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Closure of $K_1 + 2K_2$-free graphs

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Abstract

Let G and H be graphs. A *substitution* of H in G for a vertex $v \in V(G)$ is the graph $G(v \to H)$, which consists of a disjoint union of H and $G - v$ with the additional edge-set $\{xy : x \in V(H), y \in N_G(v)\}$. For a hereditary class of graphs $\mathcal{P}$, the *substitutional closure* of $\mathcal{P}$ is defined as the class $\mathcal{P}^*$ consisting of all graphs which can be obtained from graphs in P by repeated substitutions.

Let $2K_2$ be the graph consisting of two disjoint copies of the complete graphs K_2. The graph $K_1 + 2K_2$ is obtained from $2K_2$ by adding a dominating vertex. We characterize $K_1 + 2K_2$-free graphs in terms of forbidden induced subgraphs.

1 Introduction

The *neighborhood* of a vertex $x \in V(G)$ is the set $N_G(x) = N(x)$ of all vertices in G that are adjacent to x, and $N[x] = \{x\} \cup N(x)$ is the *closed neighborhood* of x.

Definition 1. *Let G and H be graphs. A* substitution *of H in G for a vertex $v \in V(G)$ is the graph $G(v \to H)$ consisting of the disjoint union of H and $G - v$ with the additional edge-set $\{xy : x \in V(H), y \in N_G(v)\}$.*

Definition 2. *For a class $\mathcal{P}$ of graphs, its* substitutional closure *$\mathcal{P}^*$ consists of all graphs that can be obtained from $\mathcal{P}$ by repeated substitutions, i.e., $\mathcal{P}^*$ is generated by the following rules:*

(S1) *$\mathcal{P} \subseteq \mathcal{P}^*$, and*

(S2) *if $G, H \in \mathcal{P}^*$ and $v \in V(G)$, then $G(v \to H) \in \mathcal{P}^*$.*

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Let $\mathrm{ISub}(G)$ be the set of all induced subgraphs of a graph G [considered up to isomorphism]. A class of graphs $\mathcal{P}$ is called *hereditary* if $\mathrm{ISub}(G) \subseteq \mathcal{P}$ for every graph $G \in \mathcal{P}$. For a set of graphs Z, the class of Z*-free* graphs consists of all graphs G such that $\mathrm{ISub}(G) \cap Z = \emptyset$. It is easy to show that if $\mathcal{P}$ is a hereditary class, then $\mathcal{P}^*$ is also a hereditary class.

Problem 1. *For a hereditary class $\mathcal{P}$ given by a set Z of forbidden induced subgraphs, find a forbidden induced subgraph characterization of the substitutional closure $\mathcal{P}^*$.*

Bertolazzi, De Simone, and Galluccio [1], and De Simone [2] noted that this problem is especially interesting in the case where $\mathcal{P}$ is a good class for the vertex packing problem, i.e., the weighted stability number can be found in polynomial time for all graphs in $\mathcal{P}$. We say that such a class is α_w*-polynomial.* The substitutional closure of a hereditary α_w-polynomial class is always α_w-polynomial. The same observation is valid for Weighted Clique Problem [and similarly defined ω_w*-polynomial* hereditary classes].

We shall use the Reducing Pseudopath Method of Zverovich [5] for constructing the substitutional closure of an arbitrary hereditary class.

Definition 3. *A set $W \subseteq V(G)$ is called* homogeneous *in a graph G if*

(H1) $2 \le |W| \le |V(G)| - 1$*, and*

(H2) $N(x) \setminus W = N(y) \setminus W$ *for all* $x, y \in W$*.*

According to (H2), a homogeneous set W defines a partition $W \cup W^+ \cup W^- = V(G)$ such that

- every vertex of W is adjacent to every vertex of W^+ [notation $W \sim W^+$], and
- every vertex of W is non-adjacent to every vertex of W^- [notation $W \not\sim W^-$].

By (H1), $W^+ \cup W^- \neq \emptyset$ for every homogeneous set W.

A graph without homogeneous sets is called *prime.* A graph H is called a *(primal) extension* of a graph G if

(E1) G is an induced subgraph of H, and

(E2) H is a prime graph.

Definition 4. *An extension H of G is* minimal *if there are no extensions of G in the set* $\mathrm{ISub}(H) \setminus \{H\}$*. We denote by* $\mathrm{Ext}(G)$ *the set of all minimal extensions of a graph G.*

In general, we can change some of the forbidden induced subgraphs for an α_w-polynomial (respectively, ω_w-polynomial) class with their substitutional closure to obtain a wider α_w-polynomial (respectively, ω_w-polynomial) class.

Problem 2. *Given a graph G, find* $\mathrm{Ext}(G)$.

Problem 2 was solved for all graphs of order at most four that have finitely many minimal extensions. For many graphs of order five it is easy to construct the set $\mathrm{Ext}(G)$, see Zverovich [4]. This method cannot be applied to the graph $K_1 + 2K_2$, also known as Butterfly. Here $2K_2$ is the graph consisting of two disjoint copies of the complete graphs K_2, and $K_1 + 2K_2$ is obtained from $2K_2$ by adding a dominating vertex. We solve Problem 2 for $G = K_1 + 2K_2$. In other words, we characterize the substitutional closure of $K_1 + 2K_2$-free graphs in terms of forbidden induced subgraphs.

For a set of graphs Z we put $\mathrm{Ext}(Z) = \bigcup\limits_{G\in Z} \mathrm{Ext}(G)$, and we define Z^o as the set of all minimal graphs in $\mathrm{Ext}(Z)$ with respect to the partial order 'to be an induced subgraph'.

Theorem 1. *If Z is the set of all minimal forbidden induced subgraphs for a hereditary class $\mathcal{P}$, then Z^o is the set of all minimal forbidden induced subgraphs for $\mathcal{P}^*$.*

2 Reducing pseudopaths

The notation $x \sim y$ (respectively, $x \not\sim y$) means that x and y are adjacent (respectively, non-adjacent). For disjoint sets X and Y, the notation $X \sim Y$ (respectively, $X \not\sim Y$) means that every vertex of X is adjacent to (respectively, non-adjacent) to every vertex of Y. In case of $X = \{x\}$ we also write $x \sim Y$ and $x \not\sim Y$ instead of $\{x\} \sim Y$ and $\{x\} \not\sim Y$, respectively.

Definition 5. *Let G be an induced subgraph of a graph H, and let W be a homogeneous set of G. We define a* reducing W-pseudopath *with respect to G in H as a sequence $R = (u_1, u_2, \ldots, u_t)$, $t \geq 1$, of pairwise distinct vertices of $V(H) \setminus V(G)$ satisfying the following conditions:*

(R1) *there exist vertices $w_1, w_2 \in W$ such that*

- **(R1a)** *$u_1 \sim w_1$, and*
- **(R1b)** *$u_1 \not\sim w_2$;*

(R2) *for each $i = 2, 3, \ldots, t$ either*

- **(R2a)** *$u_i \sim u_{i-1}$ and $u_i \not\sim W \cup \{u_1, u_2, \ldots, u_{i-2}\}$, or*
- **(R2b)** *$u_i \not\sim u_{i-1}$ and $u_i \sim W \cup \{u_1, u_2, \ldots, u_{i-2}\}$*
  [when $i = 2$, $\{u_1, u_2, \ldots, u_{i-2}\} = \emptyset$];

(R3) *for every $i = 1, 2, \ldots, t-1$*

- **(R3a)** *$u_i \sim W^+$, and*

(R3b) $u_i \not\sim W^-$;

(R4) *either*

(R4a) $u_t \not\sim x$ *for a vertex* $x \in W^+$, *or*

(R4b) $u_t \sim y$ *for a vertex* $y \in W^-$.

Theorem 2 (Zverovich [5]). *Let* H *be an extension of its induced subgraph* G*, and let* W *be a homogeneous set of* G*. Then there exists a reducing* W*-pseudopath with respect to any induced copy of* G *in* H*.*

3 The substitutional closure of $K_1 + 2K_2$-free graphs

As usual, C_n and K_n denote the cycle and the complete graph of order n, respectively.

Theorem 3. *A graph is in the substitutional closure of* $K_1 + 2K_2$*-free graphs if and only if it has no induced subgraphs* $G_1, G_2, \ldots, G_{37}$ *shown in Figure 1, Figure 2 and Figure 3.*

We simplify the pictures of some graphs G_i in Figures 1, 2 and 3 as follows. If G_i contains a vertex marked 'd' then d is adjacent to all other vertices of G_i except a unique vertex that is linked with d by a dotted line.

Proof. Let W be a homogeneous set of a graph G. We denote by $\mathcal{H}(G, W)$ the set of all graphs that can be obtained from G by adding a reducing W-pseudopath. Also, $\mathcal{H}_t(G, W)$ is the set of all graphs that can be obtained from G by adding a reducing W-pseudopath of length t.

Claim 1. *Let* W *be a homogeneous set of a graph* G*. If* $W^+ = \{x\}$ *and* $W^- = \emptyset$ *then* $\mathcal{H}(G, W)$ *is reducible to* $\mathcal{H}_1(G, W) \cup \mathcal{H}^a(G, W)$*, where* $\mathcal{H}^a(G, W)$ *consists of all reducing* W*-pseudopaths* $R = (u_1, u_2, \ldots, u_t)$ *such that* $t \geq 2$ *and each vertex* u_i*,* $i = 2, 3, \ldots, t$*, satisfies (R2a).*

Proof. Let H be a graph that obtained from G by adding a reducing W-pseudopath $R = (u_1, u_2, \ldots, u_t)$. If the statement does not hold, then $t \geq 2$ and there is a vertex u_i satisfying (R2b). We assume that i is the minimal number in $\{2, 3, \ldots, t\}$ such that u_i satisfies (R2b).

By (R2b), $u_i \sim W$ and $u_i \not\sim u_{i-1}$. The set $W \cup \{u_i\}$ induces a subgraph G' isomorphic to G [with u_i for x]. Clearly, W is a homogeneous set in G'. It follows from $u_i \not\sim u_{i-1}$ and (R4) that $(u_1, u_2, \ldots, u_{i-1})$ is a reducing W-pseudopath with respect to G'. □

Claim 2. *Every graph in* $\mathrm{Ext}(K_1 + 2K_2)$ *contains at least one of* G_1, G_6, G_{23} *(Figure 1), or* H_1, H_2, H_3, H_4, H_5 *(Figure 4) as an induced subgraph.*

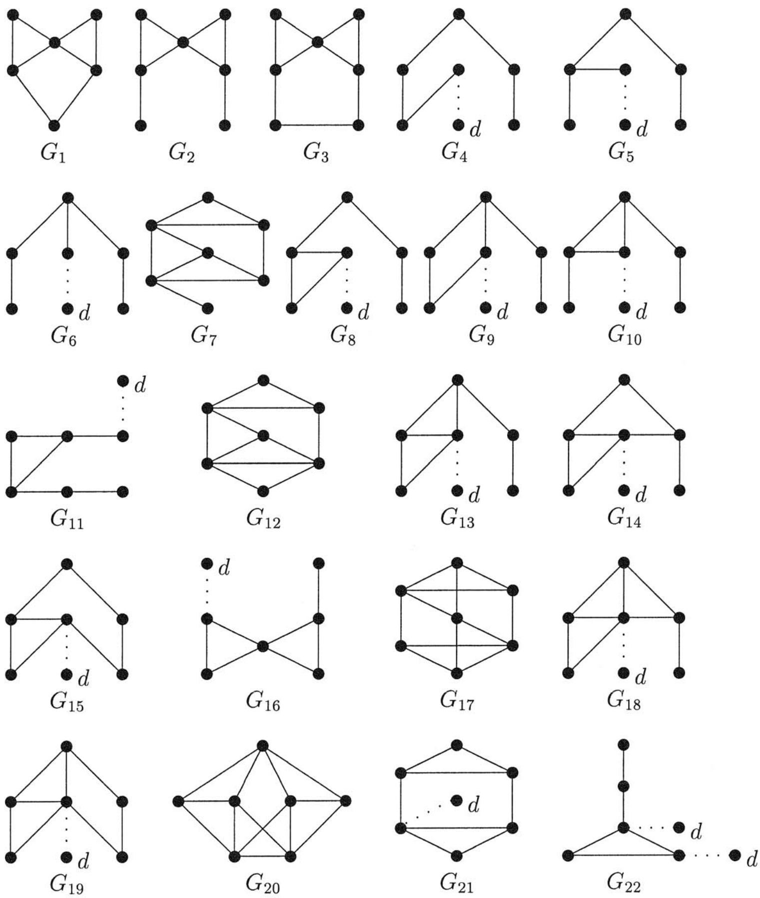

Figure 1: Forbidden induced subgraphs of order six and seven.

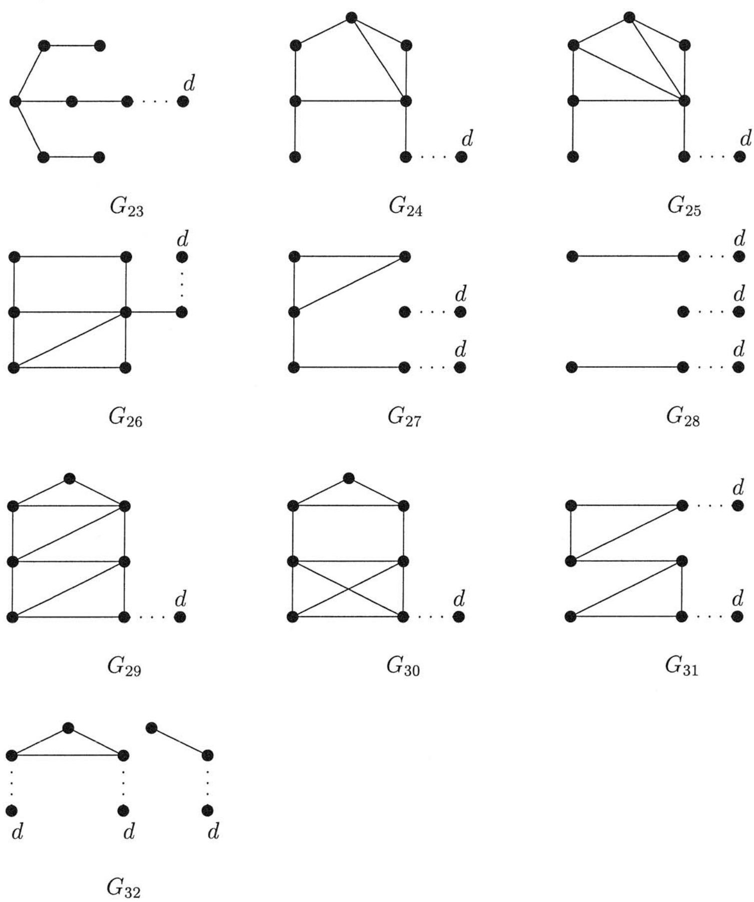

Figure 2: Forbidden induced subgraphs of order eight.

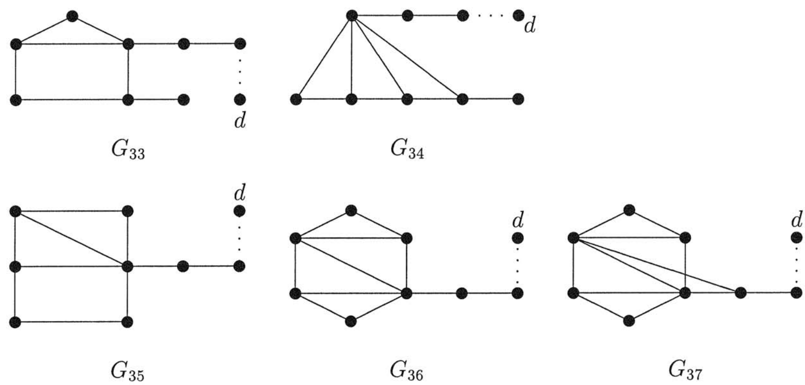

Figure 3: Forbidden induced subgraphs of order nine.

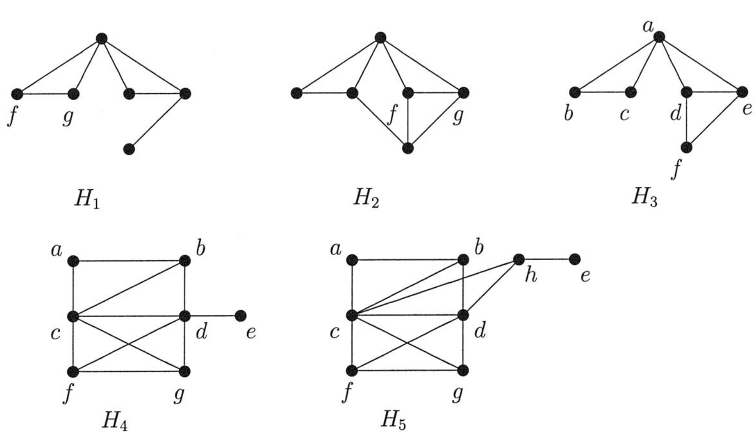

Figure 4: The graphs H_i.

Proof. Let G be a graph isomorphic to $K_1 + 2K_2$. Specifically, $V(G) = \{a, b, c, d, e\}$ and $E(G) = \{ab, cd, ea, eb, ec, ed\}$. The set $W = \{a, b, c, d\}$ is homogeneous in G. Hence every graph $H \in \mathrm{Ext}(G)$ contains a reducing W-pseudopath $R = (u_1, u_2, \ldots, u_t)$. According to Claim 1, we may restrict ourselves with $\mathcal{H}_1(G, W) \cup \mathcal{H}^a(G, W)$.

If $t = 1$ then we obtain one of G_1 (Figure 1) or H_1, H_2, H_3 (Figure 4). If $t = 2$ then we may assume that $u_2 \not\sim W$ and $u_1 \sim a$.

- If $u_1 \not\sim \{c, d\}$ then $\{a, c, d, e, u_1, u_2\}$ induces H_1. So let $u_1 \sim c$.
- If $u_1 \not\sim \{b, d\}$ then $W \cup \{e, u_1, u_2\}$ induces G_6. So let $u_1 \sim b$. By (R1b), $u_1 \not\sim d$ and $W \cup \{u_1, u_2\}$ induces H_4.

Finally, let $t \geq 3$. We have $\{u_{t-1}, u_t\} \not\sim W$, $u_t \not\sim e$, and $\{u_{t-2}, u_{t-1}\} \sim e$.

- If $u_{t-2} \not\sim \{a, b\}$ then $\{a, b, e, u_{t-1}, u_{t-2}, u_t\}$ induces H_1. Thus, $u_{t-2} = u_1$. By the symmetry, may assume that $u_{t-2} \sim \{a, c\}$.
- If $u_{t-2} \not\sim \{b, d\}$ then $W \cup \{e, u_{t-1}, u_{t-2}, u_t\}$ induces G_{23}. So let $u_{t-2} \sim b$. Since $u_{t-2} = u_1$ satisfies (R1b), $u_{t-2} \not\sim d$, and $W \cup \{u_{t-1}, u_{t-2}, u_t\}$ induces H_5.

□

Claim 3. (Zverovich and Zverovich [7]) *Let W be a homogeneous set of a graph G. If $|W| = 2$ then $\mathcal{H}(G, W)$ is reducible to $\mathcal{H}_1(G, W)$.*

Claim 4. *Every graph in* $\mathrm{Ext}(H_1)$ *contains at least one of*

$$G_1, G_2, G_3, G_4, G_5, G_7, G_9, G_{10}, G_{11}, G_{12}, G_{16}$$

(Figure 1) as an induced subgraph.

Proof. The graph H_1 has a unique homogeneous set $X = \{f, g\}$, see Figure 4. By Claim 3, we need to construct reducing X-pseudopaths of length $t = 1$ only. For H_1, there are 15 possible variants, 5 of them being redundant [there is a proper induced subgraph isomorphic to G_1]. □

Claim 5. *Every graph in* $\mathrm{Ext}(H_2)$ *contains at least one of*

$$G_1, G_{14}, G_{15}, G_{17}, G_{18}, G_{19}, G_{20}, G_{21}, G_{22}$$

(Figure 1), or H_1 (Figure 4) as an induced subgraph.

Proof. The graph H_2 has a unique homogeneous set $X = \{f, g\}$, see Figure 4. By Claim 3, we need to construct reducing X-pseudopaths of length $t = 1$ only. There are 15 possible variants, 5 of them being redundant [there is an induced subgraph isomorphic G_1], and two variants being reducible to H_1. □

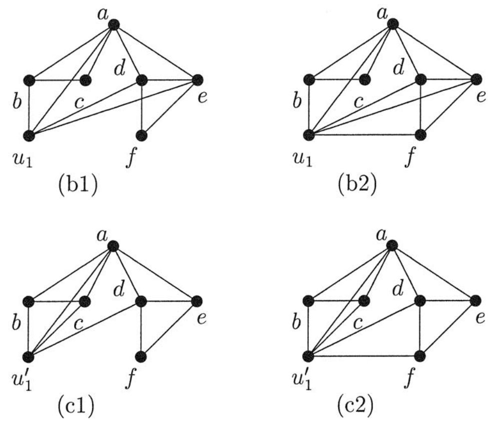

Figure 5: Cases (b1), (b2), (c1) and (c2).

Claim 6. *Every graph in* $\mathrm{Ext}(H_3)$ *contains at least one of*

$$G_1, G_8, G_{13}, G_{29}, G_{30}, G_{31}, G_{32}$$

(Figure 1 and Figure 2), or H_1 *(Figure 4) as an induced subgraph.*

Proof. The graph H_3 has two homogeneous sets, namely, $W_1 = \{b, c\}$ and $W_2 = \{d, e\}$, see Figure 4. By Claim 3, we need to construct reducing pseudopaths of length $t = 1$ only.

a) Suppose that $R = (u_1)$ is a reducing W_1-pseudopath and a reducing W_2-pseudopath simultaneously. In other words, u_1 is adjacent to exactly one of b, c, and u_1 is adjacent to exactly one of d, e. If $u_1 \not\sim a$ then $\{a, b, c, d, e, u_1\}$ induces G_1, and the proof is complete. If $u_1 \sim a$ then $\{a, b, c, d, e, f, u_1\}$ induces either G_8 [when $u_1 \not\sim f$] or G_{13} [when $u_1 \sim f$].

b) Suppose that $R_1 = (u_1)$ is a reducing W_1-pseudopath, but it is not a reducing W_2-pseudopath. We may assume that u_1 is adjacent to b and u_1 is not adjacent to c.

b$_1$) Let $u_1 \not\sim \{d, e\}$. If $u_1 \not\sim a$ then $\{a, b, c, d, e, u_1\}$ induces H_1. If $u_1 \sim a$ then $u_1 \sim f$ [by (R4)], and $\{a, b, d, e, f, u_1\}$ induces H_2.

b$_2$) Let $u_1 \sim \{d, e\}$. If $u_1 \not\sim a$ then $\{a, b, c, d, e, u_1\}$ induces H_2. If $u_1 \sim a$ then we obtain the variants (b1) and (b2) shown in Figure 5.

c) Suppose that $R_2 = (u_1')$ is a reducing W_2-pseudopath, but it is not a reducing W_1-pseudopath. We may assume that u_1' is adjacent to d and u_1' is not adjacent to e.

c$_1$) Let $u_1' \not\sim \{b, c\}$. If $u_1' \not\sim a$ then $\{a, b, c, d, e, u_1'\}$ induces H_1. If $u_1' \sim a$ then $u_1' \not\sim f$ [by (R4)], and $\{a, b, c, d, f, u_1'\}$ induces H_1.

c_2) Let $u_1' \sim \{b, c\}$. If $u_1' \not\sim a$ then $\{a, b, c, d, e, u_1'\}$ induces H_2. If $u_1' \sim a$ then we obtain the variants (c1) and (c2) shown in Figure 5.

d) It remains to consider four possible variants: (b1) and (c1); (b1) and (c2); (b2) and (c1); (b2) and (c2). Each of them has two subvariants depending on adjacency of u_1 and u_1'.

If we have (b1) & (c1) then either $\{a, c, e, f, u_1, u_1'\}$ induces H_1 [when $u_1 \not\sim u_1'$] or $V(H_3) \cup \{u_1, u_1'\}$ induces G_{29} [when $u_1 \sim u_1'$]. If we have (b1) & (c2) then either $\{a, c, e, f, u_1, u_1'\}$ induces G_1 [when $u_1 \not\sim u_1'$] or $\{b, c, d, e, f, u_1, u_1'\}$ induces G_{13} [when $u_1 \sim u_1'$]. If we have (b2) & (c1) then $V(H_3) \cup \{u_1, u_1'\}$ induces either G_{30} or G_{31}. Finally, if we have (b2) & (c2) then either $\{b, c, d, f, u_1, u_1'\}$ induces H_2 [when $u_1 \not\sim u_1'$] or $V(H_3) \cup \{u_1, u_1'\}$ induces G_{32} [when $u_1 \sim u_1'$]. □

Claim 7. *Every graph in* Ext(H_4) *contains at least one of*

$$G_1, G_6, G_{24}, G_{25}, G_{26}, G_{27}, G_{28}$$

(Figure 1 and Figure 2), or H_1, H_2 *(Figure 4) as an induced subgraph.*

Proof. The graph H_4 has a unique homogeneous set $X = \{f, g\}$ (see Figure 4). By Claim 3, we need to construct reducing X-pseudopaths $R = (u_1)$ only. Let $u_1 \sim g$ and $u_1 \not\sim f$.

Suppose that $u_1 \not\sim c$. Then the set $\{a, b, c, f, g, u_1\}$ induces G_1 or H_1. Hence we may assume that $u_1 \sim c$. Further, suppose that $u_1 \sim e$. If $|N(u_1) \cap \{a, b\}| \leq 1$ then the set $\{a, b, c, e, g, u_1\}$ induces either G_6 or H_1. Let $u_1 \sim \{a, b\}$. If $u_1 \not\sim d$ then the set $\{a, c, d, e, f, u_1\}$ induces G_1. If $u_1 \sim d$ then $V(H_4) \cup \{u_1\}$ induces G_{28}. So we may assume that $u_1 \not\sim e$.

Now we know that $u_1 \sim c$ and $u_1 \not\sim e$. It remains to consider seven possible variants. If $u_1 \not\sim \{a, b\}$ then $u_1 \not\sim d$ and $\{a, b, c, d, e, g, u_1\}$ induces G_6.

Let $u_1 \sim a$. If $u_1 \not\sim d$ then $\{a, c, d, e, f, u_1\}$ induces H_1. If $u_1 \sim d$ then $V(H_4) \cup \{u_1\}$ induces one of G_{26}, G_{27}. Let $u_1 \not\sim a$. Then $u_1 \sim b$ and $V(H_4) \cup \{u_1\}$ induces one of G_{24}, G_{25}. □

Claim 8. *Every graph in* Ext(H_5) *contains at least one of*

$$G_1, G_6, G_{23}, G_{33}, G_{34}, G_{35}, G_{36}, G_{37}$$

(Figure 1 and Figure 2), or H_1, H_2, H_4 *(Figure 4) as an induced subgraph.*

Proof. The graph H_5 has a unique homogeneous set $X = \{f, g\}$, see Figure 4. By Claim 3, we need to construct reducing X-pseudopaths $R = (u_1)$ only. Let $u_1 \sim g$ and $u_1 \not\sim f$.

Suppose that $u_1 \not\sim c$. Then the set $\{a, b, c, f, g, u_1\}$ induces one of G_1, H_1 or H_2. Therefore we may assume that $u_1 \sim c$. Further, suppose that $u_1 \sim e$. If $|N(u_1) \cap \{a, b\}| \leq 1$ then the set $\{a, b, c, e, g, u_1\}$ induces either G_6 or H_1. Let

$u_1 \sim \{a, b\}$. If $u_1 \not\sim d$ then the set $\{a, c, d, e, f, u_1\}$ induces H_1. If $u_1 \sim d$ then the set $\{a, b, c, e, f, g, u_1\}$ induces H_4. So we may assume that $u_1 \not\sim e$.

Now suppose that $u_1 \sim h$. If $u_1 \not\sim \{a, b\}$ then the set $\{a, b, c, e, h, u_1\}$ induces H_1. If $|N(u_1) \cap \{a, b\}| = 1$ then the set $\{a, b, c, e, f, g, h, u_1\}$ induces G_{23}. If $u_1 \sim \{a, b\}$ then either the set $\{a, c, d, e, f, h, u_1\}$ induces G_6 or the set $V(H_5) \cup \{u_1\}$ induces G_{37}.

Thus, $u_1 \sim c$ and $u_1 \not\sim \{e, h\}$. It remains to consider seven possible variants. If $u_1 \not\sim \{a, b\}$ then $u_1 \not\sim d$ and $\{a, b, c, d, e, g, h, u_1\}$ induces G_{23}.

Let $u_1 \sim a$. If $u_1 \not\sim d$ then $\{a, c, d, e, h, u_1\}$ induces H_1. If $u_1 \sim d$ then $V(H_5) \cup \{u_1\}$ induces one of G_{35}, G_{36}. Let $u_1 \not\sim a$. Then $u_1 \sim b$ and $V(H_5) \cup \{u_1\}$ induces one of G_{33}, G_{34}. □

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