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Existence of 2-perfect GD6CSs

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Abstract

A group divisible k-cycle system (GDkCS) of type g^n is an edge-disjiont decomposition of a complete multipartite graph $K_n(g)$ into k-cycles. Let $\mathcal{C}$ be the set of k-cycles of a GDkCS and $C(2)$ be the graph formed by joining vertices that are distance 2 apart in a k-cycle C. If the set $\mathcal{C}(2) = \{C(2) : C \in \mathcal{C}\}$ also forms an edge-disjoint decomposition of $K_n(g)$, then the GDkCS is said to be 2-perfect. The elementary necessary conditions for the existence of a 2-perfect GD6CS of type g^n are shown to be sufficient with two exceptions.

1 Introduction

Let $K_{g_1,g_2,\ldots,g_n}$ be the complete multipartite graph with vertex set $V = \bigcup_{1\le i\le n} V_i$, where vertex classes V_i are disjoint sets with $|V_i| = g_i$, $i = 1, 2, \ldots, n$, and where two vertices $x \in V_i$ and $y \in V_j$ $(1 \le i,\ j \le n)$ are joined by exactly one edge if and only if $i \ne j$. We simply denote a complete n-partite multigraph $K_{g,g,\cdots,g}$ by $K_n(g)$.

A *group divisible k-cycle system* (GDkCS) is a triple $(X,\ \mathcal{G},\ \mathcal{C})$, where X is a set (of vertices), $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$ is a partition of X into subsets (called *groups*) G_i of size $g_i = |G_i|$, $1 \le i \le n$, and $\mathcal{C}$ is a collection of edge-disjoint k-cycles which partition edges of $K_{g_1,g_2,\ldots,g_n}$ with vertex classes $G_1, G_2, \ldots, G_n$. The *group-type* (or *type*) of the GDkCS is the multiset $\{g_i : 1 \le i \le n\}$ which is usually denoted by an "exponential" notation: a group-type $1^i 2^j 3^k \cdots$ means i occurrences of 1, j occurrences of 2, etc. A GDkCS of type T will be denoted by GDkCS(T).

The *distance* 2 *graph* of a cycle C, denoted by $C(2)$, is the graph formed by joining vertices that are distance 2 apart in C. Let $(X,\ \mathcal{G},\ \mathcal{C})$ be a GDkCS with group set $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$, where $|G_i| = g_i$ for $1 \le i \le n$, and set $\mathcal{C}(2) = \{C(2) : C \in \mathcal{C}\}$. If $\mathcal{C}(2)$ forms an edge-disjoint decomposition of $K_{g_1,g_2,\ldots,g_n}$ with vertex classes $G_1, G_2, \ldots, G_n$, then the GDkCS is said to be 2-*perfect*.

A GDkCS(1^n) is referred to as a *k-cycle system* of order n, denoted by kCS(n). It is well known that a 2-perfect kCS(n) is equivalent to a quasigroup of order n (see [13]). Much work has been done for the spectrum for 2-perfect kCSs. Vital papers in this area include [3, 11, 12, 14, 15]. GDkCSs are one of the basic ingredients in the construction of other types of designs such as k-cycle systems and k-cycle covering designs (see, for example, [2], [10]), as well as being of interest in their own right. It is trivial to see that every GD3CS is 2-perfect, which is also known as a *group divisible design* (GDD) with block size three (see [9]). When $k = 5$, 2-perfect GDkCSs are also called *holey Steiner pentagon systems*. The existence of 2-perfect GD5CSs(g^n) was shown in [1] with some possible exceptions. Since no GD4CS is 2-perfect, $k = 6$ is the smallest even number for which a 2-perfect GDkCS possibly exists.

We denote a k-cycle by $(x_0, x_1, \ldots, x_{k-1})$, so that $\{x_i,\ x_{i+1}\}$ is an edge for $0 \le i \le k-1$, reducing subscripts modulo k. Note that $C(2)$ is the union of two 3-cycles when C is a cycle of length 6. If $(x_0, x_1, x_2, x_3, x_4, x_5)$ is a 6-cycle, then two 3-cycles (x_0, x_2, x_4) and (x_1, x_3, x_5) arise from this 6-cycle by joining all vertices which are distance 2 apart. Therefore, a 2-perfect GD6CS will give a GD3CS (or a GDD with block size 3) which contains an even number of triples. However, the converse is not necessarily true. Lindner et al. [11] have found the spectrum for 2-perfect GD6CSs(1^n), i.e. 2-perfect 6CSs(n), with two possible exceptions which were removed by Billington and Lindner [5].

Theorem 1.1 *There exists a 2-perfect* 6CS(n) *if and only if* $n \equiv 1$ *or* 9 (mod 12) *except for* $n = 9$.

The following known result on 2-perfect GD6CSs is contained in [5].

Theorem 1.2 *There exists a 2-perfect* GD6CS $((2m)^3)$ *for every integer* $m \ge 2$.

The main purpose of this paper is to investigate the existence spectrum for 2-perfect GD6CSs(g^n). For a complete n-partite graph $K_n(g)$ to be decomposable into edge-disjoint 6-cycles, it is clear that $n \ge 3$, the degree of each vertex, $g(n-1)$, must be even, and the number of edges, $g^2 n(n-1)/2$, should be a multiple of 6. Thus we obtain the following basic conditions for the existence of a GD6CS(g^n):

$$n \ge 3,\ g(n-1) \equiv 0 \pmod 2,\ \text{and } g^2 n(n-1) \equiv 0 \pmod{12} \tag{1.1}$$

In what follows, we employ both direct and recursive methods of construction to show that the conditions (1.1) are also sufficient for the existence of a 2-perfect GD6CS(g^n), except for $(g,\ n) \in \{(1,\ 9),\ (2,\ 3)\}$.

2 Recursive constructions

To describe our recursive constructions we require several types of auxiliary designs. We use [4] as our standard design theory reference.

A *group divisible design* (or GDD) is a triple $(X, \mathcal{G}, \mathcal{B})$ where X is a set (of *points*), $\mathcal{G}$ is a partition of X into subsets (called *groups*), and $\mathcal{B}$ is a set of subsets of X (called *blocks*) such that no block contains two distinct points of any group, but any other pairset of X occurs in exactly one block of $\mathcal{B}$.

The *type* of the GDD is the multiset $\{|G| : G \in \mathcal{G}\}$. The notation K-GDD stands for a GDD having block-sizes from a set of positive integers K. When $K = \{k\}$, we simply write k for K. We wish to remark that a k-GDD of type 1^v is essentially a *pairwise balanced design* (PBD) of index unity, denoted by $B(K, 1; v)$.

For all practical purposes, we list some existence results concerning GDDs as follows.

Lemma 2.1 [9] *There exists a* 3*-GDD of type* g^n *if and only if* $n \geq 3$, $(n-1)g \equiv 0 \pmod 2$ *and* $n(n-1)g^2 \equiv 0 \pmod 6$.

Lemma 2.2 [7] *Let* g, w *and* $t \geq 3$ *be positive integers. There exists a* 3*-GDD of type* $g^t w^1$ *if and only if* $w \leq g(t-1)$, $gt \equiv w - g \equiv 0 \pmod 2$ *and* $gt(g(t-1)-w) \equiv 0 \pmod 3$.

Lemma 2.3 [8] *Let* v *be an integer and* $B(K) = \{v : \text{a } B(K, 1; v) \text{ exists}\}$. *Then* (1) $v \in B(\{3, 4, 5\})$ *if* $v \geq 3$ *and* $v \notin \{6, 8\}$; (2) $v \in B(\{4, 6, 7, 9\})$ *if* $v \equiv 0$ *or* $1 \pmod 3$, $v \geq 4$ *and* $v \notin \{10, 12, 15, 18, 19, 24, 27, 75\}$.

The significance of GDDs to our constructions for 2-perfect GD6CSs is seen in the following constructions. The first one is a modification of Wilson's fundamental construction for GDDs (see [16]).

Construction 2.4 (Weighting) *Suppose that there exist a* K*-GDD of type* m^n *and a* 2*-perfect* GDkCS(g^h) *for each* $h \in K$. *Then there exists a* 2*-perfect* GDkCS$((mg)^n)$.

The following PBD construction is a special case of Construction 2.4 .

Construction 2.5 (PBD) *Suppose that there exist a* $B(K, 1; v)$ *and a* 2*-perfect* GDkCS (g^h) *for each* $h \in K$. *Then there exists a* 2*-perfect* GDkCS(g^v).

We also have the following product construction for 2-perfect GD6CSs.

Construction 2.6 (Inflation) *Let* $m \geq 2$ *be an integer. If there exists a* 2*-perfect* GD6CS(g^n), *then there exists a* 2*-perfect* GD6CS$((mg)^n)$.

Proof. Let $(X, \mathcal{G}, \mathcal{C})$ be the given 2-perfect GD6CS(g^n). Give weight m to every vertex of X, that is, replace every vertex $x \in X$ by $\{x\} \times I_m$, where the set $I_m = \{1, 2, \ldots, m\}$. By Lemma 2.1, there exists a 3-GDD of type m^3 based on set $I_3 \times I_m$ with $\{i\} \times I_m$, $i = 1, 2, 3$, as its groups. Let $\mathcal{B}$ be the set of blocks of the GDD. For any 6-cycle $C = (x_0, x_1, x_2, x_3, x_4, x_5)$ of $\mathcal{C}$ and any block $B = \{(1, a), (2, b), (3, c)\}$ of $\mathcal{B}$, construct a 6-cycle $C_B = ((x_0, a), (x_1, b), (x_2, c), (x_3, a), (x_4, b), (x_5, c))$. Write $X^* = X \times I_m$, $\mathcal{G}^* = \{G \times I_m : G \in \mathcal{G}\}$, and $\mathcal{C}^* = \{C_B : C \in \mathcal{C}, B \in \mathcal{B}\}$. Then $(X^*, \mathcal{G}^*, \mathcal{C}^*)$ is a 2-perfect GD6CS$((mg)^n)$. □

We can "fill in the groups" of an existing 2-perfect GDkCS to obtain ones with more groups.

Construction 2.7 (1) *Suppose that there exist a 2-perfect* GDkCS$((mg)^n)$ *and a 2-perfect* GDkCS(g^m). *Then there exists a 2-perfect* GDkCS(g^{mn}).
(2) *Suppose that there exist a 2-perfect* GDkCS$((mg)^n)$ *and a 2-perfect* GDkCS(g^{m+1}). *Then there exists a 2-perfect* GDkCS(g^{mn+1}).
(3) *Suppose that there exist a 2-perfect* GDkCS$(g^n(mg)^1)$ *and a 2-perfect* GDkCS(g^m). *Then there exists a 2-perfect* GDkCS(g^{m+n}).

Before stating the next recursive constructions, we first generalize the concept of a GDkCS(g^n). Let $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$ be the vertex class set of $K_n(g)$, where $|G_i| = g$ for $1 \le i \le n$. Let $\mathcal{H} = \{H_1, H_2, \ldots, H_t\}$, where H_i $(1 \le i \le t)$ is a set of nh_i vertices of $K_n(g)$ such that $|H_i \cap G_j| = h_i$ for $1 \le j \le n$, and $H_i \cap H_j = \emptyset$ for $1 \le i < j \le n$. The notation $K_n(g) \setminus \cup_{1 \le i \le t} K_n(h_i)$ stands for the graph obtained by removing $K_n(h_i)$, $i = 1, 2, \ldots, t$, from $K_n(g)$, that is, the graph has vertex set $X = \cup_{1 \le i \le n} G_i$ and edge set $\{\{x, y\} : x \in G_i, y \in G_j, 1 \le i < j \le n$, and $|\{x, y\} \cap H_i| \le 1$ for $1 \le i \le n\}$. A *holey group divisible k-cycle system* having *hole* set $\mathcal{H}$ and *group* set $\mathcal{G}$ is a quadruple $(X, \mathcal{H}, \mathcal{G}, \mathcal{C})$, where $\mathcal{C}$ is a collection of edge-disjoint k-cycles which partition the edges of $K_n(g) \setminus \cup_{1 \le i \le t} K_n(h_i)$. If $(X, \mathcal{H}, \mathcal{G}, \mathcal{C}(2))$ is also an edge-disjoint decomposition of $K_n(g) \setminus \cup_{1 \le i \le t} K_n(h_i)$, then $(X, \mathcal{H}, \mathcal{G}, \mathcal{C})$ is said to be 2-*perfect*.

When $\mathcal{H} = \{H_1, H_2, \ldots, H_t\}$ is a partition of X, we simply denote the system by HGDkCS(n, T), where T is the multiset $\{h_i : 1 \le i \le t\}$. As with GDkCSs, we use an "exponential" notation to describe T. If $\mathcal{H} = \{H_1\}$, the system is called *incomplete* GDkCS, denoted by IGDkCS$((g, h_1)^n)$. An IGDkCS$((g, 0)^n)$ is just a GDkCS(g^n).

The following construction is an extension of Construction 2.7.

Construction 2.8 *Suppose that there exist a 2-perfect* IGDkCS$((mg + h, h)^n)$ *and a 2-perfect* GDkCS$(g^m h^1)$. *Then there exists a 2-perfect* GDkCS$(g^{mn}(nh)^1)$.

The technique of "filling in holes" is simple but useful.

Construction 2.9 *Suppose that there exist a 2-perfect* HGDkCS$(n, g^m w^1)$, *a 2-perfect* IGDkCS$((g+1, 1)^n)$, *a 2-perfect* IGDkCS$((w+1, 1)^n)$, *and a 2-perfect* GDkCS $((g + 1)^n)$. *Then there exists a 2-perfect* GDkCS$((mg + w + 1)^n)$.

In order to apply Construction 2.9, we need to build more families of 2-perfect HGDkCSs later. The following construction for 2-perfect HGDkCSs is a modification of the PBD construction for *mutually orthogonal Latin squares* (see [6]).

Construction 2.10 *Suppose that there exist a K-*GDD *of type T and a 2-perfect* HGDkCS$(n, 1^t)$ *for each $t \in K$. Then there exists a 2-perfect* HGDkCS(n, T).

3 Direct constructions

In our direct constructions of some systems, we shall adopt the standard approach of using a finite abelian group G to generate the set of cycles for a given system. That is, instead of listing all of the cycles of the system, we shall list a set of base (or starter) cycles and generate the others by an additive subgroup of G. The notation "$+d$ mod g" written behind the base cycles denotes that all elements of the base cycles should be taken cyclically by adding d (mod g) to them, while the infinite point, if it occurs in the base block, is always fixed. Moreover, we assume that the reader is familiar with Bose's mixed difference method. If not, the reader is referred to [4].

Lemma 3.1 *There exists a 2-perfect* GD6CS(2^n) *for any* $n \in \{4, 6, 7, 9\}$.

Proof. For each stated n, we take the vertex set $X = Z_{2n}$ and the group set $\mathcal{G} = \{\{j, j+n\} : j \in Z_n\}$. For $n \in \{4, 6, 7\}$, all of the cycles of the system are listed below. For $n = 9$, we only list the required base cycles.

$n = 4$: (1, 2, 4, 5, 6, 0), (1, 3, 2, 5, 7, 6), (1, 4, 3, 5, 0, 7),
(2, 7, 4, 6, 3, 0).

$n = 6$: (1, 2, 0, 7, 8, 11), (1, 4, 6, 7, 5, 0), (1, 5, 9, 7, 4, 3),
(1, 6, 2, 7, 10, 8), (1, 9, 11, 7, 3, 10), (2, 3, 6, 8, 9, 4),
(2, 5, 3, 8, 0, 9), (2, 10, 5, 8, 4, 11), (3, 11, 6, 9, 10, 0),
(4, 5, 6, 10, 11, 0).

$n = 7$: (1, 2, 3, 4, 5, 6), (1, 3, 6, 2, 11, 12), (1, 4, 7, 3, 9, 0),
(1, 5, 10, 4, 2, 13), (1, 7, 13, 3, 0, 11), (1, 9, 4, 13, 12, 10),
(2, 5, 11, 10, 13, 8), (2, 7, 5, 13, 0, 12), (2, 10, 7, 11, 8, 0),
(3, 5, 8, 9, 13, 11), (3, 8, 6, 11, 9, 12), (4, 6, 9, 10, 8, 12),
(4, 8, 7, 6, 10, 0), (5, 9, 7, 12, 6, 0).

$n = 9$: (1, 2, 3, 4, 7, 9), (1, 5, 9, 15, 12, 17), (1, 6, 8, 11, 16, 12),
(1, 7, 14, 8, 0, 11), +3 mod 18.

□

Lemma 3.2 *There exists a 2-perfect* GD6CS($2^3 4^1$).

Proof. $(X, \mathcal{G}, \mathcal{C})$ is such a system, where $X = Z_{10}$, $\mathcal{G} = \{\{0, 5\}, \{1, 6\}, \{2, 7\}, \{3, 4, 8, 9\}\}$ and $\mathcal{C}$ contains six 6-cycles:

(1, 2, 3, 5, 7, 4), (6, 0, 3, 1, 5, 4), (0, 7, 3, 6, 2, 4),
(1, 7, 8, 0, 2, 9), (0, 1, 8, 5, 6, 9), (7, 6, 8, 2, 5, 9).

□

Lemma 3.3 *There exists a 2-perfect* GD6CS(3^n) *for* $n \in \{5, 9\}$.

Proof. Take $X = Z_{3n}$, $\mathcal{G} = \{\{j, j+n, j+2n\} : j \in Z_n\}$. We list all of the cycles of the system for $n = 5$ and the base cycles for $n = 9$ below.

$n = 5:$ (1, 2, 3, 4, 5, 8), (1, 3, 7, 6, 4, 12), (1, 4, 13, 6, 12, 5),
(1, 7, 10, 6, 8, 0), (1, 9, 2, 6, 14, 13), (1, 10, 9, 3, 0, 14),
(2, 4, 10, 3, 6, 0), (2, 5, 3, 14, 11, 13), (2, 8, 9, 6, 5, 14),
(2, 10, 13, 9, 0, 11), (3, 11, 7, 5, 9, 12), (4, 7, 13, 5, 11, 8),
(4, 11, 10, 8, 12, 0), (7, 8, 14, 12, 11, 9), (7, 14, 10, 12, 13, 0).

$n = 9:$ (1, 2, 4, 8, 3, 13), (1, 4, 18, 24, 5, 12), +1 mod 27.

□

Lemma 3.4 *There exists a 2-perfect* GD6CS(5^9).

Proof. We take the vertex set $X = Z_{45}$ and the group set $\mathcal{G} = \{\{j,\ j+9,\ j+18,\ j+27,\ j+36\} : j \in Z_9\}$. The required base cycles are listed below.

(0, 40, 15, 37, 26, 3), (0, 6, 35, 23, 40, 8), (1, 23, 25, 11, 24, 34),
(1, 4, 33, 0, 20, 39), (0, 28, 44, 42, 21, 11), (2, 32, 31, 7, 6, 40),
(0, 13, 5, 26, 29, 19), (0, 38, 19, 34, 3, 44) (0, 15, 7, 13, 38, 43)
(1, 5, 44, 30, 2, 42), +3 mod 45.

□

Lemma 3.5 *There exists a 2-perfect* GD6CS(6^n) *for any* $n \in \{5,\ 8\}$.

Proof. For each stated n, we take the vertex set $X = Z_{6n}$ and the group set $\mathcal{G} = \{\{j,\ j+n,\ j+2n,\ j+3n,\ j+4n,\ j+5n\} : j \in Z_n\}$. Then the required base cycles are listed below.

$n = 5:$ (1, 2, 4, 10, 22, 8), (1, 4, 8, 21, 12, 20), +1 mod 30.

$n = 8:$ (0, 13, 34, 1, 3, 30), (1, 8, 31, 12, 24, 35), (1, 4, 14, 3, 23, 5),
(1, 18, 4, 39, 6, 11), (1, 2, 5, 14, 15, 20), (1, 7, 43, 18, 38, 40),
(1, 23, 16, 42, 36, 32), +2 mod 48.

□

Lemma 3.6 *There exists a 2-perfect* IGD6CS $((g+w,\ w)^n)$ *where* $(g, w, n) \in \{(2,1,9),\ (4,1,9),\ (6,4,3)\}$.

Proof. For each stated triple (g, w, n), we take the vertex set $X = (Z_g \cup \{\infty_i : i \in Z_w\}) \times Z_n$, the group set $\mathcal{G} = \{(Z_g \cup \{\infty_i : i \in Z_w\}) \times \{j\} : j \in Z_n\}$, and the hole $H = \{\infty_i : i \in Z_w\} \times Z_n$. The required base cycles are listed below.

$(g,\ w,\ n) = (2,\ 1,\ 9)$:
((0,0), (1,3), (0,2), (0,8), (1,1), (1,2)), ((1,1), (1,3), (1,0), (0,1), (1,4), (1,8)),
((∞_0,1), (1,2), (1,3), (∞_0,4), (0,2), (1,7)), ((∞_0,1), (1,3), (1,5), (∞_0,2), (1,7), (1,6)),
((∞_0,1), (1,4), (0,6), (∞_0,8), (1,0), (1,5)), ((∞_0,2), (1,1), (1,4), (∞_0,6), (0,0), (0,4)),
((∞_0,2), (1,6), (0,1), (∞_0,0), (1,5), (0,8)), ((∞_0,3), (1,2), (0,6), (∞_0,0), (0,4), (1,5)),
(+1 mod 2, +3 mod 9).

$(g,\ w,\ n) = (4,\ 1,\ 9)$:
((1,1), (3,2), (1,5), (2,1), (0,8), (3,5)), ((1,1), (0,2), (0,5), (0,1), (1,8), (2,5)),
((∞_0,0), (1,1), (1,2), (∞_0,3), (2,1), (3,2)), ((∞_0,0), (1,3), (1,5), (∞_0,8), (2,3), (3,5)),
(+1 mod 4, +1 mod 9).

$(g,\ w,\ n) = (6,\ 4,\ 3)$:
((0,0), (1,1), (5,2), (3,0), (5,1), (4,2)),
((∞_0,0), (1,1), (1,2), (∞_1,0), (2,1), (3,2)), ((∞_2,0), (1,1), (3,2), (∞_3,0), (2,1), (5,2)),
((∞_0,1), (1,2), (1,0), (∞_1,1), (4,2), (3,0)), ((∞_2,1), (1,2), (2,0), (∞_3,1), (2,2), (5,0)),
((∞_0,2), (1,1), (1,0), (∞_1,2), (2,1), (4,0)), ((∞_2,2), (1,1), (2,0), (∞_3,2), (2,1), (5,0)),
(+1 mod 6, —).

□

Lemma 3.7 *There exists a 2-perfect* HGD6CS(9, 1^3).

Proof. Take the vertex set $X = Z_3 \times Z_9$, the group set $\mathcal{G} = \{Z_3 \times \{j\} : j \in Z_9\}$, and the hole set $\mathcal{H} = \{\{i\} \times Z_9 : i \in Z_3\}$. The required base cycles are listed as follows:

((1,1), (2,2), (0,3), (1,4), (2,5), (0,6)), ((1,1), (2,3), (0,2), (1,6), (2,7), (0,4)),
((1,1), (2,4), (0,8), (1,7), (2,6), (0,3)), ((0,0), (1,1), (2,5), (0,7), (1,8), (2,2)),
((1,1), (2,6), (0,0), (1,5), (2,4), (0,2)), ((1,1), (2,7), (0,5), (1,0), (2,3), (0,8)),
((1,1), (2,8), (0,4), (1,3), (2,0), (0,7)), ((1,1), (2,0), (0,6), (1,2), (2,8), (0,5)),
((1,2), (2,4), (0,0), (1,8), (2,3), (0,5)), ((1,2), (2,5), (0,8), (1,6), (2,4), (0,7)),
((1,2), (2,7), (0,3), (1,6), (2,8), (0,0)), ((1,3), (2,7), (0,0), (1,4), (2,6), (0,5)),
(+1 mod 3, —).

□

4 Existence results

In this section, we shall establish our main results concerning 2-perfect GD6CSs(g^n).

Lemma 4.1 *Let g and n be positive integers satisfying $g \equiv 0 \pmod 6$ and $n \geq 3$. Then there exists a 2-perfect* GD6CS(g^n).

Proof. We first consider the case $g = 6$. The result for $n \in \{3, 5, 8\}$ was provided in Theorem 1.2 and Lemma 3.5. For $n \in \{4, 6\}$, take a 2-perfect GD6CS(2^n) from Lemma 3.1; the result then follows from Construction 2.6 with weight 3. For any other value of n, we have $n \in B(\{3, 4, 5\})$ by Lemma 2.3; then apply Construction 2.5 to obtain the desired system.

For the case $g \geq 12$, we write $g = 6x$ where $x \geq 2$, give weight x to each vertex of a 2-perfect GD6CS(6^n), and apply Construction 2.6 to get the result. ∎

Lemma 4.2 *Let g and n be positive integers satisfying $g \equiv 3 \pmod 6$, $n \equiv 1 \pmod 4$ and $n \geq 5$. Then there exists a 2-perfect* GD6CS(g^n).

Proof. When $g \geq 9$, write $g = 3x$ where x is odd and $x \geq 3$. Since a 2-perfect GD6CS(3^n) can be inflated to form a 2-perfect GD6CS($(3x)^n$) by Construction 2.6, we need only handle the case $g = 3$ as follows.

For $n \in \{5, 9\}$, the result was given in Lemma 3.3. For $n \equiv 1, 9 \pmod{12}$ and $n \geq 13$, give weight 3 to each vertex of a 2-perfect GD6CS(1^n) which exists by Theorem 1.1, then apply Construction 2.6 to get the result. For $n \equiv 5 \pmod{12}$ and $n \geq 17$, write $n = 4(3x+1)+1$ where $x \geq 1$, and take a 2-perfect GD6CS(12^{3x+1}) from Lemma 4.1. Since there exists a 2-perfect GD6CS(3^5), we can apply Construction 2.7 (2) to obtain the desired system. ∎

Lemma 4.3 *Let g and n be positive integers satisfying $g \equiv 2, 4 \pmod 6$, $n \equiv 0, 1 \pmod 3$, $n \geq 3$ and $(g, n) \neq (2, 3)$. Then there exists a 2-perfect* GD6CS(g^n).

Proof. We first deal with the case $g = 2$. The result for $n \in \{4, 6, 7, 9\}$ was provided in Lemma 3.1. For any other value of $n \notin \{10, 12, 15, 18, 19, 24, 27, 75\}$, since $n \in B(\{4, 6, 7, 9\})$ by Lemma 2.3, we can apply Construction 2.5 to obtain the desired system. The result for $n \in \{10, 19\}$ follows from applying Construction 2.7 (2) with 2-perfect GD6CSs (2^4) and (6^t) from Lemma 4.1, where $t \in \{3, 6\}$. Since there exist 2-perfect GD6CSs (2^6) and (12^4) from Lemma 4.1, we can apply Construction 2.7 (1) to get a 2-perfect GD6CS(2^{24}). For $n \in \{12, 18, 27, 75\}$, take a 2-perfect GD6CS($(2x)^3$) from Theorem 1.2, where $x = n/3$, and fill the groups of this system with 2-perfect GD6CSs (2^x) to get the result. The only remaining case is $n = 15$. For this case, take a 2-perfect GD6CS($2^3 4^1$) from Lemma 3.2 and a 2-perfect IGD6CS($(10, 4)^3$) from Lemma 3.6, then apply Construction 2.8 to get a 2-perfect GD6CS($2^9 12^1$). The conclusion comes from filling the long group with a 2-perfect GD6CS(2^6).

When $g \geq 4$, the result for $n = 3$ is contained in Theorem 1.2. For $n \geq 4$, we can inflate a 2-perfect GD6CS(2^n) to establish the result. ∎

Lemma 4.4 *Let g and n be positive integers satisfying $g \equiv 1, 5 \pmod 6$, $n \equiv 1, 9 \pmod{12}$, $g > 1$ and $n \geq 9$. Then there exists a 2-perfect* GD6CS(g^n).

Proof. For the stated value of n except for $n = 9$, we take a 2-perfect GD6CS(1^n) from Theorem 1.1, then apply Construction 2.6 to obtain the desired 2-perfect GD6CS(g^n).

Now we deal with the remaining case $n = 9$. The result for $g = 5$ was provided in Lemma 3.4. When $g \equiv 1 \pmod 6$, we first write $g = 2x + 1$ and apply Construction 2.10 with a 3-GDD of type 2^x (see Lemma 2.1) and a 2-perfect HGD6CS(9, 1^3) (see Lemma 3.7) to create a 2-perfect HGD6CS(9, 2^x). Since there exist a 2-perfect IGD6CS$((3,1)^9)$ from Lemma 3.6 and a 2-perfect GD6CS(3^9) from Lemma 4.2, we can get the desired result by applying Construction 2.9. When $g \equiv 5 \pmod 6$ and $g \geq 11$, write $g = 2x + 5$ and take a 3-GDD of type 2^x4^1 from Lemma 2.2, then apply Construction 2.10 to get a 2-perfect HGD6CS(9, 2^x4^1). Note that there exist a IGD6CS$((5,1)^9)$ by Lemma 3.6. So the conclusion follows from Construction 2.9. This completes the proof. □

Combining the above results with Theorem 1.1, we have proved

Theorem 4.5 *The necessary conditions for the existence of a 2-perfect* GD6CS(g^n)*, that is,* $n \geq 3$*,* $g(n-1) \equiv 0 \pmod 2$*, and* $g^2n(n-1) \equiv 0 \pmod{12}$*, are also sufficient, except for* $(g,\ n) \in \{(1,9),\ (2,3)\}$*.*

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