The Domination number of (K_p, P_5)-free graphs

Igor É. Zverovich*

RUTCOR–Rutgers Center for Operations Research
Rutgers, The State University of New Jersey
640 Bartholomew Road
Piscataway, NJ 08854-8003
USA
igor@rutcor.rutgers.edu

Abstract

We prove that, for each $p \geq 1$, there exists a polynomial time algorithm for finding a minimum dominating set in the class of all (K_p, P_5)-free graphs.

Let G be a graph with vertex-set $V(G)$ and edge-set $E(G)$. The notation $x \sim y$ (respectively, $x \not\sim y$) means that vertices $x, y \in V(G)$ are adjacent (respectively, non-adjacent). Moreover, if $X \subseteq V(G)$ and $y \in V(G)\backslash X$, we write $y \sim X$ (respectively, $y \not\sim X$) to indicate that y is adjacent (respectively, non-adjacent) to all vertices in X. The *neighborhood* of a vertex $x \in V(G)$ is the set $N(x) = N_G(x) = \{y \in V(G) : x \sim y\}$; the *closed neighborhood* of x is $N[x] = \{x\} \cup N(x)$. Similarly, for a set $X \subseteq V(G)$, $N(X) = \bigcup\limits_{x \in X} N(x)$ and $N[X] = X \cup N(X)$. We use the notation P_n and K_n for a path and a complete graph of order $n \geq 1$, respectively.

A set $D \subseteq V(G)$ is a *domination* set in a graph G if every vertex of $V(G)\backslash D$ is adjacent to a vertex of D. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set in G. A dominating set G in G is *minimum* if $|D| = \gamma(G)$. For a set $X \subseteq V(G)$ we say that X *dominates* $N[X]$.

Let $\mathcal{Z}$ be a set of graphs. A graph G is called $\mathcal{Z}$*-free* if G does not contain any graph of $\mathcal{Z}$ as an induced subgraph. It is well known (see Bertossi [1], Johnson [3], and Korobitsin [4]) that the problem of finding a minimum dominating set is NP-complete for both P_5-free graphs and K_p-free graphs ($p \geq 3$). We prove that this problem can be solved in polynomial time for (K_p, P_5)-free graphs.

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Definition 1. *For $n \geq m \geq 1$ we define a graph $H = S(n,m)$ as follows: $V(H) = A \cup B$, where $A = \{u_1, u_2, \ldots, u_n\}$ and $B = \{v_1, v_2, \ldots, v_m\}$ are disjoint sets, and*

$$E(H) = \{u_i u_j : i, j \in \{1, 2, \ldots, n\}, i \neq j\} \cup \{u_i v_i : i = 1, 2, \ldots, m\}.$$

Any graph $S(n,m)$ (n and m are not fixed) will be called a simple split graph *(Figure 1).*

All graph of the form $S(n,m)$ are split graphs in sense of Földes and Hammer [2].

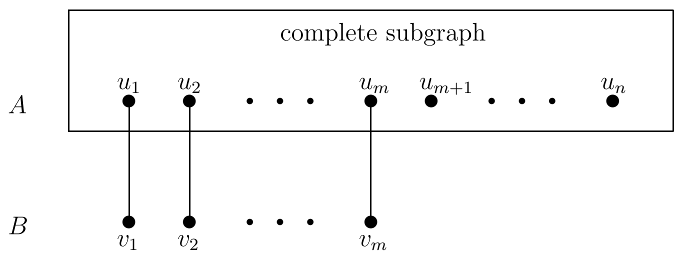

Figure 1. The simple split graph $S(m,n)$

Theorem 1. *For each $p \geq 1$, there exists a polynomial time algorithm for finding a minimum dominating set in the class of all (K_p, P_5)-free graphs.*

Proof. Let G be an arbitrary (K_p, P_5)-free graph. Without loss of generality we may assume that G is connected. Let us choose a subset $D \subseteq V(G)$ such that

(1) D induces a simple split graph;

(2) D dominates the largest number of vertices among all subsets satisfying (1).

We denote by H the subgraph of G induced by D. We shall assume as in Definition 1 that $V(H) = D = A \cup B$, where $A = \{u_1, u_2, \ldots, u_n\}$ and $B = \{v_1, v_2, \ldots, v_m\}$ are as in 1.

Suppose that D is not a dominating set in G. We consider a vertex x at distance two from D. Clearly, D does not dominate by x. There exists a vertex $w \notin D$ such that $w \sim x$ and $N(w) \cap D \neq \emptyset$.

Claim 1. $N(w) \cap A \neq \emptyset$.

Proof. Suppose that $N(w) \cap A = \emptyset$. Since $N(w) \cap D \neq \emptyset$, there exists a vertex $v_i \in B$ $(1 \leq i \leq m)$ which is adjacent to w.

If there is a vertex $u_j \in A \backslash \{u_i\}$ then the set $\{u_j, u_i, v_i, w, x\}$ induces P_5, a contradiction. Therefore $n = i = 1$, i.e., $A = \{u_1\}$ and $B = \{v_1\}$. The set $D' = \{u_1, v_1, w, x\}$ induces $P_4 \cong S(2,2)$, $D \subset D'$ and D' dominates more vertices than D. This contradicts to the choice of D. $\square$

We introduce a partition of A:

$A_0 = \{u_i : u_i \not\sim w \text{ and } i \in \{m+1, m+2, \ldots, n\}\}$,

$A_1 = \{u_i : u_i \sim w \text{ and } i \in \{m+1, m+2, \ldots, n\}\}$,

$A_{00} = \{u_i : u_i \not\sim w, v_i \not\sim w \text{ and } i \in \{1, 2, \ldots, m\}\}$,

$A_{01} = \{u_i : u_i \not\sim w, v_i \sim w \text{ and } i \in \{1, 2, \ldots, m\}\}$,

$A_{10} = \{u_i : u_i \sim w, v_i \not\sim w \text{ and } i \in \{1, 2, \ldots, m\}\}$,

$A_{11} = \{u_i : u_i \sim w, v_i \sim w \text{ and } i \in \{1, 2, \ldots, m\}\}$,

and a partition of B:

$B_{00} = \{v_i : u_i \in A_{00}\}$,

$B_{01} = \{v_i : u_i \in A_{01}\}$,

$B_{10} = \{v_i : u_i \in A_{10}\}$,

$B_{11} = \{v_i : u_i \in A_{11}\}$.

According to Claim 1, the vertex w is adjacent to a vertex $u \in A$.

Claim 2. $A_{00} = B_{00} = \emptyset$.

Proof. It sufficient to show that $A_{00} = \emptyset$. If there is a vertex $u_i \in A_{00}$ then the set $\{v_i, u_i, u, w, x\}$ induces P_5, a contradiction. $\square$

We put

$$A' = A_{10} \cup A_{11} \cup A_1 \cup \{w\} = (A \cap N(w)) \cup \{w\},$$

$$B' = B_{10} \cup \{x\} = (B \backslash N(w)) \cup \{x\},$$

and

$$D' = A' \cup B'.$$

It is clear that D' induces a simple split graph H'. The vertex $u \in N(w) \cap A \subseteq A'$ dominates A. The vertex $w \in D'$ dominates the set $B_{01} \cup B_{11}$. By Claim 2, $B = B_{01} \cup B_{10} \cup B_{11}$. Since $B_{10} \subseteq D'$, D' dominates D.

The set D' dominates x, but D does not. It follows from the choice of D that there exists a vertex y that is dominated by D and not dominated by D'. Clearly, $y \notin D \cup D'$.

Claim 3. $y \not\sim A$.

Proof. Since $A_{10} \cup A_{11} \cup A_1 \subseteq D'$ and D' does not dominate y, $y \not\sim A_{10} \cup A_{11} \cup A_1$.

By Claim 2, $A = (A_0 \cup A_{01}) \cup (A_{10} \cup A_{11} \cup A_1)$. Suppose that there exists a vertex $u_i \in A_0 \cup A_{01}$ that is adjacent to y. We have $y \not\sim \{w, x\} \subseteq D'$. Recall that the vertex $u \in A$ is adjacent to w, therefore $u \notin A_0 \cup A_{01}$ and $u \neq u_i$. Then the set $\{y, u_i, u, w, x\}$ induces P_5, a contradiction. Thus, $y \not\sim A_0 \cup A_{01}$ and $y \not\sim A$. ☐

Claim 4. (i) $y \sim B$,

(ii) $B_{10} = \emptyset$, *and*

(iii) $B = B_{01} \cup B_{11} \neq \emptyset$.

Proof. (i) The vertex y is dominated by $D = A \cup B$. By Claim 3, $y \not\sim A$. Therefore y is adjacent to a vertex $v_i \in B$. If there exists a vertex $v_j \in B \backslash \{v_i\}$ that is non-adjacent to y, then the set $\{v_j, u_j, u_i, v_i, y\}$ induces P_5, a contradiction. Hence $y \sim B$.

(ii) By Claim 2, $B = B_{01} \cup B_{10} \cup B_{11}$. Since $B_{10} \subseteq D'$ and D' does not dominate y, $y \not\sim B_{10}$. According to (i), $y \sim B_{10}$. Hence $B_{10} = \emptyset$.

(iii) It follows from (ii) that $B = B_{01} \cup B_{11}$. The vertex y is dominated by the set $D' = A' \cup (B_{01} \cup B_{11})$. By Claim 3, $y \not\sim A$. Thus, $B_{01} \cup B_{11} \neq \emptyset$. ☐

Since $B_{10} = \emptyset$, we also have $A_{10} = \emptyset$. Now we consider two possible cases.

Case 1: $|B| \geq 2$.

Claim 5. $|A| = |B| = 2$.

Proof. Since $|B| \geq 2$, B contains two distinct vertices v_i and v_j. Then $u_i, u_j \in A$ and $|A| \geq 2$. If there exists a vertex $u_k \in A \backslash \{u_i, u_j\}$ then the set $\{v_i, y, v_j, u_j, u_k\}$ induces P_5, a contradiction. It follows that $|A| = 2$ and $|B| = 2$. ☐

By Claim 5, we may assume that $A = \{u_1, u_2\}$ and $B = \{v_1, v_2\}$. By Claim 1, w is adjacent to either u_1 or u_2. We may assume that $w \sim u_2$.

We consider the set $D_1 = \{y, v_2, u_2, w, x\}$. Recall that $x \not\sim \{u_2, v_2\} \subseteq D$, $y \not\sim \{w, x\} \subseteq D'$, and $y \not\sim u_2 \in A$. Since D_1 does not induce P_5, v_2 is adjacent to w and D_1 induces a subgraph $H_1 \cong S(3, 2)$. The set D_1 dominates x, but D does not. Hence there exists a vertex z that is dominated by D and not dominated by D_1. Clearly, $z \notin D \cup D_1$ and z is adjacent to a vertex of $\{u_1, v_1\} = D \backslash D_1$.

If $z \sim u_1$ then the set $\{z, u_1, u_2, v_2, y\}$ induces P_5, a contradiction. So $z \not\sim u_1$. Then $z \sim v_1$ and the set $\{z, v_1, y, v_2, u_2\}$ induces P_5, a contradiction.

Case 2: $|B| = 1$.

Let $B = \{v_1\}$. By Claim 4, $y \sim v_1$.

Claim 6. $w \sim v_1$.

Proof. Suppose that $w \not\sim v_1$. The set $\{x, w, u_1, v_1, y\}$ can not induce P_5, therefore $w \not\sim u_1$. Recall that the vertex w is adjacent to the vertex $u \in A$. We have $u \neq u_1$ and the set $\{w, u, u_1, v_1, y\}$ induces P_5, a contradiction. □

The set $D_2 = \{u_1, v_1, w, x\}$ induces a subgraph H_2 isomorphic to either $S(2, 2)$ (when $w \not\sim u_1$) or $S(3, 1)$ (when $w \sim u_1$). Since D_2 dominates x and D does not, there exists a vertex $z \notin D \cup D_2 \cup \{y\}$ and $z \not\sim \{u_1, v_1, w, x\}$. If $z \sim y$ then the set $\{z, y, v_1, w, x\}$ induces P_5, a contradiction. So $z \not\sim y$. Since D dominates z, z is adjacent to a vertex $u_i \in A\backslash\{u_1\}$. Then the set $\{z, u_i, u_1, v_1, y\}$ induces P_5, a contradiction.

Thus, we have shown that D is a dominating set in G. Since G does not contain K_p as an induced subgraph, $n = |A| \leq p-1$. So $\gamma(G) \leq |D| = |A|+|B| \leq 2|A| \leq 2(p-1)$. Since $\gamma(G)$ is bounded above, a minimum domination set can be found by considering all t-subsets of $V(G)$ with $t \leq 2(p-1)$ in polynomial time. □

Corollary 1. *If G is a (K_p, P_5)-free graph, then a minimum dominating set of G can be found in time $O(n^p)$.*

Proof. The statement follows from the proof of Theorem 1. □

Corollary 2. *If G is a P_5-free graph and the largest clique in G has size ω, then $\gamma(G) \leq 2\omega$.*

Proof. The statement follows from the proof of Theorem 1. □

Corollary 3. *Each (K_p, P_5)-free graph contains a dominating set D with $|D| \leq 2p-2$ that induces a simple split graph.*

However, a (K_p, P_5)-free graph may have a minimum dominating set that does not induces a simple split graph. For example, any minimum dominating set in C_5 [a 5-cycle] induces $O_2 = \overline{K_2}$ [a graph with two non-adjacent vertices]. Clearly, O_2 is not a simple split graph.

Corollary 4. *The domination number of a P_5-free k-colorable graph can be found in polynomial time.*

Corollary 5. *The domination number of a P_5-free planar graph can be found in polynomial time.*

By a result of Földes and Hammer [2], the class of all split graphs coincides with the class of all $(2K_2, C_4, C_5)$-free graphs. Since $2K_2$ is an induced subgraph of P_5, a split graph must be P_5-free.

Corollary 6. *The domination number of a K_p-free split graph can be found in polynomial time.*

Open Problem 1. *Find the maximum value of q such that the domination number in the class of all (K_p, P_q)-free graphs can be found in polynomial time.*

For a graph G, the class of all G-free graphs is called a *monogenic class*. Korobitsin [4] proved that the domination number of a graph in a monogenic class $\mathcal{P}$ of all G-free graphs can be found in polynomial time if G is an induced subgraph of P_4 with (possibly) isolated vertices. Otherwise the problem is NP-complete. For bigenic classes, the problem is much more complicated. A *bigenic class* is a hereditary class with exactly two minimal forbidden induced subgraphs.

Open Problem 2. *Let* $S = \{\{G_1, G_2\}$: none of G_1, G_2 is an induced subgraph of the other $\}$. *Find a partition* $S_1 \cup S_2 = S$ *such that*

- *for each pair* $\{G_1, G_2\} \in S_1$, *the domination number in the bigenic class of all* $\{G_1, G_2\}$*-free graphs can be found in polynomial time, and*
- *for each pair* $\{G_1, G_2\} \in S_2$, *the domination number problem in the bigenic class of all* $\{G_1, G_2\}$*-free graphs is NP-complete.*

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