Some results on 2-homogeneous graphs*

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Abstract

We consider 2-homogeneous graphs, introduced by Nomura [J. Combin. Theory ser. B 60 (1994)], and discuss the elementary properties. Moreover, the diameter of 2-homogeneous graphs with even girth is bounded, and some restrictions of 2-homogeneous graphs with odd girth are obtained.

1 Introduction

Let Γ be a finite connected graph of diameter d with the usual metric ∂ on the vertex set $V\Gamma$ of Γ. The *girth* of a graph Γ is the minimal length of all the circuits in Γ, denoted by g. For vertices $u, v \in V\Gamma$, let

$$\Gamma_i(u) = \{x \in V\Gamma \mid \partial(u,x) = i\}, \;\; D^i_j(u,v) = \Gamma_i(u) \cap \Gamma_j(v).$$

The family $\{D^i_j(u,v)\}_{0\leq i,j\leq d}$ is called the *intersection diagram* of Γ with respect to u and v.

For any vertex x and a subset Y of $V\Gamma$, let $e(x,Y)$ denote the number of edges connecting x and Y.

A connected graph Γ is said to be 2-*homogeneous* if $x \in D^r_s(u,v)$ and $x' \in D^r_s(u',v')$ imply $e(x, D^i_j(u,v)) = e(x', D^i_j(u',v'))$ for all integers r, s, i, j whenever $\partial(u,v) = \partial(u',v') = 2$.

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A connected graph Γ is said to be *distance-regular* if, for all vertices u and v with distance i, the parameters

$$c_i = e(v, \Gamma_{i-1}(u)), a_i = e(v, \Gamma_i(u)), b_i = e(v, \Gamma_{i+1}(u))$$

depend only on i rather than the individual choice of u and v with $\partial(u,v) = i$. The parameters c_i, a_i, b_i are called the *intersection numbers*, and $\{b_0, b_1, \cdots, b_{d-1}; c_1, c_2, \cdots, c_d\}$ is called the *intersection array* of Γ, denoted by $i(\Gamma)$. For more information about distance-regular graphs, we refer readers to [1], [2], [3].

Nomura [5] determined all 2-homogeneous bipartite graphs. In this paper, we discuss the elementary properties of 2-homogeneous graphs, and bound the diameter of 2-homogeneous graphs of even girth. For odd girth case, some restrictions on intersection numbers are obtained.

2 Main results

Proposition 2.1 *A 2-homogeneous graph Γ is distance-regular.*

Proof. If Γ is a complete graph, our result follows. Now assume that $d \geq 2$. Let u, x be two vertices at distance i. First we consider case $i \geq 2$. Take a shortest path $(u, w, v, \cdots, x)$ connecting u and x. Then we have $x \in D^i_{i-2}(u,v)$ and

$$\begin{aligned} e(x, \Gamma_{i+1}(u)) &= e(x, D^{i+1}_{i-1}(u,v)), \\ e(x, \Gamma_i(u)) &= e(x, D^i_{i-2}(u,v)) + e(x, D^i_{i-1}(u,v)), \\ e(x, \Gamma_{i-1}(u)) &= e(x, D^{i-1}_{i-3}(u,v)) + e(x, D^{i-1}_{i-2}(u,v)) + e(x, D^{i-1}_{i-1}(u,v)). \end{aligned}$$

Now we consider case $i = 1$. Take $v \in \Gamma_1(x) \cap \Gamma_2(u)$. Then

$$\begin{aligned} e(x, \Gamma_2(u)) &= 1 + e(x, D^2_1(u,v)) + e(x, D^2_2(u,v)), \\ e(x, \Gamma_1(u)) &= e(x, D^1_1(u,v)) + e(x, D^1_2(u,v)). \end{aligned}$$

Assume $i = 0$. Take $v \in \Gamma_2(u)$. Then

$$e(u, \Gamma_1(u)) = e(u, D^1_1(u,v)) + e(u, D^1_2(u,v)) + e(u, D^1_3(u,v)).$$

Since Γ is 2-homogeneous, the right sides do not depend on the choice of u and x with distance i. Hence Γ is distance-regular. ■

Lemma 2.2 *Let Γ be a 2-homogeneous graph of diameter $d \geq 2$. Then there exist constants $\gamma_1, \gamma_2, \cdots, \gamma_d$ such that $|\Gamma_{r-1}(u) \cap D^1_1(x,y)| = \gamma_r$ for all $u \in V\Gamma$ and $x, y \in \Gamma_r(u)$ with $\partial(x,y) = 2$ ($r = 1, 2, \cdots, d$).*

Proof. It is clear that $\gamma_1 = 1$. Suppose $r > 1$. Since Γ is 2-homogeneous, for any $r \geq 2$, there exist constants $\delta_2, \delta_3, \cdots, \delta_d$ such that

$$|\Gamma_1(u) \cap \Gamma_{r-1}(x) \cap \Gamma_{r-1}(y)| = \delta_r.$$

We conclude that $\gamma_r = \dfrac{\delta_r\delta_{r-1}\cdots\delta_2}{c_{r-1}c_{r-2}\cdots c_2}$. Let N be the number of paths of length $r-1$ from u to $D_1^1(x,y)$. Let $(u = u_r, u_{r-1}, \cdots, u_1)$ be a path of length $r-1$ connecting u and some $u_1 \in D_1^1(x,y)$. Thus we have $u_i \in D_i^i(x,y)$ for all i. It is clear that $e(u_i, D_{i-1}^{i-1}(x,y)) = \delta_i$. So $N = \delta_r\delta_{r-1}\cdots\delta_2$. On the other hand, for a fixed vertex $z \in \Gamma_{r-1}(u)\cap\Gamma_1(x)\cap\Gamma_1(y)$, there are $c_{r-1}c_{r-2}\cdots c_2$ paths of length $r-1$ connecting z and u. Thus $N = |\Gamma_{r-1}(u)\cap\Gamma_1(x)\cap\Gamma_1(y)|c_{r-1}c_{r-2}\cdots c_2$, and so our conclusion holds. Hence γ_r is a constant for all $r = 1, 2, \cdots, d$. ■

The following results generalize Nomura's results in [4].

Lemma 2.3 *Let Γ be a 2-homogeneous graph of even girth and valency $k \geq 3$, and let*

$$\gamma_i = |\Gamma_{i-1}(u)\cap\Gamma_1(x)\cap\Gamma_1(y)|,$$

where $x, y \in \Gamma_i(u)$ at distance 2. Then the following hold.

(*i*) $c_2 \geq 2$;

(*ii*) $(k-2)(\gamma_2 - 1) = (c_2 - 1)(c_2 - 2)$;

(*iii*) $\gamma_i(c_{i+1} - 1) = c_i(c_2 - 1),\ 0 < i < d$;

(*iv*) $(c_2 - 1)(\gamma_i - 1) = (c_i - 1)(\gamma_2 - 1),\ 0 < i < d$.

Proof. (*i*) Since the girth of Γ is even, there exists a positive integer r such that

$$1 = c_1 = c_2 = \cdots = c_r < c_{r+1} \text{ and } a_1 = a_2 = \cdots = a_r = 0.$$

We conclude that $\gamma_r > 0$. Pick $w \in \Gamma_{r-1}(u)$, then we can choose two distinct vertices $x, y \in \Gamma_r(u)\cap\Gamma_1(w)$ by $b_{r-1} = k-1 \geq 2$. Note that $\partial(x,y) = 2$ by $a_1 = 0$. It is clear that $w \in \Gamma_{r-1}(u)\cap\Gamma_1(x)\cap\Gamma_1(y)$, and so $\gamma_r > 0$. Thus our conclusion is valid. Take $z \in \Gamma_{r+1}(u)$. We can choose two distinct vertices $x', y' \in \Gamma_r(u)\cap\Gamma_1(z)$ by $c_{r+1} > 1$. By $\gamma_r > 0$, there exists $v \in \Gamma_{r-1}(u)\cap\Gamma_1(x')\cap\Gamma_1(y')$. Thus $x', y' \in \Gamma_1(v)\cap\Gamma_1(z)$, and so $c_2 \geq 2$.

The proof of (*ii*), (*iii*), (*iv*) is similar to that of the case when Γ is bipartite, and will be omitted. ■

Theorem 2.4 *Let Γ be a 2-homogeneous graph of valency $k \geq 3$ and even girth. If $\gamma_2 > 1$, then $d \leq 5$. If $\gamma_2 = 1$, then $c_i = i$ and so $d \leq k$.*

Proof. First we consider the case $\gamma_2 > 1$. By (*ii*) of Lemma 2.3,

$$k = \frac{(c_2-1)(c_2-2)}{\gamma_2 - 1} + 2.$$

If $d \geq 3$, we have $c_3 = (c_2(c_2-1)/\gamma_2)+1$ by *(iii)* of Lemma 2.3. If $d \geq 4$, we get $c_4 = \dfrac{c_2(c_2^2-2c_2+2\gamma_2)}{\gamma_2+c_2\gamma_2-c_2}$. If $d \geq 5$, then $c_5 = \dfrac{c_2^4-3c_2^3+c_2^2+3\gamma_2c_2^2-2\gamma_2c_2+\gamma_2^2}{\gamma_2c_2^2+\gamma_2^2-c_2^2}$. If $d \geq 6$, then

$$b_5 = k - c_5 - a_5 = \frac{(c_2-\gamma_2)(c_2-\gamma_2-\gamma_2^2)}{(\gamma_2-1)(c_2^2\gamma_2+\gamma_2^2-c_2^2)} - a_5 \geq 1.$$

So $\gamma_2(c_2-1)(2c_2-2\gamma_2-c_2\gamma_2) \geq a_5 \geq 0$, which is impossible. Hence $d \leq 5$. If $\gamma_2 = 1$, then $\gamma_i = 1$ and $c_i = i$ for all $1 \leq i \leq d$ by Lemma 2.3. ■

Corollary 2.5 *Let Γ be a 2-homogeneous graph of valency $k \geq 3$ and diameter d. Suppose all the odd circuits are of length $2d+1$. Then one of the following holds.*

(i) $d \leq 5$.

(ii) $i(\Gamma) = \{k, k-1, k-2, \cdots, k-d+1; 1, 2, \cdots, d-1, d\}$.

Proof. Assume $d \geq 6$. We conclude that $c_d \geq 2$. Suppose not. Then Γ is a Moore graph, and so $d = 2$. This is impossible. So Γ is of even girth. By Theorem 2.4, the desired result follows. ■

Proposition 2.6 *Let Γ be a 2-homogeneous graph of odd girth $g \geq 5$ and valency $k \geq 3$. Then $c_i = 1$ or $b_{i-2} = 1$ for each $i = 2, 3, \cdots, d$.*

Proof. If there exists some i such that $c_i > 1$, then we claim that $b_{i-2} = 1$. Suppose not. Assume $b_{i-2} \geq 2$. Take vertices x and y at distance $i-2$. Since $b_{i-2} \geq 2$, we can choose two distinct vertices $x', y' \in \Gamma_{i-1}(x) \cap \Gamma_1(y)$. We have $\partial(x', y') = 2$ by $g \geq 5$, and so $\gamma_{i-1} \geq 1$. Let u, w be two vertices at distance i. By $c_i > 1$, we can take two distinct vertices $x, y \in \Gamma_1(w) \cap \Gamma_{i-1}(u)$. We get $\partial(x, y) = 2$ by $g \geq 5$. Since $\gamma_{i-1} \geq 1$, there exists $v \in \Gamma_1(x) \cap \Gamma_1(y) \cap \Gamma_{r-2}(u)$. Thus $v, w \in \Gamma_1(x) \cap \Gamma_1(y)$, so $c_2 \geq 2$, which contradicts to $g \geq 5$. Hence $b_{i-2} = 1$. ■

Definition 2.1 *Let Γ be a distance-regular graph. For any two vertices x, y at distance i, the induced subgraph on $\Gamma_{i-1}(x) \cap \Gamma_1(y)$ is called the c_i-graph of Γ, and the induced subgraph on $\Gamma_{i+1}(x) \cap \Gamma_1(y)$ is called the b_i-graph of Γ.*

Proposition 2.7 *Let Γ be a 2-homogeneous graph of girth 3. If $c_2 = 1$, then c_i-graph of Γ is a clique or b_{i-2}-graph is a clique, for all $i = 1, 2, \cdots, d$.*

Proof. If there exists some i such that both of c_i-graph and b_{i-2}-graph of Γ are not cliques. Let $\partial(u, v) = i-2$. Then there exist two vertices $x, y \in \Gamma_{i-1}(u) \cap \Gamma_1(v)$ at distance 2, and so $\gamma_{r-1} \geq 1$. Take two vertices u_1, v_1 at distance i, then there exist $x', y' \in \Gamma_{i-1}(u_1) \cap \Gamma_1(v_1)$ at distance 2. By $\gamma_{r-1} \geq 1$, there exits $v' \in \Gamma_1(x') \cap \Gamma_1(y') \cap \Gamma_{r-2}(u_1)$. Thus $c_2 \geq 2$, a contradiction. Hence the desired result follows. ■

Corollary 2.8 *Let Γ be a 2-homogeneous graph of girth 3. If $c_2 = 1$, then $c_i \leq a_1 + 1$ or $b_{i-2} \leq a_1 + 1$.*

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