent ways using repeatedly partial corona. We give a non-symmetric definition favoring G_r over H_p.

- Let $P_0 = G_r, V_0 = VP_0$.
- Let $Q_{i+1} = P_i \bullet_{V_i} H_p, U_{i+1} = VQ_{i+1} - VP_i, i = 0, 1, 2, ...$
- Let $P_i = Q_i \bullet_{U_i} G_r, V_i = VP_i - VQ_i, i = 1, 2, ...$

The graphs form an embedded sequence of rooted graphs

$$P_0 \subset Q_1 \subset P_1 \subset Q_2 \subset ...$$

whose limit graph is denoted by $G_r * H_p$. It is not hard to see that $H_p * G_r = G_r * H_p$. The concept of the limit graph can be found, for instance in [1, 5].

Example 4.1 *Consider $K_3 * K_3$. Label the vertices in the first copy of K_3 by $r, 0, 1$ and the vertices in the second copy by $r', 0'$ and $1'$. Let r and r' be the corresponding roots. The vertices,ordered by distance from the root are given by:*

$$V(K_3 * K_3) = \{\epsilon, 0, 1, 0', 1', 00', 01', 10', 11', 0'0, 0'1, 1'0, 1'1, ...\}$$

They can be encoded as binary strings with an additional bit of information, say "1" or "0" that tells whether the lefmost digit is followed by a prime (') or not!).

$$V(K_3 * K_3) = \{0, 00, 01, 10, 11, 000, 001, 010, 011, 100, 101, 110, 111, ...\}$$

In the latter encoding the distance from the root is given by $d-1$ where d is the length of the binary string. The product looks like a tree of triangles. It is not hard to see that the distance sequence at each vertex is given by:

$$1, 4, 8, 16, 32, ...$$

Proposition 4.2 *If G and H are vertex transitive, then $G * H$ is also vertex transitive.*

Proof. Follows from the definition. Any map from one vertex of $G * H$ to another can be extended to an automorphism of the whole graph. ■

In the proof of the next theorem the following two graphs are very useful. They are both obtained as limit graphs similarly to the graph $G_r * H_p$.

- Let $P_0 = G_r, V_0 = VP_0 - r$.
- Let $Q_{i+1} = P_i \bullet_{V_i} H_p, U_{i+1} = VQ_{i+1} - VP_i, i = 0, 1, 2, ...$
- Let $P_i = Q_i \bullet_{U_i} H_p, V_i = VP_i - VQ_i, i = 1, 2, ...$

Let $S(G_r, H_p)$ be limit graph obtained via the above process. Its root vertex is the r in the first copy of G_r which has nothing attached to it. Note that in general $S(G_r, H_p)$ is not isomorphic to $S(H_p, G_r)$.

Lemma 4.3 *The graph $S(G_r, H_p)$ is isomorphic to $G_r \bullet_{V(G)-r} S(H_p, G_r)$*

Proof. Follows from the definition of these graphs. ■

We have:

Theorem 4.4

$$1/(\Delta(G_r * H_p; x) - 1 = 1/\Delta(G_r; x) - 1 + 1/\Delta(H_p; x) - 1$$

Proof. Let $F_1(x)$ denote the growth function of $S(G_r, H_p)$ and let $F_2(x)$ denote the growth function of $S(H_p, G_r)$. Observe that the intersection of $S(G_r, H_p)$ and $S(H_p, G_r)$ is a single vertex, namely the root, while its union is the graph $F(G_r, H_p)$. This gives rise to the first equation:

$$\Delta(G_r * H_p; x) = F_1(x) + F_2(x) - 1$$

The other two equations follow from Lemma 4.3 and Proposition 2.4:

$$F_1(x) = 1 + (\Delta(G_r; x) - 1)F_2(x)$$

$$F_2(x) = 1 + (\Delta(H_p; x) - 1)F_1(x)$$

By elimination of $F_1(x)$ and $F_2(x)$ one obtains the formula stated in the theorem. ■

The above theorem was used for groups in several places, see for instance [19]. In the special case when $G = H$ we get:

Corollary 4.5

$$\Delta(G * G; x) = \frac{\Delta(G; x)}{2 - \Delta(G; x)}$$

We can easily show that:

Corollary 4.6

$$\frac{1}{\Delta(G_1 * G_2 * ... * G_k; x)} - 1 = \frac{1}{\Delta(G_1; x)} - 1 + \frac{1}{\Delta(G_2; x)} - 1 + ... + \frac{1}{\Delta(G_k; x)} - 1$$

Proof. By induction on k. ■

In the special case when $G_i = G$ the formula reduces to:

Corollary 4.7

$$\Delta(G * G * ... * G; x) = \frac{\Delta(G; x}{k - (k-1)\Delta(G; x)}$$

If we apply this to the complete graph K_n, we get:

Corollary 4.8

$$\Delta(K_n * K_n; x) = \frac{1 + (n-1)x}{1 - (n-1)x}$$

In our Example 4.1 we have $n = 3$ and $\Delta(K_3 * K_3; x) = \frac{1+2x}{1-2x}$. If we choose $n = 2$ we get $\Delta(K_2 * K_2; x) = \frac{1+x}{1-x}$ which is a function that we have seen already in Example 3.6. This is not a surprise since one can easily verify that $K_2 * K_2 = C_\infty$. The $k-$fold free product of K_2 by itself has the generating function:

Corollary 4.9

$$\Delta(K_2 * K_2 * ... * K_2; x) = \frac{1 + x}{1 - (k-1)x}$$

We have calculated the growth of a $k-$way infinite tree. If $k = 2m$ is even, $K_2 * K_2 * ... * K_2$ can be written as the product m copies of C_∞. This is the standard Cayley graph for the free group F_m with m generators. Its growth series is $\frac{1+x}{1-(2k-1)x}$; see [9].

Proposition 4.10 *If G and H are growth regular, then $G * H$ is also growth regular.*

PROOF. Follows from Theorem 4.4 since the growth function of the product is determined by the growth functions of the two factors. ■

Finally, for any (finite) graph G we may generate an (infinite) vertex transitive graph by a construction that is similar to the multiple free product. We assume that G has a finite number, say m, of vertex orbits. Let $v_1, v_2, ..., v_m$ be representatives of the respecitve vertex orbits $[v_1], [v_2], ..., [v_m]$. Then $F(G_{v_1}, G_{v_2}, ..., G_{v_m})$ is constructed as follows.

- Let $F_0 = G, V_0(j) = V(G) - [v_j], j = 1, 2, ..., m.$
- Let $F_{i+1} = F_i \bullet_{V_i(1)} G_{v_1} \bullet_{V_i(2)} G_{v_2} ... \bullet_{V_i(m)} G_{v_m}, i = 0, 1, 2, ...$
- Let $V_{i+1} = V(F_{i+1}) - V(F_i), i = 0, 1, 2, ..., j = 1, 2, ..., m$
- Let $V_{i+1}(j) \subset V_{i+1}$ denote the vertices whose leftmost label does not belong to $[v_j]$ for $i = 0, 1, 2, ..., j = 1, 2, ..., m$

In other words F_{i+1} is obtained from F_i by attaching at vertex v, belonging to the orbit $[v_j]$, a copy of each G_{v_i} except for i = j. The resulting limit graph [1, 5] $F(G_{v_1}, G_{v_2}, ..., G_{v_m})$ is vertex transitive, called the *free transitive closure of G*. Its growth function can be easily computed by setting up and solving a system of linear equations similar to the one in the proof of Theorem 4.4. Let $m_{i,j}(x) = \Delta(G, [v_j]_{v_i}; x)$ denote the growth function defined by $[v_j]$ in G_{v_i} and let $M(x) = (m_{i,j}(x)), E = (1)_{(i,j)}$ and I (identity matrix)) be $m \times m$ matrices. The growth function is given by

$$e^T(E - M(x)(E - I))^{-1}((m-1)I - (m-2)M(x))e - m + 1,$$

where $e = (1, 1, ..., 1)^T$.

The construction of $F(G_{v_1}, G_{v_2}, ..., G_{v_m})$ makes sense for any equivalence relation on $V(G)$ with m equivalence classes. However, if the classes are not vertex orbits then the formula for growth function does not apply as the functions $m_{i,j}(x)$ are not well-defined. On the other hand, if the classes are defined by the full automorphism group they depend on the graph alone and we may shorten notation $F(G_{v_1}, G_{v_2}, ..., G_{v_m}) = F(G)$. If G is itself vertex-transitive we have $F(G) = G$.

Example 4.11 *Let D be the Dürer graph, i.e. the generalized Petersen graph $P(6, 2)$. Clearly D has two vertex orbits, one composed of a hexagon, and another consisting of two triangles. A simple calculation shows that*

$$m_{1,1}(x) = 1 + 2x + 2x^2 + x^3$$
$$m_{1,2}(x) = x + 4x^2 + x^3$$
$$m_{2,1}(x) = x + 4x^2 + x^3$$
$$m_{2,2}(x) = 1 + 2x + 2x^3 + x^4$$

Note that $F(D)$ is 6-valent, 1-connected, vertex-transitive, planar graph whose growth function is given by:

$$\Delta(F(D); x) = \frac{1 + 5x + 10x^2 + 9x^3 - 2x^4 - 4x^5 - x^6}{1 - x - 12x^2 - 10x^3 + 2x^4 + 4x^5 + x^6}$$

$F(D)$ is obtained by gluing infinitely many copies of D. Each vertex of the outer hexagon of D is glued to a vertex of a triangle of separate copy of D (see the left part of Figure 3). D can be embedded in the plane so that either a triangle bounds the infinite outer face, or a pentagon. The former, called a triangular patch*, has the outer face of size 3, while the latter, called a* pentagonal patch*, has a pentagonal outer face. To each of the two patches we have to glue six triangular and six pentagonal patches, as indicated in central and right part of Figure 3.*

5 Strong Product

Strong product is similar to the Cartesian product. Here we denote it by $\boxtimes$. If G and H are graphs their strong product $G \boxtimes H$ is defined on the vertex set $V(G)\Box V(H)$ and two vertices (v, u) and (v', u') are adjacent in $G \boxtimes H$ if and only if either v is adjacent to v' and u is adjacent to u' or $v = v'$ and u is adjacent to u' or v is adjacent to v' and $u = u'$.

It is possible to visualize the strong product in terms of direct product of group presentations and their Cayley graphs. If $\langle X|R\rangle$ is a presentation for group Γ and $\langle Y|S\rangle$ is a presentation for the group Σ, we may take the following presentation for their direct product $\Gamma\Box\Sigma$: $\langle X, Y, Z|R, S, [X, Y], Z = XY\rangle$. This means that we add new generators Z and relators, so that $z = xy$ for $z \in Z$, $x \in X$ and $y \in Y$. Then $C(\Gamma\Box\Sigma, X \cup Y \cup Z) = C(\Gamma, X) \boxtimes C(\Gamma, Y)$.

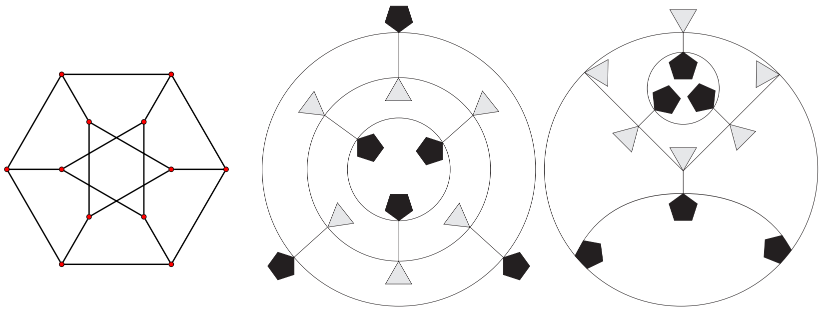

Figure 3: The Dürer graph (left) is planar and can be drawn in the plane in two distinct ways (center and right) forming the so-called triagonal and pentagonal patch. Small pentagons and triangles indicate the gluing process that yields the limit graph $F(D)$.

By $\boxtimes$ we denote not only the strong product of graphs but also the *Hadamard product* of series. If $g_1(x) = \sum_{i=0}^{\infty} \gamma_1(i)x^i$ and $g_1(x) = \sum_{i=0}^{\infty} \gamma_2(i)x^i$ then let $g_1 \boxtimes g_2(x) = \sum_{i=0}^{\infty} \gamma_1(i)\gamma_2(i)x^i$.

Theorem 5.1

$$\Gamma(G \boxtimes H, (r,p); x) = \Gamma(G_r; x) \boxtimes \Gamma(H_p; x)$$

In other words:

$$\frac{\Delta(G \boxtimes H, (r,p); x)}{1-x} = \frac{\Delta(G_r; x)}{1-x} \boxtimes \frac{\Delta(H_p; x)}{1-x}$$

Proof. Follows easily from the observation that for any positive integer d the distance between any two vertices of $G \boxtimes H$ is at most d if and only if the distance in both projections is at most d. ■

There is a program by Marko Petkovšek based on the work [17] included in the system Vega [20] for calculating the Hadamard product of any two rational functions.

Corollary 5.2 *Let G and H be growth regular then their strong product $G \boxtimes H$ is growth regular.*

6 Lexicographic Product, Tensor Product and more

Lexicographic product is not commutative. For many of its properties, see [12]. The edges that project to the first factor from a complete bipartite graph. This makes the growth function easy to compute. The distance between any two vertices that project to different vertices in the first factor is determined by the distance in the

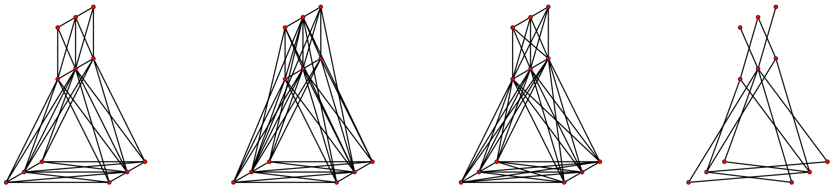

Figure 4: Strong product, two lexicographic products and the tensor product of the two graphs from Figure 1.

first factor. On the other hand, any two vertices projecting to the same vertex of the first factor are at distance two in the lexicographic product. This can be summarized in the form of the following result.

Proposition 6.1

$$\Delta(G_r[H_p]; x) = 1 + |V(H)|(\Delta(G_r; x) - 1) + (|V(H)| - 1)x^2$$

We may use this result to recompute the growth in the regular complete $n-$partite graph $K_{n(r)} = K_n[\bar{K}_r]$, compare Example 2.2.

Since $|V(H)| = \Delta(H_p; 1)$ we may express the growth in the lexicographic product in terms of growth functions of its factors:

$$\Delta(G_r[H_p]; x) = 1 + |\Delta(H_p; 1)|(\Delta(G_r; x) - 1) + (|\Delta(H_p; 1)| - 1)x^2$$

The proposition has the following immediate consequence.

Corollary 6.2 *Let G be growth regular and H finite then the lexicographic product $G[H]$ is growth regular.*

The last product we mention is the so-called tensor product of graphs. In [12] it is called the direct product. It is the product in the category of graphs and non-degenerate graph maps. However, its behavior in terms of growth is most surprising.

Proposition 6.3 *The growth function of the tensor product of the graphs G_r and H_p cannot be expressed in terms of the growth functions of the two factors.*

Proof. The growth functions of C_3 and P_3 rooted in the central vertex are equal and are given by $1 + 2x$. However, the tensor products $C_3 \times C_3$ and $C_3 \times P_3$ have different growth functions: $1 + 4x + 4x^2$ and $1 + 4x + 2x^2 + 2x^3$. ∎

In view of this negative result we may introduce the notion of a *growth compatibility.* If a function on graphs has the property that the growth of the resulting graph is determined by the growth of its arguments then such a function is called growth compatible. Cartesian product, free product, strong product and lexicographic product are growth compatible functions with two arguments. Tensor product is not growth compatible.

There are functions of one argument, such as the subdivision graph $S(G)$, the line graph $L(G)$, etc. One could explore growth compatibility of such functions. Clearly, $S(G)$ is not growth compatible. However, if restricted to trees subdivision is growth compatible. For any tree T we have:

$$\Delta(S(T)_v;x) = ((1+x)\Delta(T_v;x^2)-1)/x$$

In particular, this applies to a k-way infinite tree.

$$\Delta(S(K_2 * K_2 * ... * K_2)_v;x) = \frac{1+kx+x^2}{1-(k-1)x^2}$$

Here $S(K_2 * K_2 * ... * K_2)$ is rooted at a k-valent vertex v.

If we want to root S(T) in a new vertex of valence two, we must generalize the notion of a rooted graph. In general we may root a graph G in any subset U of its vertex set. Here we assume that the growth at each vertex $u \in U$ is equal to $x^0 = 1$. For any vertex $v \in V(G) - U$ the growth is given by x^d, where d is the (minimum) distance from v to U. We also restrict our attention to the sets composed of two adjacent vertices. If $e = \{u, v\}$ is an edge of a tree T we may denote by e the corresponding vertex of valence 2 in $S(T)$. In such a case we obtain

$$\Delta(S(T)_e;x) = (1+x)\Delta(T_e;x^2)-1$$

In particular

$$\Delta(S(K_2 * K_2 * ... * K_2)_e;x) = \frac{1+2x+(k-1)x^2}{1-(k-1)x^2}$$

Finally, a refinement is in order. For instance, if we know the growth in $G, S(G)$, and $S(S(G))$ can we determine the growth in $S(S(S(G)))$? An interesting question along these lines was addressed in [18] where the iteration of the so-called truncation $T(G) = L(S(G))$ was performed ad infinitum.

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