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Some equitably 2 and 3-colourable cube decompositions

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Abstract

Let G be a graph in which each vertex has been coloured using one of k colours, say $c_1, c_2, \ldots, c_k$. If a 3-dimensional cube Q in G has n_i vertices coloured c_i, $i = 1, 2, \ldots, k$, and $|n_i - n_j| \leq 1$ for any $c_i, c_j \in \{c_1, c_2, \ldots, c_k\}$, then Q is said to be equitably k-coloured. A cube decomposition $\mathcal{Q}$ of a graph G is equitably k-colourable if the vertices of G can be coloured so that every cube in $\mathcal{Q}$ is equitably k-coloured. For $k = 2$ and 3, we completely settle the existence problem for equitably k-colourable cube decompositions of K_v, $K_v - F$ and $K_{m,n}$.

1 Introduction

Let G and H be graphs. A G-decomposition of H is a set $\mathcal{G} = \{G_1, G_2, \ldots, G_p\}$ such that G_i is isomorphic to G for $1 \leq i \leq p$ and $\mathcal{G}$ partitions the edge set of H. If, for each $G_i \in \mathcal{G}$, the vertex set of G_i has been coloured using k colours such that n_i vertices receive colour c_i, for $1 \leq i \leq k$, then G_i is said to be *equitably k-coloured* if $|n_i - n_j| \leq 1$ for any $c_i, c_j \in \{c_1, c_2, \ldots, c_k\}$. A G-decomposition is said to be *equitably k-coloured* if G_i is k-coloured for $1 \leq i \leq p$.

Most commonly, $H = K_v$, the complete graph on v vertices. Other common choices for H are $K_v - F$, the complete graph on v vertices with the edges of a 1-factor removed, and $K_{m,n}$, the complete bipartite graph with m vertices in one part and n in the other. In this paper we consider each of these graphs with G being a 3-dimensional cube, denoted Q, and k equal to 2 and 3. When $k = 2$, each cube in the decomposition has four vertices of each of the two colours. Similarly, when $k = 3$, each cube in an equitably 3-coloured cube decomposition has three vertices of one colour, three vertices of a second colour and two vertices of a third colour.

Previous work on equitably k-coloured G-decompositions has focused on having G isomorphic to an m-cycle. In [3], the existence question for equitably 2-coloured m-cycle decompositions of K_v and $K_v - F$ is completely settled for $m \in \{4, 5, 6\}$. The same problem, but with $k = 3$, is completely settled in [2]. The necessary and sufficient conditions for the existence of an equitably 2-coloured 3 or 5-cycle decomposition of the complete multipartite graph with all parts of the same size are presented in [7]. Also presented in [7] are the necessary and sufficient conditions for the existence of an equitably 2-coloured 4 or 6-cycle decomposition of the complete multipartite graph with p parts and n_i vertices in part i, for $1 \leq i \leq p$. Of course, a 4-cycle is a 2-dimensional cube. Thus, [2] and [3] completely settle the existence problem for equitable 2 and 3-coloured 2-cube decompositions.

Kotzig began work on Q-decompositions when he gave such a decomposition of K_{16} in [5]. In 1981, he went on to show in [6] that a Q-decomposition of K_v is possible only if $v \equiv 1, 16 \pmod{24}$. In the same paper, he proved sufficiency for the case $v \equiv 1 \pmod{24}$. It was not until 1994 that sufficiency was proven for $v \equiv 16 \pmod{24}$ in [4]. Furthermore, in [1], Adams, Bryant and El-Zanati completely settle the existence question for a Q-decomposition of λK_v, the lambda-fold complete graph. In this paper, for $k = 2$ and 3, we completely settle the existence question for equitably k-colourable Q-decompositions of K_v, $K_v - F$ and $K_{m,n}$. Our main result is given below.

Main Theorem

- There exist equitably 2-colourable Q-decompositions of K_v or $K_v - F$ if and only if $v \equiv 16 \pmod{24}$ or $v \equiv 2 \pmod{6}$ respectively.
- There exist equitably 3-colourable Q-decompositions of K_v or $K_v - F$ if and only if $v \equiv 1, 16 \pmod{24}$ or $v \equiv 2 \pmod{6}$ respectively.
- There exist equitably 2- or 3-colourable Q-decompositions of $K_{m,n}$, where $m \leq n$, if and only if $3 \mid m$, $3 \mid n$, $12 \mid mn$ and $4 \leq m \leq n$.

We now introduce some nototation to be used throughout this paper. The cube with vertex set $\{a, b, c, d, e, f, g, h\}$ and edge set $\{\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}, \{e, f\}, \{f, g\}, \{g, h\}, \{h, e\}, \{a, e\}, \{b, f\}, \{c, g\}, \{d, h\}\}$ is denoted (a, b, c, d, e, f, g, h). $K_{m,n}$ is used to denote the complete bipartite graph with m vertices in one part and n in the other. $K_m \setminus K_n$ denotes the complete graph on m vertices with a hole of size n, while $K_l \setminus (K_m + K_n)$ is used to denote the complete graph on l vertices with two disjoint holes of sizes m and n. Finally, we use the colours black and white when looking at equitable 2-colourings and black, white and grey when considering equitable 3-colourings.

2 Equitably 2-colourable Q-decompositions of K_v

From [6] and [4] we have the following existence result for uncoloured Q-decompositions.

Lemma 2.1 ([6], [4]) *There exists a Q-decomposition of K_v if and only if $v \equiv 1, 16 \pmod{24}$.*

An equitably 2-coloured cube must have four vertices coloured black and four coloured white. Each vertex also has degree three. These observations are used to prove Lemma 2.2.

Lemma 2.2 *An equitably 2-coloured Q-decomposition of K_v can exist only if v is even.*

Proof. Assume that v is odd and that an equitably 2-coloured Q-decomposition of K_v exists. We now seek a contradiction. Colour b vertices of K_v black. The degree sum of the black vertices in K_v is $b(v-1)$. A Q-decomposition of K_v has $v(v-1)/24$ cubes each containing four black vertices, each with degree three. Hence, we may express the degree sum of the black vertices as $v(v-1)/2$. Equating these two expressions, we find that $b = v/2 \notin \mathbb{Z}$.

We now present some existence results necessary for the proof of Theorem 2.9. All existence results were found by hand but have been checked using a distinct method.

Lemma 2.3 *There exists an equitably 2-coloured Q-decomposition of $K_{6,6}$.*

Proof. Let the vertex set of $K_{6,6}$ be $\cup_{i=1,2}\{0_i, 1_i, \ldots, 5_i\}$. For $i = 1, 2$, colour the vertices 0_i, 1_i and 2_i black and the vertices 3_i, 4_i and 5_i white. A suitable decomposition of $K_{6,6}$ is given by: $(0_1, 0_2, 1_1, 4_2, 3_2, 3_1, 1_2, 4_1)$, $(1_1, 2_2, 2_1, 3_2, 5_2, 4_1, 0_2, 5_1)$, $(2_2, 0_1, 1_2, 5_1, 3_1, 5_2, 2_1, 4_2)$.

Lemma 2.4 *If $m, n \equiv 0 \pmod 6$, there exists an equitably 2-colourable Q-decomposition of $K_{m,n}$.*

Proof. Let x and y be positive integers. Take a copy of $K_{x,y}$ and replace each vertex by six new vertices, colouring three black and three white. By Lemma 2.3, we can place an equitably 2-coloured Q-decomposition of $K_{6,6}$ on each set of vertices arising from an edge of $K_{x,y}$. The result is an equitably 2-coloured Q-decomposition of $K_{6x,6y}$.

Lemma 2.5 *There exists an equitably 2-coloured Q-decomposition of K_{16}.*

Proof. Let the vertex set of K_{16} be $\mathbb{Z}_{16}$. Colour the vertices $0, 1, \ldots, 7$ black and $8, 9, \ldots, 15$ white. A suitable decomposition of K_{16} is given by:

$(0,2,12,4,3,10,8,11)$, $(1,0,8,6,5,13,9,14)$, $(6,0,9,7,13,15,1,8)$,
$(0,10,6,12,14,1,11,7)$, $(4,15,2,13,8,5,9,3)$, $(0,5,4,7,11,12,14,15)$,
$(1,7,5,2,13,10,11,14)$, $(1,3,6,4,12,15,9,10)$, $(6,2,3,5,15,8,14,10)$,
$(7,2,4,3,13,11,9,12)$.

Lemmas 2.6 and 2.7 are needed for the construction of a Q-decomposition of $K_{40} \setminus K_{16}$, given in Lemma 2.8.

Lemma 2.6 *There exists an equitably 2-coloured Q-decomposition of $K_{28} \setminus (K_4 + K_{16})$.*

Proof. Let the vertex set of $K_{28} \setminus (K_4 + K_{16})$ be $\{0_1, 1_1, \dots, 7_1\} \cup \{0_2, 1_2, 2_2, 3_2\} \cup \{0_3, 1_3, \dots, 15_3\}$. Vertices with the subscript 2 are in the hole of size four while those with subscript 3 are in the hole of size sixteen. Colour the vertices 0_1, 1_1, 2_1, 3_1, 0_2, 1_2, 0_3, $1_3, \dots, 7_3$ black and the remaining vertices white. A suitable decomposition of $K_{28} \setminus (K_4 + K_{16})$ is given by:

$$(0_1,7_1,4_1,3_1,6_1,1_1,2_1,5_1),\quad (0_1,3_2,5_1,1_2,2_2,4_1,0_2,1_1),\quad (2_1,3_2,7_1,1_2,2_2,6_1,0_2,3_1),$$
$$(0_1,1_1,4_1,5_1,2_1,3_1,6_1,7_1),\quad (0_1,4_1,1_2,9_3,0_2,8_3,6_1,2_1),\quad (5_1,1_1,3_2,1_3,2_2,0_3,3_1,7_1),$$
$$(0_1,0_3,5_1,8_3,1_3,4_1,9_3,1_1),\quad (2_1,0_3,7_1,8_3,1_3,6_1,9_3,3_1),\quad (0_2,9_3,3_2,0_3,1_3,2_2,8_3,1_2),$$
$$(0_1,4_3,6_1,2_3,3_3,5_1,10_3,4_1),\quad (1_1,3_3,6_1,6_3,2_3,7_1,13_3,2_2),\quad (2_1,3_3,3_2,6_3,7_3,2_2,15_3,7_1),$$
$$(3_1,4_3,2_2,5_3,7_3,3_2,12_3,5_1),\quad (0_2,4_3,7_1,5_3,7_3,4_1,14_3,6_1),\quad (1_2,2_3,3_2,5_3,6_3,5_1,11_3,4_1),$$
$$(5_3,2_1,11_3,0_1,1_1,14_3,2_2,10_3),\quad (6_3,0_2,13_3,0_1,3_1,15_3,4_1,12_3),\quad (7_3,1_1,15_3,0_1,1_2,13_3,5_1,14_3),$$
$$(2_3,2_1,13_3,3_1,0_2,10_3,3_2,14_3),\quad (3_3,0_2,11_3,3_1,1_2,12_3,7_1,10_3),\quad (4_3,1_1,11_3,1_2,2_1,12_3,6_1,15_3).$$

Lemma 2.7 *Let H be a graph with six parts, labelled H_1, $H_2, \dots, H_6$, each containing four vertices. Let there be an edge connecting each vertex of part H_i to each vertex in parts H_4, H_5 and H_6, for $1 \le i \le 3$. Furthermore, let there be a copy of K_4 placed on the vertices in H_3 and also on the vertices in H_6. Then there exists an equitably 2-coloured Q-decomposition of H.*

Proof. Let the vertex set of H be $\cup_{i=1,2,\dots,6}\{0_i, 1_i, 2_i, 3_i\}$, with 0_i and 1_i coloured black and 2_i and 3_i coloured white, for $1 \le i \le 6$. A suitable decomposition of H is given by:

$$(0_1,2_6,3_3,0_4,1_5,3_2,3_5,0_3),\quad (0_1,3_6,2_3,1_4,2_5,2_2,0_5,1_3),\quad (0_1,0_6,1_3,2_4,3_5,2_2,1_5,2_3),$$
$$(0_1,1_6,0_3,3_4,0_5,3_2,2_5,3_3),\quad (0_3,0_5,1_1,1_4,2_4,2_1,2_6,2_2),\quad (1_3,3_5,1_1,0_4,3_4,3_1,1_6,2_2),$$
$$(2_3,2_5,0_2,0_4,3_4,1_1,0_6,3_2),\quad (3_3,1_5,1_2,1_4,2_4,1_1,3_6,3_2),\quad (2_1,3_4,1_2,0_4,1_4,0_2,2_4,3_1),$$
$$(2_1,3_5,0_2,1_5,2_5,1_2,0_5,3_1),\quad (2_1,1_6,1_2,0_6,3_6,0_2,2_6,3_1),\quad (0_3,2_3,1_3,3_3,3_6,1_6,2_6,0_6),$$
$$(0_3,1_3,3_6,2_6,0_6,1_6,3_3,2_3).$$

Lemma 2.8 *There exists an equitably 2-coloured Q-decomposition of $K_{40} \setminus K_{16}$.*

Proof. Take a copy of K_7 with vertex set $\{a, b, c, d, e, f, g\}$. Replace each vertex except g with four new vertices, in each case colouring two new vertices black and two white. Replace g with sixteen new vertices, colouring eight black and eight white.

By Lemma 2.6, we can place an equitably 2-coloured Q-decomposition of $K_{28} \setminus (K_4 + K_{16})$ on the set of vertices arising from the edges $\{a, b\}$, $\{a, c\}$, $\{a, g\}$, $\{b, c\}$, $\{b, g\}$ and $\{c, g\}$ in K_7, with the hole of size four on the vertices that replaced c and the hole of size sixteen on the vertices that replaced g. Apply the same decomposition to the vertices arising from the edges $\{d, e\}$, $\{d, f\}$, $\{d, g\}$, $\{e, f\}$, $\{e, g\}$ and $\{f, g\}$. The hole of size four is on the vertices that replaced f, while the hole of size sixteen remains on the vertices that replaced g.

Finally, we place the equitably 2-coloured Q-decomposition of the graph H given in Lemma 2.7 on the remaining edges. We let the vertices replacing a correspond to H_1, those replacing b correspond to H_2 and so on, where H_i is described in Lemma 2.7.

Theorem 2.9 *There exists an equitably 2-colourable Q-decomposition of K_v if and only if $v \equiv 16 \pmod{24}$.*

Proof. The necessary condition results from the combination of Lemmas 2.2 and 2.1. We now prove sufficiency.

Let $v = 24x + 16$, where x is a non-negative integer. Arrange the vertices of K_v into x groups of twenty-four vertices and one group of sixteen vertices. Within each group, colour half the vertices black and half white. Let the i^{th} group of twenty-four vertices be denoted by the set V_i, $1 \leq i \leq x$. Label the remaining sixteen vertices $\infty_1, \infty_2, \ldots, \infty_{16}$.

By Lemma 2.5, we can place an equitably 2-coloured Q-decomposition of K_{16} on the vertices $\infty_1, \infty_2, \ldots, \infty_{16}$. Furthermore, by Lemma 2.4, we can place an equitably 2-coloured Q-decomposition of $K_{24,24}$ on $V_i \cup V_j$, for $1 \leq i < j \leq x$. Finally, by Lemma 2.8, we can place an equitably 2-coloured Q-decomposition of $K_{40} \setminus K_{16}$ on $V_i \cup \{\infty_1, \infty_2, \ldots, \infty_{16}\}$, for $1 \leq i \leq x$, where the hole of size sixteen is on the vertices $\infty_1, \infty_2, \ldots, \infty_{16}$. The result is an equitably 2-coloured Q-decomposition of K_v, where $v \equiv 16 \pmod{24}$.

3 Equitably 2-colourable Q-decompositions of $K_v - F$

Finding equitably 2-colourable Q-decompositions of $K_v - F$ is considerably easier than the problem considered in the previous section. In fact, we require only one additional existence result.

Lemma 3.1 *There exists an equitably 2-coloured Q-decomposition of $K_8 - F$.*

Proof. Let the vertex set of $K_8 - F$ be $\mathbb{Z}_8$. Colour vertices 0, 1, 2 and 3 black and vertices 4, 5, 6 and 7 white. Let the edges of F be $\{0, 4\}$, $\{1, 5\}$, $\{2, 6\}$ and $\{3, 7\}$. A suitable decomposition of $K_8 - F$ is given by: $(0, 1, 2, 7, 5, 4, 3, 6)$, $(2, 0, 3, 5, 4, 6, 1, 7)$.

Theorem 3.2 *There exists an equitably 2-colourable Q-decomposition of $K_v - F$ if and only if $v \equiv 2 \pmod 6$, $v \geq 8$.*

Proof. Obviously, $v \geq 8$ as a cube has eight vertices. Furthermore, only when $v \equiv 2 \pmod 6$ is the number of edges in $K_v - F$ divisible by twelve and the degree of each vertex divisible by three. We now prove that these conditions are also sufficient.

Let $v = 6x + 2$, where x is a positive integer. Arrange the vertices of $K_v - F$ into x groups of six vertices and one group of two vertices. Within each group, colour half the vertices black and half white. Let the i^{th} group of six vertices be denoted by the set V_i, for $1 \leq i \leq x$. Label the vertices in V_i $0_i, 1_i, \ldots, 5_i$, where 0_i, 1_i and 2_i are the black vertices. Let the edges $\{0_i, 4_i\}$, $\{1_i, 5_i\}$ and $\{2_i, 6_i\}$ be contained in F. Furthermore, label the remaining two vertices ∞_1 and ∞_2 and let the edge $\{\infty_1, \infty_2\}$ be contained in F also.

By Lemma 2.3, we can place an equitably 2-coloured Q-decomposition of $K_{6,6}$ on $V_i \cup V_j$, for $1 \leq i < j \leq x$. Furthermore, by Lemma 3.1, we can place an equitably 2-coloured Q-decomposition of $K_8 - F$ on the vertices $V_i \cup \{\infty_1, \infty_2\}$, for $1 \leq i \leq x$. The result is an equitably 2-coloured Q-decomposition of $K_v - F$, where $v \equiv 2 \pmod 6$.

4 Equitably 3-colourable Q-decompositions of K_v

An equitably 3-coloured cube must have three vertices each of two colours and two vertices of the remaining colour. In this case, we are no longer restricted to considering decompositions of graphs with a vertex set of even size. Once again, we begin by establishing some existence results. It should be noted that the decomposition in Lemma 4.5 was found using a computational search.

Lemma 4.1 *There exists an equitably 3-coloured Q-decomposition of $K_{6,6}$.*

Proof. Let the vertex set of $K_{6,6}$ be $\cup_{i=1,2}\{0_i, 1_i, \ldots, 5_i\}$. For $i = 1, 2$, colour the vertices 0_i and 1_i black, vertices 2_i and 3_i white and vertices 4_i and 5_i grey. A suitable decomposition of $K_{6,6}$ is given by: $(0_1, 0_2, 1_1, 3_2, 2_2, 2_1, 4_2, 4_1)$, $(3_1, 3_2, 2_1, 5_2, 4_2, 5_1, 1_2, 0_1)$, $(4_1, 5_2, 5_1, 0_2, 1_2, 1_1, 2_2, 3_1)$.

Lemma 4.2 *If $m, n \equiv 0 \pmod 6$, there exists an equitably 3-colourable Q-decomposition of $K_{m,n}$.*

Proof. The proof mirrors that in Lemma 2.4.

Lemma 4.3 *There exists an equitably 3-coloured Q-decomposition of $K_{9,12}$.*

Proof. Let the vertex set of $K_{9,12}$ be $\{0_1, 1_1, \ldots, 8_1\} \cup \{0_2, 1_2, \ldots, 11_2\}$. Colour the vertices 0_1, 1_1, 2_1, 0_2, 1_2, 2_2 and 3_2 black. Colour the vertices 3_1, 4_1, 5_1, 4_2, 5_2, 6_2 and 7_2 white. Finally, colour the vertices 6_1, 7_1, 8_1, 8_2, 9_2, 10_2 and 11_2 grey. A suitable decomposition of $K_{9,12}$ is given by:

$(0_2, 0_1, 1_2, 3_1, 4_1, 4_2, 6_1, 8_2)$, $(1_2, 1_1, 2_2, 5_1, 4_1, 5_2, 8_1, 10_2)$, $(3_2, 0_1, 2_2, 4_1, 5_1, 6_2, 7_1, 11_2)$,
$(5_2, 3_1, 6_2, 6_1, 7_1, 9_2, 1_1, 0_2)$, $(4_2, 5_1, 7_2, 7_1, 8_1, 8_2, 2_1, 1_2)$, $(7_2, 4_1, 6_2, 8_1, 6_1, 9_2, 2_1, 3_2)$,
$(11_2, 8_1, 9_2, 0_1, 2_1, 0_2, 5_1, 5_2)$, $(8_2, 7_1, 10_2, 0_1, 1_1, 3_2, 3_1, 7_2)$, $(10_2, 6_1, 11_2, 1_1, 2_1, 2_2, 3_1, 4_2)$.

Lemma 4.4 *There exists an equitably 3-coloured Q-decomposition of K_{16}.*

Proof. Let the vertex set of K_{16} be $\cup_{i=1,2,3}\{0_i, 1_i, \ldots, 4_i\} \cup \{\infty\}$. Colour vertices with subscript 1 black, vertices with subscript 2 white and vertices with subscript 3 grey. Finally, colour the vertex ∞ black. A suitable decomposition of K_{16} can be found by developing the following two starter cycles modulo 5, leaving the subscripts and the point ∞ fixed: $(\infty, 0_1, 1_1, 1_2, 0_3, 3_3, 0_2, 4_2)$, $(0_1, 2_1, 4_3, 0_3, 1_2, 1_3, 2_2, 4_1)$.

Lemma 4.5 *There exists an equitably 3-coloured Q-decomposition of K_{25}.*

Proof. Let the vertex set of K_{25} be $\mathbb{Z}_{25}$. Colour the vertices 0, 1, 3, 8, 11, 17, 18, 20 and 21 black. Colour the vertices 2, 5, 9, 12, 13, 15, 19 and 22 white. Finally, colour the remaining eight vertices grey. A suitable decomposition of K_{25} is given by:

(14, 17, 24, 18, 22, 0, 23, 12), (10, 18, 20, 19, 23, 1, 24, 13), (11, 19, 21, 15, 24, 2, 20, 14),
(12, 15, 22, 16, 20, 3, 21, 10), (13, 16, 23, 17, 21, 4, 22, 11), (12, 17, 22, 19, 24, 8, 3, 6),
(13, 18, 23, 15, 20, 9, 4, 7), (14, 19, 24, 16, 21, 5, 0, 8), (10, 15, 20, 17, 22, 6, 1, 9),
(11, 16, 21, 18, 23, 7, 2, 5), (0, 1, 3, 7, 10, 12, 9, 14), (1, 2, 4, 8, 11, 13, 5, 10),
(2, 3, 0, 9, 12, 14, 6, 11), (3, 4, 1, 5, 13, 10, 7, 12), (4, 0, 2, 6, 14, 11, 8, 13),
(0, 12, 8, 14, 19, 4, 20, 23), (1, 13, 9, 10, 15, 0, 21, 24), (2, 14, 5, 11, 16, 1, 22, 20),
(3, 10, 6, 12, 17, 2, 23, 21), (4, 11, 7, 13, 18, 3, 24, 22), (0, 16, 19, 18, 20, 5, 9, 6),
(1, 17, 15, 19, 21, 6, 5, 7), (2, 18, 16, 15, 22, 7, 6, 8), (3, 19, 17, 16, 23, 8, 7, 9),
(4, 15, 18, 17, 24, 9, 8, 5).

Theorem 4.6 *There exists an equitably* 3*-colourable* Q*-decomposition of* K_v *if and only if* $v \equiv 1, 16 \pmod{24}$.

Proof. The necessary conditions follow from Lemma 2.1. We prove sufficiency by considering two cases.

Case 1. $v \equiv 1 \pmod{24}$

Let $v = 24x + 1$, where x is a non-negative integer. Arrange the vertices into x groups of twenty-four vertices and one "left-over" vertex. Within each group of twenty-four vertices, colour eight black, eight white and eight grey. The remaining vertex, labelled ∞, is coloured black. Let the i^{th} group of twenty-four vertices be denoted by the set V_i, for $1 \le i \le x$.

By Lemma 4.5, we can place an equitably 3-coloured Q-decomposition of K_{25} on the vertex set $V_i \cup \{\infty\}$, for $1 \le i \le x$. By Lemma 4.2, we can place an equitably 3-coloured Q-decomposition of $K_{24,24}$ on $V_i \cup V_j$, for $1 \le i < j \le x$. The result is an equitably 3-coloured Q-decomposition of K_v, where $v \equiv 1 \pmod{24}$.

Case 2. $v \equiv 16 \pmod{24}$

Let $v = 24x + 16$, where x is a non-negative integer. Arrange the vertices as for Theorem 2.9. Colour and label each group of twenty-four vertices as in Case 1. Furthermore, partition the 24 vertices in the set V_i into 4 groups of six vertices such that each group contains 2 vertices of each colour. For each set V_i, denote these "subgroups" by V_{i1}, V_{i2}, V_{i3} and V_{i4}. Label the vertices within the group of size sixteen $\infty_1, \infty_2, \dots, \infty_{16}$. Colour these new vertices such that $\infty_1, \infty_2, \dots, \infty_6$ are black, $\infty_7, \infty_8, \dots, \infty_{11}$ are white and $\infty_{12}, \infty_{13}, \dots, \infty_{16}$ are grey.

By Lemma 4.4, we can place an equitably 3-coloured Q-decomposition of K_{16} on the vertices $\infty_1, \infty_2, \dots, \infty_{16}$. Furthermore, by Lemma 4.5, we can place an equitably 3-coloured Q-decomposition of K_{25} on the vertex set $V_i \cup \infty_1$, for $1 \le i \le x$. By Lemma 4.2, we can place an equitably 3-coloured Q-decomposition of $K_{6,6}$ on $V_{ik} \cup \{\infty_2, \infty_3, \infty_7, \infty_8, \infty_{12}, \infty_{13}\}$ for $1 \le i \le x$ and $1 \le k \le 4$ and an equitably 3-coloured Q-decomposition of $K_{24,24}$ on $V_i \cup V_j$, for $1 \le i < j \le x$. We complete the equitably 3-coloured Q-decomposition of K_v by placing an equitably 3-coloured Q-decomposition of $K_{9,12}$, which exists by Lemma 4.3, on the set of vertices $V_{i1} \cup V_{i2} \cup \{\infty_4, \infty_5, \infty_6, \infty_9, \infty_{10}, \infty_{11}, \infty_{14}, \infty_{15}, \infty_{16}\}$ and $V_{i3} \cup V_{i4} \cup \{\infty_4, \infty_5, \infty_6, \infty_9, \infty_{10}, \infty_{11}, \infty_{14}, \infty_{15}, \infty_{16}\}$, for $1 \le i \le x$, with the obvious vertex partition.

5 Equitably 3-colourable Q-decompositions of $K_v - F$

Lemma 5.1 *There exists an equitably 3-coloured Q-decomposition of $K_8 - F$.*

Proof. Let the vertex set of $K_8 - F$ be $\mathbb{Z}_8$. Colour the vertices 0, 1 and 2 black, vertices 3, 4 and 5 white and vertices 6 and 7 grey. Let the edges of F be $\{0,4\}$, $\{1,5\}$, $\{2,6\}$ and $\{3,7\}$. A suitable decomposition of $K_8 - F$ is given by: $(0,1,2,7,5,4,3,6)$, $(2,0,3,5,4,6,1,7)$.

Theorem 5.2 *There exists an equitably 3-colourable Q-decomposition of $K_v - F$ if and only if $v \equiv 2 \pmod 6$, $v \geq 8$.*

Proof. The necessary conditions are clearly the same as for Theorem 3.2. To prove sufficiency, let $v = 6x + 2$, where x is a positive integer. Arrange the vertices of $K_v - F$ into x groups of six vertices and one group of two vertices. Let the i^{th} group of six vertices be denoted by the set V_i, for $1 \leq i \leq x$. Label the vertices in V_i $0_i, 1_i, \ldots, 5_i$, where 0_i and 1_i are coloured black, 2_i and 3_i are coloured white and 4_i and 5_i are coloured grey. Let the edges $\{0_i, 3_i\}$, $\{1_i, 4_i\}$ and $\{2_i, 5_i\}$ be contained in F. Furthermore, label the remaining two vertices ∞_1 and ∞_2, where ∞_1 is coloured black and ∞_2 is coloured white, and let the edge $\{\infty_1, \infty_2\}$ be contained in F also.

By Lemma 4.1, we can place an equitably 2-coloured Q-decomposition of $K_{6,6}$ on $V_i \cup V_j$, for $1 \leq i < j \leq x$. Furthermore, by Lemma 5.1, we can place an equitably 2-coloured Q-decomposition of $K_8 - F$ on the vertices $V_i \cup \{\infty_1, \infty_2\}$, for $1 \leq i \leq x$. The result is an equitably 3-coloured Q-decomposition of $K_v - F$, where $v \equiv 2 \pmod 6$.

6 Equitably 2 and 3-colourable Q-decompositions of $K_{m,n}$

Lemma 6.1 ([4]) *There exists a Q-decomposition of $K_{m,n}$, where $m \leq n$, if and only if $3 \mid m$, $3 \mid n$, $12 \mid mn$ and $4 \leq m \leq n$.*

Theorem 6.2 *There exists an equitably 2-colourable Q-decomposition of $K_{m,n}$, where $m \leq n$, if and only if $3 \mid m$, $3 \mid n$, $12 \mid mn$ and $4 \leq m \leq n$.*

Proof. This follows immediately from Lemma 6.1 by colouring the m vertices in one part black and the n vertices in the other part white.

Theorem 6.3 *There exists an equitably 3-colourable Q-decomposition of $K_{m,n}$, where $m \leq n$, if and only if $3 \mid m$, $3 \mid n$, $12 \mid mn$ and $4 \leq m \leq n$.*

Proof. The necessary conditions follow from Lemma 6.1. We now prove sufficiency. Within each part colour one third of the vertices with each of the three colours. We consider two cases.

Case 1: $m \equiv n \equiv 0 \pmod 6$.

This follows immediately from Lemma 4.2.

Case 2: $m \equiv 0 \pmod{12}$ and $n \equiv 3 \pmod 6$ (or vice versa).

Let $m = 12x$ and $n = 6y + 3$. Arrange the vertices of the first part into x groups of twelve vertices. Arrange the vertices of the second part into $y - 1$ groups of six vertices and one group of nine vertices. Within each group, colour one third of the vertices with each of the three colours. In the first part, let the i^{th} group of twelve vertices be denoted by the set U_i, for $1 \leq i \leq x$. In the second part, let the i^{th} group of six vertices be denoted by the set V_i, for $1 \leq i \leq y - 1$, and let the group of nine vertices be denoted by the set W.

By Lemma 4.2, we can place an equitably 3-coloured Q-decomposition of $K_{12,6}$ on $U_i \cup V_j$, for $1 \leq i \leq x$ and $1 \leq j \leq y - 1$. Furthermore, by Lemma 4.3, we can place an equitably 3-coloured Q-decomposition of $K_{12,9}$ on $U_i \cup W$, for $1 \leq i \leq x$. Thus we have an equitably 3-coloured Q-decomposition of $K_{m,n}$.

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