Total domination good vertices in graphs

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Abstract

A set S of vertices in a graph G is a total dominating set of G if every vertex of G is adjacent to some vertex in S. The total domination number $\gamma_t(G)$ is the minimum cardinality of a total dominating set of G. A vertex that is contained in some minimum total dominating set of a graph G is a good vertex, otherwise it is a bad vertex. We determine for which triples (x, y, z) there exists a connected graph G with $\gamma_t(G) = x$ and with y good vertices and z bad vertices, and we give graphs realizing these triples.

1 Introduction

Let G be a graph without isolated vertices, and let v be a vertex of G. A set $S \subseteq V(G)$ is a *total dominating set* if every vertex in $V(G)$ is adjacent to a vertex in S. Every graph without isolated vertices has a total dominating set, since $S = V(G)$ is such a set. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set. A total dominating set of cardinality $\gamma_t(G)$ we call a $\gamma_t(G)$-set.

Total domination in graphs was introduced by Cockayne, Dawes and Hedetniemi [3] and is now well studied in graph theory (see, for example, [5] and [10]).

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The literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi and Slater [7, 8].

For notation and graph theory terminology, we in general follow [2, 7]. Specifically, let $G = (V, E)$ be a graph with vertex set V of order n and edge set E. For any vertex $v \in V$, the *open neighborhood of* v is the set $N(v) = \{u \in V \mid uv \in E\}$, and its *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, its *open neighborhood* is the set $N(S) = \cup_{v \in S} N(v)$ and its *closed neighborhood* is the set $N[S] = N(S) \cup S$. For $S \subseteq V$, we denote the subgraph induced by S by $\langle S \rangle$.

The *private neighborhood* $\text{pn}(v, S)$ of $v \in S$ is defined by $\text{pn}(v, S) = N(v) - N(S - \{v\})$. Equivalently, $\text{pn}(v, S) = \{u \in V \mid N(u) \cap S = \{v\}\}$. Each vertex in $\text{pn}(v, S)$ is called a *private neighbor* of v. The *external private neighborhood* $\text{epn}(v, S)$ *of* v *with respect to* S consists of those private neighbors of v in $V - S$. Thus, $\text{epn}(v, S) = \text{pn}(v, S) \cap (V - S)$.

A *leaf* of a tree is a vertex of degree 1, while a *support vertex* is a vertex adjacent to a leaf. A *double star* is a tree that contains exactly two vertices that are not end-vertices; necessarily, these two vertices are adjacent. If the one central vertex of a double star is adjacent to r leaves and the other central vertex to s leaves, then we denote the double star by $S(r, s)$.

We call a vertex that is contained in some minimum total dominating set of a graph G is a *good vertex*, otherwise it is a *bad vertex*. Let $g(G)$ (respectively, $b(G)$) denote the number of good (respectively, bad) vertices in a graph G. Note that for any graph G of order n without an isolated vertex, $g(G) + b(G) = n$.

Fricke, Haynes, Hedetniemi, Hedetniemi and Laskar [6] defined a graph G to be γ_t*-excellent* if every vertex of G is a good vertex, i.e., if $g(G) = n$. Henning [11] provided a constructive characterization of γ_t-excellent trees. Cockayne, Henning and Mynhardt [4] characterized the set of vertices of a tree that are contained in all, or in no, respectively, minimum total dominating sets of the tree. Haynes and Henning [9] studied graphs having unique minimum total dominating sets, i.e., graphs G for which $g(G) = \gamma_t(G)$ and $b(G) = n - \gamma_t(G)$. They provided three equivalent conditions for a tree to have a unique minimum total dominating set and gave a constructive characterization of such trees.

Let (x, y, z) be a triple of integers. If there exists a connected graph G such that $\gamma_t(G) = x$, $g(G) = y$, and $b(G) = z$, then we shall call G a *realization of* (x, y, z) and we call the triple (x, y, z) *realizable*. Our aim is to determine which triples (x, y, z) are realizable, and to find a realization of each realizable triple.

2 Known Results

The decision problem to determine the total domination number of a graph is known to be NP-complete. Hence it is of interest to determine upper bounds on the total domination number of a graph. Cockayne et al. [3] obtained the following upper bound on the total domination number of a connected graph in terms of the order of the graph.

Theorem 1 (Cockayne et al. [3]) *If G is a connected graph of order $n \geq 3$, then $\gamma_t(G) \leq 2n/3$.*

A large family of graphs attaining the bound in Theorem 1 can be established using the following transformation of a graph. The 2-*corona* of a graph H is the graph of order $3|V(H)|$ obtained from H by attaching a path of length 2 to each vertex of H so that the resulting paths are vertex disjoint. The 2-corona of a connected graph has total domination number two-thirds its order. The following characterization of connected graphs of order at least 3 with total domination number exactly two-thirds their order is obtained in [1].

Theorem 2 (Brigham et al. [1]) *Let G be a connected graph of order $n \geq 3$. Then $\gamma_t(G) = 2n/3$ if and only if G is C_3, C_6 or the 2-corona of some connected graph.*

The following property of minimal total dominating sets is established in [3].

Proposition 3 (Cockayne et al. [3]) *If S is a minimal total dominating set of a connected graph $G = (V, E)$, then each $v \in S$ has at least one of the following two properties:*
P_1 : There exists a vertex $w \in V - S$ such that $N(w) \cap S = \{v\}$;
P_2 : $\langle S - \{v\} \rangle$ contains an isolated vertex.

In [10], the following property of minimum total dominating sets in graphs is established.

Theorem 4 (Henning [10]) *If G is a connected graph of order $n \geq 3$ and $G \not\cong K_n$, then G has a minimum total dominating set S that maximizes the number of edges in $\langle S \rangle$ and such that every vertex of S has property P_1 or is adjacent to a vertex of degree 1 in $\langle S \rangle$ that has property P_1.*

The following characterization of trees that have a unique minimum total dominating set is proven in [9].

Theorem 5 (Haynes, Henning [9]) *Let T be a tree of order $n \geq 2$. Then T has a unique $\gamma_t(T)$-set if and only if T has a $\gamma_t(T)$-set S for which every vertex $v \in S$ is a support vertex or satisfies $|pn(v, S)| \geq 2$.*

3 Preliminary Results

Our aim in this section is to establish a few preliminary results that we will need in subsequent sections.

Every graph G with no isolated vertex satisfies $\gamma_t(G) \geq 2$ and $b(G) \geq 0$. Since every vertex in a $\gamma_t(G)$-set is a good vertex, $g(G) \geq \gamma_t(G)$. This yields the following observation.

Observation 6 *If (x, y, z) is a realizable triple, then $y \geq x \geq 2$ and $z \geq 0$.*

The next result establishes a lower bound on the number of bad vertices in a graph.

Theorem 7 *If G is a graph with no isolated vertex satisfying $g(G) = \gamma_t(G) + k$, then $b(G) \geq \frac{2}{3}(\gamma_t(G) - 2k)$.*

Proof. Let S be a minimum dominating set of $G = (V, E)$ in which every vertex has property P_1 or is adjacent to a vertex of degree 1 in $\langle S \rangle$ that has property P_1. Such a $\gamma_t(G)$-set exists by Theorem 4. Let $A = \{v \in S \mid v \text{ does not have property } P_1\}$. Thus each vertex of A is adjacent to a vertex of degree 1 in $\langle S \rangle$ that has property P_1.

CLAIM: $|A| \leq (\gamma_t(G) + k)/3$.

PROOF. Let $A_1 = \{v \in A \mid v \text{ is adjacent to exactly one vertex of degree 1 in } \langle S \rangle \text{ that has property } P_1\}$ and let $A_2 = A - A_1$. For $i = 1, 2$, let A'_i be the set of vertices of degree 1 in $\langle S \rangle$ that have property P_1 and are adjacent to a vertex of A_i. Then, $|A'_1| = |A_1|$ and $|A'_2| \geq 2|A_2|$. Hence,

$$\gamma_t(G) = |S| \geq |A_1| + |A'_1| + |A_2| + |A'_2| \geq 2|A_1| + 3|A_2|,$$

and so $|A_2| \leq (\gamma_t(G) - 2|A_1|)/3$. We show next that $|A_1| \leq k$. Let $v \in A'_1$ and let $u \in A_1$ be the neighbor of v in S. Further, let $w \in \text{epn}(v, S)$. Since v is the only neighbor of u of degree 1 in $\langle S \rangle$, $(S - \{u\}) \cup \{w\}$ is a $\gamma_t(G)$-set, and so w is a good vertex. Since there are exactly k good vertices in $V - S$, $|A_1| = |A'_1| \leq k$. Hence, $|A| = |A_1| + |A_2| \leq |A_1| + (\gamma_t(G) - 2|A_1|)/3 = (\gamma_t(G) + |A_1|)/3 \leq (\gamma_t(G) + k)/3$, as desired. □

Let C denote the set of vertices in $V - S$ that are adjacent to at least one vertex of $S - A$, i.e., $C = N(S - A) \cap (V - S)$. Since every vertex in $S - A$ has at least one external private neighbor in $V - S$, it follows that $|C| \geq |S - A| = \gamma_t(G) - |A|$. Thus, by the above claim, $|C| \geq (2\gamma_t(G) - k)/3$. On the other hand, $|C| \leq |V - S| = n - \gamma_t(G) = b(G) + g(G) - (g(G) - k) = b(G) + k$. Hence, $b(G) + k \geq (2\gamma_t(G) - k)/3$, and so $b(G) \geq (2\gamma_t(G) - 4k)/3$. This completes the proof of Theorem 7. □

By Theorem 1, every connected graph G of order $n \geq 3$ satisfies $\gamma_t(G) \leq 2n/3$. Theorem 2 provides a characterization of those connected graphs G satisfying $\gamma_t(G) = 2n/3$. We shall need a characterization of connected graphs G of order $n \geq 5$ satisfying $\gamma_t(G) = (2n - 1)/3$. For this purpose, we define a family $\mathcal{G}$ of graphs as follows.

Let G be the graph of order $3|V(H)| - 1$ obtained from a non-trivial connected graph H by attaching a path of length 1 (a pendant vertex) to a specified vertex of H and attaching a path of length 2 to every other vertex of H so that the resulting paths are vertex disjoint. Let $\mathcal{G}$ be the family of all such graphs G.

Theorem 8 *Let G be a connected graph of order $n \geq 5$. Then $\gamma_t(G) = (2n - 1)/3$ if and only if $G = C_5$ or $G \in \mathcal{G}$.*

Proof. The sufficiency is straightforward to verify. To prove the necessity, let $x = (2n-1)/3$ and suppose that G is a connected graph of order $n \geq 5$ such that $\gamma_t(G) = x$. Then, $x \geq 3$. By Theorem 4, G has a minimum total dominating set S that maximizes the number of edges in $\langle S \rangle$ and such that every vertex of S has property P_1 or is adjacent to a vertex of degree 1 in $\langle S \rangle$ that has property P_1.

Let $A = \{v \in S \mid v \text{ does not have property } P_1\}$ and let $B = S - A$. Further, let $|A| = a$ and $|B| = b$, and so $a = x - b$. Since each vertex of A is adjacent to a vertex of degree 1 in $\langle S \rangle$ that has property P_1, $|B| \geq |A|$. Thus, $b \geq a = x - b$, and so $b \geq x/2$. Since x is odd, $b \geq (x+1)/2$.

Each vertex of B has a private neighbor in $V - S$, and so $|V - S| \geq b$. Hence, $(3x+1)/2 = n = |S| + |V - S| \geq x + (x+1)/2 = (3x+1)/2$. Thus we have equality throughout this inequality chain. This implies that $b = (x+1)/2$ (and so $a = (x-1)/2$), each vertex of B has exactly one private neighbor in $V - S$, and $V - S$ consists entirely of these b external private neighbors of vertices of B.

Let $A = \{u_1, \ldots, u_a\}$. For $i = 1, \ldots, a$, let v_i be a vertex of degree 1 in $\langle S \rangle$ that has property P_1 and is adjacent to u_i. Necessarily, the vertices $v_1, \ldots, v_a$ are distinct. This accounts for $2a = x - 1$ vertices of S. Furthermore, each v_i has an external private neighbor in $V - S$, and hence $\deg_G v_i = 2$. Let v denote the remaining vertex of S. Then, v has property P_1 and all its neighbors in $\langle S \rangle$ belong to the set A. For notational convenience, we may assume that v is adjacent to u_1 (and possibly to other vertices of A). Hence G contains a spanning subgraph that is isomorphic to $P_5 \cup (a-1)P_3$. For $i = 1, \ldots, a$, let w_i denote the external private neighbor of v_i, i.e., $\text{epn}(v_i, S) = \{w_i\}$, and let $\text{epn}(v, S) = \{w\}$.

If $x = 3$, then either w is adjacent to w_1, in which case $G = C_5$, or w is a leaf, in which case $G = P_5 \in \mathcal{G}$. Hence we may assume in what follows that $x \geq 5$. We proceed further with the following claim.

Claim: Each vertex of $V - S$ has degree 1 in G.

Proof. Suppose $\deg w \geq 2$. Then, w is adjacent to a vertex w_i for some i, $1 \leq i \leq a$. If $i \neq 1$, then $(S - \{u_i, v\}) \cup \{w_i\}$ is a total dominating set of G of cardinality $|S| - 1 < \gamma_t(G)$, which is impossible. Thus, w is not adjacent to w_i for any i with $2 \leq i \leq a$. Similarly, w_1 is not adjacent to w_i for any i with $2 \leq i \leq a$. Hence ww_1 is an edge. But then since G is connected and $x \geq 5$, u_1 must be adjacent to some other vertex of A, and so $(S - \{u_1, v, v_1\}) \cup \{w, w_1\}$ is a total dominating set of G of cardinality $|S| - 1$, which is impossible. Hence, $\deg_G w = 1$. Similarly, $\deg_G w_1 = 1$.

Suppose, now, that there is an edge w_iw_j in G where $2 \leq i < j \leq a$. Thus, $u_i, v_i, w_i, w_j, v_j, u_j$ is a path P_6. If u_i and u_j both have degree 1 in G, then $(S - \{u_i, u_j\}) \cup \{w_i, w_j\}$ is a minimum total dominating set of G whose induced subgraph contains more edges than the subgraph induced by S. This contradicts our choice of S. Hence, at least one of u_i and u_j has degree at least 2.

If u_iu_j is an edge and if this is the only edge in $\langle A \rangle$ incident with u_i or u_j, then, since G is connected, there is some edge of $\langle V - S \rangle$, different from w_iw_j, incident with w_i or w_j. We may assume w_iw_k is an edge where $2 \leq k \leq a$ and $k \notin \{i, j\}$. But then G must contain a spanning subgraph H that is isomorphic to $P_5 \cup P_9 \cup (a-4)P_3$.

It follows that $\gamma_t(G) \le \gamma_t(H) \le 3 + 5 + 2(a - 4) = 2a = x - 1 < \gamma_t(G)$, which is impossible. Hence, there must be a vertex of A, different from u_i and u_j, adjacent to u_i or u_j. If u_1u_i is an edge, then $(S - \{u_i, u_j, v_i\}) \cup \{w_i, w_j\}$ is a total dominating set of G of cardinality $|S| - 1$, which is impossible. Hence, u_1 is not adjacent to u_i. Similarly, u_1 is not adjacent to u_j. Hence at least one of u_i and u_j is adjacent to a vertex u_ℓ where $2 \le \ell \le a$ and $\ell \notin \{i, j\}$. But then G must contain a spanning subgraph H that is isomorphic to $P_5 \cup P_9 \cup (a - 4)P_3$, which as shown earlier is impossible. We deduce therefore that there can be no edge w_iw_j in G where $2 \le i < j \le a$. The desired result follows. □

By the above claim, each vertex of $V - S$ has degree 1 in G. Since G is connected, it follows that $\langle A \cup \{v\}\rangle$ is connected, and so $G \in \mathcal{G}$. □

4 Realizable Triples

Our aim in this section is to determine which triples (x, y, z) are realizable, and to find a realization of each realizable triple. By Observation 6, $y \ge x \ge 2$ and $z \ge 0$. We consider three possibilities depending on whether $y < 3x/2$ or $y \ge 3x/2$ with $x \ge 2$ even or $y \ge (3x + 1)/2$ with $x \ge 3$ odd.

4.1 $y < 3x/2$

In this subsection, we consider the case when $x \ge 2$ and $y < 3x/2$.

Note that the bound in Theorem 7 is only meaningful for $k \le \frac{1}{2}\gamma_t(G)$ since $b(G) \ge 0$ for any graph G. Thus as an immediate consequence of Theorem 7, we have the following results.

Corollary 9 *If G is a graph satisfying $\gamma_t(G) = x$, $g(G) = y$, and $b(G) = z$ where $2 \le x \le y < 3x/2$, then $z \ge 2x - 4y/3$.*

Corollary 10 *All triples (x, y, z) of integers with $2 \le x \le y < 3x/2$ and $z < 2x - 4y/3$ are not realizable.*

Hence in what follows in this subsection we restrict our attention to values of z where $z \ge 2x - 4y/3$.

Observation 11 *The triple $(3, 4, 1)$ is not realizable.*

Proof. Suppose G is a graph of order 5 with $\gamma_t(G) = 3$. Let $S = \{u, v, w\}$ be a $\gamma_t(G)$-set. We may assume that v is adjacent to u and w. By Proposition 3, each of u and w has an external private neighbor, say u' and w' respectively. Now either $u'w' \in E(G)$, in which case $G = C_5$, or $u'w' \notin E(G)$, in which case $G = P_5$. Hence the only triples $(3, y, z)$ with $y + z = 5$ are $(3, 5, 0)$ and $(3, 3, 2)$. The desired result follows. □

We show next that all triples (x, y, z) satisfying $2 \le x \le y < 3x/2$ and $z \ge 2x - 4y/3$ are realizable except for the triple $(3, 4, 1)$. For this purpose, we prove three lemmas.

Lemma 12 *The triple $(x, (3x-1)/2, z)$ of integers where $x \geq 5$ is an odd integer and $z \geq 1$ is realizable.*

Proof. Let $k = (x-1)/2 \geq 2$ and let G be obtained from the disjoint union of a 2-corona of a path P_k on k vertices and a star $K_{1,z}$ by adding at least two edges joining the central vertex of the star to vertices of the path P_k. Since every minimum total dominating set of a graph contains all its support vertices, it follows that $\gamma_t(G) = 2k+1 = x$. Furthermore, each of the z leaves of the star $K_{1,z}$ is a bad vertex in G, while each vertex of the 2-corona is a good vertex. Hence, $\gamma_t(G) = x$, $g(G) = 3k+1 = (3x-1)/2$, and $b(G) = z$. □

Lemma 13 *The triple (x, x, z) of integers where $x \geq 2$ and $z \geq 2x/3$ is realizable.*

Proof. Suppose $x = 2$. Then, $z \geq 2$. Let G be a double star $S(\lfloor z/2 \rfloor, \lceil z/2 \rceil)$. Then the two central vertices of G form a unique $\gamma_t(G)$-set, and so $\gamma_t(G) = g(G) = 2 = x$ and $b(G) = z$. Hence we may assume that $x \geq 3$. We now consider three possibilities depending on whether x is congruent to 0, 1 or 2 modulo 3.

Let $\ell \geq 1$ be an integer. Let T_0 be the tree of order 3ℓ obtained from the disjoint union of ℓ stars $K_{1,2}$ by adding $\ell - 1$ edges joining central vertices of the stars. Let T_1 be the tree of order $3\ell+1$ obtained from the disjoint union of $\ell-1$ stars $K_{1,2}$ and a star $K_{1,3}$ by adding $\ell-1$ edges joining central vertices of the stars. Let T_2 be the tree of order $3\ell+2$ obtained from the disjoint union of $\ell-1$ stars $K_{1,2}$ and a star $K_{1,4}$ by adding $\ell-1$ edges joining central vertices of the stars. For $i = 0, 1, 2$, let $\mathcal{G}_i$ be the family of trees obtained from the tree T_i by adding vertices and edges as follows: for each leaf v of T_i, add at least one new vertex adjacent to v.

If $x = 3\ell + i$ for $i \in \{0, 1, 2\}$, then $z \geq 2\ell + i$. Hence since T_i has order x and $2\ell + i$ leaves, there exists a tree G_i in $\mathcal{G}_i$ of order $x + z$ (so the total number of new vertices added to T_i to produce G_i is z). Since every minimum total dominating set of a graph contains all its support vertices, it follows that for each $i = 0, 1, 2$, $S_i = V(T_i)$ is a $\gamma_t(G_i)$-set. By construction, each vertex $v \in S_i$ is a support vertex of G_i or satisfies $|\text{pn}(v, S_i)| \geq 2$. Hence, by Theorem 5, S_i is a unique $\gamma_t(G_i)$-set, and so $\gamma_t(G_i) = g(G_i) = x$ and $b(G_i) = z$. □

For integers (x, x, z) with $x \geq 2$ and $z \geq 2x/3$, let $G_{x,z}$ be the tree constructed in the proof of Lemma 13 that satisfies $\gamma_t(G_{x,z}) = x = g(G_{x,z})$ and $b(G_{x,z}) = z$.

Lemma 14 *All triples (x, y, z) of integers where $2 \leq x < y \leq (3x-2)/2$ and $z \geq 2x - 4y/3$ are realizable.*

Proof. Let $y = x + k$. Then, $1 \leq k \leq (x-2)/2$. Let $x' = x - 2k$. Then, $x' \geq 2$ and $z \geq 2x'/3$. We now consider the tree $T = G_{x',z}$. Let S be the unique $\gamma_t(T)$-set of T (and so $|S| = x'$) and let v be a support vertex in the subgraph $\langle S \rangle$. Let G be obtained from the disjoint union of T and a 2-corona of a path P_k by adding an edge joining v and a vertex of the path P_k. Then, $\gamma_t(G) = 2k + x' = x$. Furthermore, each of the z leaves in the tree T is a bad vertex in G, while each vertex of the 2-corona is a good vertex in G. Hence, $\gamma_t(G) = x$, $g(G) = 3k + x' = x + k = y$, and $b(G) = z$. □

An immediate consequence of Corollary 9, Observation 11, and Lemmas 12, 13, and 14 now follows.

Theorem 15 *Let (x, y, z) be a triple of integers where $2 \leq x \leq y < 3x/2$. Then (x, y, z) is realizable if and only if $z \geq 2x - 4y/3$ and $(x, y, z) \neq (3, 4, 1)$.*

4.2 x even and $y \geq 3x/2$

In this subsection, we consider the case when $x \geq 2$ is even and $y \geq 3x/2$.

Lemma 16 *All triples (x, y, z) of integers where $x \geq 4$ is even, $y \geq 3x/2$, and $z \geq 0$ are realizable.*

Proof. Let $k = x/2 \geq 2$. For $i = 1, 2, \ldots, k$, since $y \geq 3k$ we can choose integers $n_i \geq 2$ such that $\sum_{i=1}^{k} n_i = y - k$. For $i = 1, 2, \ldots, k$, let F_i be obtained from K_{n_i} by adding a new vertex u_i and joining it to a vertex of K_{n_i}. Let F be the connected graph obtained from the disjoint union of the k graphs $F_1, F_2, \ldots, F_k$ by adding the edges $u_i u_{i+1}$ for $i = 1, \ldots, k-1$. If $z = 0$, let $G = F$, while if $z \geq 1$, let G be obtained from F by adding z new vertices and joining each of these z new vertices to each of the vertices $\{u_1, u_2, \ldots, u_k\}$. Then, $\gamma_t(G) = 2k = x$. Furthermore, each of the z vertices added to F when constructing G is a bad vertex in G, while all other vertices of G are good vertices. Hence, $\gamma_t(G) = x$, $g(G) = y$, and $b(G) = z$. □

Observation 17 *All triples $(2, y, 0)$ of integers where $y \geq 2$ are realizable.*

Proof. The complete graph $G = K_y$ is a realization of $(2, y, 0)$. □

Observation 18 *The triple $(2, 3, 1)$ is not realizable.*

Proof. It is straightforward to check that if G is a connected graph of order 4, then either $G = P_4$, in which case $b(G) = 2$, or $G \neq P_4$, in which case $b(G) = 0$. □

Lemma 19 *All triples $(2, 3, z)$ where $z \geq 2$ are realizable.*

Proof. Let G be obtained from a C_4 by attaching $z - 1$ leaves to a vertex v of the cycle. Then, v and any one of its two neighbors on the 4-cycle totally dominate G, and so $\gamma_t(G) = 2$. However, neither any leaf of G nor the vertex not adjacent to v belong to any $\gamma_t(G)$-set. Thus, $g(G) = 3$, while $b(G) = z$. □

Lemma 20 *All triples $(2, y, z)$ where $y \geq 4$ and $z \geq 1$ are realizable.*

Proof. Let F be the graph obtained from a complete graph K_{y-1} by subdividing one edge uv exactly once. Let w be the resulting vertex of degree 2 adjacent to u and v. Let G be obtained from F by adding z new vertices and joining each new vertex to both u and w. Then, the vertex u together with any vertex of $V(F) - \{u, v\}$ totally dominates G, as does the set $\{v, w\}$. However, none of the z vertices (of degree 2 that were added to F to produce G) belong to a $\gamma_t(G)$-set. Thus, $\gamma_t(G) = 2$, $g(G) = y$, and $b(G) = z$. □

We summarize the results in this subsection as follows.

Theorem 21 *All triples (x, y, z) of integers where $x \geq 2$ is even, $y \geq 3x/2$ and $z \geq 0$ are realizable, except for the triple $(2, 3, 1)$.*

4.3 x odd and $y \geq (3x+1)/2$

In this subsection, we consider the case when $x \geq 3$ is odd and $y \geq (3x+1)/2$.

Lemma 22 *All triples (x, y, z) of integers where $x \geq 5$ is odd, $y \geq (3x+1)/2$ and $z \geq 1$ are realizable.*

Proof. Let $k = (x-1)/2 \geq 2$. For $i = 1, \ldots, k$, since $y \geq 3k+2$ we can choose integers $n_i \geq 2$ such that $\sum_{i=1}^{k} n_i = y - k - 1$. For $i = 1, 2, \ldots, k$, let F_i be obtained from K_{n_i} by adding a new vertex u_i and joining it to a vertex of K_{n_i}. Let F be the connected graph obtained from the disjoint union of the k graphs $F_1, F_2, \ldots, F_k$ by adding a new vertex w and adding the edges wu_i for $i = 1, 2, \ldots, k$. Let G be obtained from F by adding $z \geq 1$ new vertices and joining them to w. Then, $\gamma_t(G) = 2k + 1 = x$. Furthermore, each of the z vertices added to F when constructing G is a bad vertex in G, while all other vertices of G are good vertices. Hence, $\gamma_t(G) = x$, $g(G) = y$, and $b(G) = z$. □

Lemma 23 *All triples $(x, (3x+1)/2, 0)$ where $x \geq 5$ is an odd integer are not realizable.*

Proof. Let G be a connected graph of order $n = (3x+1)/2$ with $\gamma_t(G) = x$, where $x \geq 5$ is an odd integer. Then, by Theorem 8, $G \in \mathcal{G}$. However, then $g(G) \in \{n-1, n-2\}$ and $b(G) \in \{1, 2\}$. The desired result now follows. □

Lemma 24 *All triples $(x, y, 0)$ of integers where $x \geq 5$ is odd and $y \geq (3x+3)/2$ are realizable.*

Proof. Let $k = (x-3)/2 \geq 1$. For $i = 1, \ldots, k$, since $y \geq 3k+6$ we can choose integers $n_i \geq 2$ such that $\sum_{i=1}^{k} n_i = y - k - 6$. For $i = 1, \ldots, k$, let F_i be obtained from K_{n_i} by adding a new vertex u_i and joining it to a vertex v_i of K_{n_i}. Let F be the connected graph obtained from the disjoint union of the k graphs $F_1, \ldots, F_k$ by adding a new vertex w and adding the edges wu_i for $i = 1, \ldots, k$. Let G be obtained from F by adding a 5-cycle C and joining one of its vertices to the vertex w. Then any total dominating set of G must contain at least two vertices from F_i for each $i = 1, \ldots, k$ and at least three vertices from $V(C) \cup \{w\}$. Hence, $\gamma_t(G) \geq 2k + 3 = x$. However, the set $\{u_1, \ldots, u_k, v_1, \ldots, v_k\} \cup \{a, b, c\}$ where a, b, c are any three consecutive vertices on the 5-cycle, is a total dominating set of G, and so $\gamma_t(G) \leq x$. Consequently, $\gamma_t(G) = x$. In fact, it is not difficult to check that every vertex of G belongs to some $\gamma_t(G)$-set. Hence, $\gamma_t(G) = x$, $g(G) = y$, and $b(G) = 0$. □

Observation 25 *All triples $(3, y, z)$ of integers where $y \geq 5$ and $z \geq 0$ are realizable.*

Proof. Let F be the graph of order y obtained from a complete graph K_{y-2} by subdividing one edge uv exactly twice. Let u' and v' denote the resulting new vertices where u, u', v', v is a new path joining u and v. If $z = 0$, let $G = F$, while if $z \geq 1$, then let G be the graph obtained from F by adding z new vertices and joining each new vertex to both u' and v'. Clearly, $\gamma_t(G) \geq 3$. The sets $\{t, u, u'\}$ and $\{t, v, v'\}$ where $t \in V(F) - \{u, u', v, v'\}$ both totally dominate G, and so $\gamma_t(G) \leq 3$. Consequently, $\gamma_t(G) = 3$. However, none of the z vertices (of degree 2 that were added to F to produce G) belong to a $\gamma_t(G)$-set. Thus, $\gamma_t(G) = 3$, $g(G) = |V(F)| = y$, and $b(G) = z$. □

We summarize the results in this subsection as follows.

Theorem 26 *All triples (x, y, z) of integers where $x \geq 3$ is odd, $y \geq (3x+1)/2$, and $z \geq 0$ are realizable, except for those triples $(x, (3x+1)/2, 0)$ where $x \geq 5$ is odd.*

4.4 Summary

As a consequence of Theorems 15, 21, and 26 we have the following characterization of all triples (x, y, z) that are realizable.

Theorem 27 *All triples (x, y, z) of integers where $2 \leq x \leq y$ and $z \geq 0$ are realizable except for the triples*

(*a*) $(2, 3, 1)$,
(*b*) $(3, 4, 1)$,
(*c*) $(x, (3x + 1)/2, 0)$ *where $x \geq 5$ is odd, and*
(*d*) (x, y, z) *where $y < 3x/2$ and $z < 2x - 4y/3$.*

References

[1] R.C. Brigham, J.R. Carrington and R.P. Vitray, Connected graphs with maximum total domination number. *J. Combin. Math. Combin. Comput.* **34** (2000), 81–95.

[2] G. Chartrand and L. Lesniak, *Graphs & Digraphs: Third Edition*, Chapman & Hall, London (1996).

[3] E.J. Cockayne, R.M. Dawes and S.T. Hedetniemi, Total domination in graphs. *Networks* **10** (1980), 211–219.

[4] E.J. Cockayne, M.A. Henning and C.M. Mynhardt, Vertices contained in every minimum total dominating set of a tree. *Discrete Math.*, to appear.

[5] O. Favaron, M.A. Henning, C.M. Mynhardt and J. Puech, Total domination in graphs with minimum degree three. *J. Graph Theory* **34(1)** (2000), 9–19.

[6] G.H. Fricke, T.W. Haynes, S.S. Hedetniemi, S.T. Hedetniemi and R.C. Laskar, Excellent trees. *Bull. Inst. Combin. Appl.* **34** (2002), 27–38.

[7] T.W. Haynes, S.T. Hedetniemi and PJ. Slater (eds), *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc. New York, 1998.

[8] T.W. Haynes, S.T. Hedetniemi and P.J. Slater (eds), *Domination in Graphs: Advanced Topics*, Marcel Dekker, Inc. New York, 1998.

[9] T.W. Haynes and M.A. Henning, Trees with unique minimum total dominating sets. *Discussiones Mathematicae Graph Theory*, to appear.

[10] M.A. Henning, Graphs with large total domination number. *J. Graph Theory* **35(1)** (2000), 21–45.

[11] M.A. Henning, Total domination excellent trees. *Discrete Math.*, to appear.

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