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Independence and 2-domination in trees

Mostafa Blidia   Mustapha Chellali

Department of Mathematics
University of Blida
B.P. 270, Blida, Algeria
mblidia@hotmail.com   m_chellali@yahoo.com

Odile Favaron

L.R.I., URM 8623
Bât. 490, Université de Paris-Sud
91405-Orsay cedex, France
of@lri.fr

Abstract

In a graph a vertex is said to dominate all its neighbours. A 2-dominating set of a graph $G = (V, E)$ is a dominating set S that dominates every vertex of $V - S$ at least twice. The 2-domination number of G, which is the minimum cardinality of a 2-dominating set of G, is denoted by $\gamma_2(G)$ and the independence number by $\beta(G)$. We compare the value of γ_2 and β in trees and give bounds on these two parameters in terms of the order and the number of leaves of the tree. More precisely we show that for a nontrivial tree T, $\beta(T) \leq \gamma_2(T) \leq 3\beta(T)/2$, $(n + \ell + 2)/3 \leq \gamma_2(T) \leq (n + \ell)/2$ and $\beta(T) \leq (n + \ell - 1)/2$. We characterize the trees achieving equality in each bound.

1 Introduction

For notation and graph theory terminology, we in general follow [1, 6]. In a graph $G = (V, E)$, the *neighbourhood* of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$. If S is a subset of vertices, its neighbourhood is $N(S) = \cup_{v \in S} N(v)$. The order of a graph G is denoted by $n(G)$ and abbreviated to n if there is no confusion. The *degree* of a vertex v of G, denoted by $\deg_G(v)$, is the order of its neighbourhood. A vertex of degree one is called a *pendant vertex* or a *leaf* and its neighbour is called a *support* vertex. If v is a support vertex of a tree T then L_v will denote the set of the leaves attached at v. A support vertex v is called *strong* if $|L_v| > 1$. We also denote the set of leaves of T by $L(T)$ and let $|L(T)| = \ell(T)$. The trivial tree is K_1. A tree T is a *double star* if it contains exactly two vertices that are not leaves. A double star with respectively p and q leaves attached at each support vertex is denoted by

$S_{p,q}$. A *subdivided star* SS_q is obtained from a star $K_{1,q}$ by subdividing each edge by exactly one vertex. The *corona* $G \circ K_1$ of a graph G is obtained from G by adding a leaf at each of its vertices.

A subset $S \subseteq V$ is a *dominating set* of G if for every vertex v of $V - S$, $|N(v) \cap S| \geqslant 1$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. The *domination independence number* $i(G)$ and the *independence number* $\beta(G)$ are respectively the minimum and the maximum cardinality of a set that is both dominating and independent. A graph G is *well covered* if $i(G) = \beta(G)$. The class of well covered trees, denoted here by $\mathcal{WCT}$, consists of the coronas of all trees. We are interested in dominating sets S with the additional property that S dominates each vertex in $V - S$ at least twice. Formally, a subset S of V is a 2-*dominating set* of G if for every vertex $v \in V - S$, $|N(v) \cap S| \geqslant 2$. The 2-*domination number* $\gamma_2(G)$ is the minimum cardinality of a 2-dominating set of G. Note that 2-domination in graphs is a type of multiple domination in which each vertex not in S (S is a dominating set) is dominated at least k times for a fixed positive integer k. Multiple domination was introduced by Fink and Jacobson [5]. For any parameter $\mu(G)$ associated to a graph property $\mathcal{P}$, we refer to a set of vertices with Property $\mathcal{P}$ and cardinality $\mu(G)$ as a $\mu(G)$-set. For a comprehensive treatment of domination in graphs, see [6, 7].

In this paper we study some properties of $\beta(G)$ and $\gamma_2(G)$ when G is a tree. In general the independence number and the 2-domination number are not comparable and the difference $\beta(G) - \gamma_2(G)$ as well as the ratio $\beta(G)/\gamma_2(G)$ can be arbitrarily large. This can be seen for instance on the complete bipartite graph $K_{p,q}$ with $4 \leq p < q$ for which $\beta = q$ and $\gamma_2 = 4$. We show that every nontrivial tree T satisfies $\beta(T) \leq \gamma_2(T) \leq 3\beta(T)/2$ and we give sharp upper bounds on $\gamma_2(T)$ and $\beta(T)$ and a lower bound on $\gamma_2(T)$ as a function of $n(T) + \ell(T)$. For each bound we characterize the class of extremal trees. Note that a lower bound of a similar form was previously given on $\beta(T)$ by the third author.

Theorem A (Favaron [3]) *Every tree T with n vertices and ℓ leaves satisfies $\beta(T) \geq (n + \ell)/3$ with equality if and only if the tree is well covered.*

We begin by some straightforward observations.

Observation 1

- *For a star $K_{1,p}$ with $p \geq 2$, $\beta(K_{1,p}) = \gamma_2(K_{1,p}) = p$.*
- *For a double star $S_{p,q}$ with $p \geq 2$ and $q \geq 2$, $\beta(S_{p,q}) = \gamma_2(S_{p,q}) = p + q$.*
- *For a double star $S_{1,q}$ with $q \geq 1$, $\beta(S_{1,q}) = 1 + q$ and $\gamma_2(S_{1,q}) = 2 + q$.*
- *For a subdivided star SS_q, $\beta(SS_q) = \gamma_2(SS_q) = q + 1$.*
- *For an odd path P_{2q+1}, $\beta(P_{2q+1}) = \gamma_2(P_{2q+1}) = q + 1$.*
- *For an even path P_{2q}, $\beta(P_{2q}) = q$ and $\gamma_2(P_{2q}) = q + 1$.*

Observation 2 *In a graph G, there exists a $\beta(G)$-set containing all the leaves of G. Moreover if u is a strong support vertex then every $\beta(G)$-set contains L_u. A leaf of a graph G is contained in every $\gamma_2(G)$-set.*

Observation 2 can be completed by the following lemma.

Lemma 3 *Let G be a graph different from a star. Let u be a support vertex of G having only one nonpendant neighbour and let $G' = G - (L_u \cup \{u\})$. Then $\beta(G') = \beta(G) - |L_u|$, there exists a $\gamma_2(G)$-set not containing u and $\gamma_2(G') \leq \gamma_2(G) - |L_u|$. If moreover u is a strong support vertex, then $\gamma_2(G') = \gamma_2(G) - |L_u|$.*

Proof. Let v be the unique nonpendant neighbour of u. If I is a $\beta(G)$-set, then $I - L_u$ is a $\beta(G')$-set and conversely if I' is a $\beta(G')$-set, then $I' \cup L_u$ is a $\beta(G)$-set. Hence $\beta(G') = \beta(G) - |L_u|$.

Let S be a $\gamma_2(G)$-set. By Observation 2, $L_u \subset S$. If $u \in S$ then $(S - \{u\}) \cup \{v\}$ is another $\gamma_2(G)$-set not containing u. Let D be a $\gamma_2(G)$-set not containing u. Then $D - L_u$ is a 2-dominating set of G' and thus $\gamma_2(G') \leq \gamma_2(G) - |L_u|$. If moreover $|L_u| \geq 2$, let conversely D' be a $\gamma_2(G')$-set. Then $D' \cup L_u$ is a 2-dominating set of G and thus $\gamma_2(G) \leq \gamma_2(G') + |L_u|$, implying equality. ■

2 Comparison between $\gamma_2(T)$ and $\beta(T)$

We show in this section that for every tree T, the 2-domination number is bounded below by the independence number.

Theorem 4 *If T is a tree then $\gamma_2(T) \geqslant \beta(T)$.*

Proof. We proceed by induction on the order of T. Clearly the result holds for $n = 1, 2$ establishing the basis cases. Let $n \geqslant 3$ and assume that for every tree T' of order $n' < n$ we have $\gamma_2(T') \geqslant \beta(T')$. Let T be a tree of order n. If T is a star then $\gamma_2(T) = \beta(T) = n - 1$ and hence the result is valid. So assume that T is not a star and let u be a support vertex of T for which the subgraph induced by $V(T) - L_u \cup \{u\}$ is a tree (for instance, u is the support vertex of a leaf of maximum eccentricity). Let $T' = T - L_u \cup \{u\}$. Since T is not a star, T' has order at least two and u has a unique neighbour in T'. By Lemma 3, $\gamma_2(T') \leqslant \gamma_2(T) - |L_u|$ and $\beta(T') = \beta(T) - |L_u|$. By the induction hypothesis applied to T', we have $\gamma_2(T') \geqslant \beta(T')$ implying that $\gamma_2(T) \geqslant \beta(T)$. ■

In order to characterize the trees with equal 2-domination and independence numbers we introduce the familly $\mathcal{F}$ of all trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_k$ ($k \geq 1$) of trees, where T_1 is a star $K_{1,t}$ ($t \geqslant 2$) of center vertex w, $T = T_k$, and, if $k \geq 2$, T_{i+1} is obtained recursively from T_i by one of the three operations defined below. Put $A(T_1) = L_w$.

- **Operation $\mathcal{O}_1$** : Add a star $K_{1,p}$, $p \geqslant 1$, centered at a vertex x and join x by an edge to a leaf y of T_i. Put $A(T_{i+1}) = A(T_i) \cup L_x$.
- **Operation $\mathcal{O}_2$** : Add a star $K_{1,p}$, $p \geqslant 1$, centered at a vertex x and join x by an edge to a nonpendant vertex y of $A(T_i)$. Put $A(T_{i+1}) = A(T_i) \cup L_x$.

- **Operation** $\mathcal{O}_3$: Add a star $K_{1,p}$, $p \geqslant 2$, centered at a vertex x and join x by an edge to a vertex y of $V(T_i) - A(T_i)$. Put $A(T_{i+1}) = A(T_i) \cup L_x$.

We also define $\mathcal{F}_1$ as the subfamily of $\mathcal{F}$ consisting of trees constructed from T_1 by recursively applying Operation $\mathcal{O}_1$. For instance an odd path P_{2q+1} belongs to $\mathcal{F}_1$ since it is obtained from $K_{1,2}$ by applying $q-1$ times $\mathcal{O}_1$ with each p equal to 1.

We give in the following lemma some properties of the trees in $\mathcal{F}$. The first ones will be of use in this section, the other ones in Sections 3 and 4.

Lemma 5 *1. If T is a tree of $\mathcal{F}$ then $A(T)$ is both the unique $\gamma_2(T)$-set and the unique $\beta(T)$-set.*

2. If T is obtained from T_1 by applying $r \geq 0$ times Operations $\mathcal{O}_2$ or $\mathcal{O}_3$ and any number of times Operation $\mathcal{O}_1$, then $|A(T)| = (n(T) + \ell(T) - r - 1)/2$. If moreover $r \geq 1$ (that is if $T \in \mathcal{F} - \mathcal{F}_1$), then $n(T) + \ell(T) \geq 3r + 7$ with equality if and only if T is a subdivided star SS_q with $q \geq 3$ or the double star $S_{2,2}$.

Proof. 1. Let T be a tree of $\mathcal{F}$. From the way in which T is constructed, $A(T)$ is both a 2-dominating set and an independent set. Thus $\beta(T) \geqslant |A(T)| \geqslant \gamma_2(T)$. The equality follows from Theorem 4.

To show that $A(T)$ is both the unique $\gamma_2(T)$-set and the unique $\beta(T)$-set, we proceed by induction on k where $k-1$ is the number of operations performed to construct T from T_1. If $k = 1$, then $T = K_{1,t}$ with $t \geq 2$ and so $A(T)$ is the unique $\gamma_2(T)$-set and the unique $\beta(T)$-set. This establishes the basis case. Assume now that $k \geqslant 2$ and the result holds for all trees of $\mathcal{F}$ that can be constructed from a sequence of at most $k-2$ operations. Let T be a tree of $\mathcal{F}$ constructed by $k-1$ operations, $T' = T_{k-1}$, x the center of the star $K_{1,p}$ added to T' to get T, and y the neighbour of x in T'. Then $\beta(T) = |A(T)| = \beta(T') + |L_x|$ and $\gamma_2(T) = |A(T)| = \gamma_2(T') + |L_x|$. By induction hypothesis applied to T', we know that $A(T')$ is the unique $\gamma_2(T')$-set and the unique $\beta(T')$-set.

Suppose that $A(T) = A(T') \cup L_x$ is not the unique $\gamma_2(T)$-set and let D be a second $\gamma_2(T)$-set. Then D must contain x for otherwise $A(T')$ and $D - L_x$ are two different $\gamma_2(T')$-sets, a contradiction. Also $y \notin D$ and y is dominated by x and by exactly one vertex of $V(T') \cap D$ (else $D - (L_x \cup \{x\})$ would be a 2-dominating set of T of order less than $\gamma_2(T')$). Then $(D \cap V(T')) \cup \{y\}$ is a 2-dominating set of T' of order $|D| - (|L_x| + 1) + 1 = |A(T)| - |L_x| = |A(T')|$. By the unicity of the $\gamma_2(T')$-set, $(D \cap V(T')) \cup \{y\}$ is the independent set $A(T')$. This contradicts the fact that y has a neighbour in $V(T') \cap D$. Consequently, $A(T)$ is a unique $\gamma_2(T)$-set.

Let I be a $\beta(T)$-set. If L_x is not contained in I then by Observation 2, $|L_x| = 1$, T is constructed from T' by Operation $\mathcal{O}_1$ or $\mathcal{O}_2$, and $y \in A(T')$. If u is the leaf of T adjacent to x, $u \notin I$, $x \in I$ and $y \notin I$. The set $I_1 = (I - \{x\}) \cup \{u\}$ is a $\beta(T)$-set such that $I_1 \cap V(T')$ is a $\beta(T')$-set and $I_1 \cap V(T') = I \cap V(T')$. By the unicity of the $\beta(T')$-set, $I \cap V(T') = A(T')$ contradicting $y \in A(T')$ but $y \notin I$. Therefore $L_x \subset I$ and $I - L_x = I \cap V(T')$ is a $\beta(T')$-set. By the unicity of the $\beta(T')$-set, $I - L_x = A(T')$ and thus $I = A(T)$.

2. If a tree T is obtained from a tree T' of $\mathcal{F}$ by an operation $\mathcal{O}_1$, then $n(T) = n(T') + p + 1$, $\ell(T) = \ell(T') + p - 1$, $\gamma_2(T) = \gamma_2(T') + p$ and thus $\gamma_2(T) - (n(T) + \ell(T))/2 = \gamma_2(T') - (n(T') + \ell(T'))/2$. If T is obtained from T' of $\mathcal{F}$ by an operation $\mathcal{O}_2$ or $\mathcal{O}_3$, then $n(T) = n(T') + p + 1$, $\ell(T) = \ell(T') + p$, $\gamma_2(T) = \gamma_2(T') + p$ and thus $\gamma_2(T) - (n(T) + \ell(T))/2 = \gamma_2(T') - (n(T') + \ell(T') + 1)/2$. Hence the value of $\gamma_2 - (n + \ell)/2$ is unchanged in each application of $\mathcal{O}_1$ and decreases by 1/2 in each of the $r \geq 0$ applications of $\mathcal{O}_2$ or $\mathcal{O}_3$. Since for $T_1 = K_{1,t}$ with $t \geq 2$, $\gamma_2(T_1) = (n(T_1) + \ell(T_1) - 1)/2$, we get $|A(T)| = (n(T) + \ell(T) - r - 1)/2$. We note in particular that if T is in $\mathcal{F}_1$ then $r = 0$ and $\beta(T) = \gamma_2(T) = (n(T) + \ell(T) - 1)/2$.

Each operation $\mathcal{O}_3$ increases n by $p + 1 \geq 3$ and ℓ by $p \geq 2$. Hence if the tree T of $\mathcal{F}$ is constructed from $T_1 = K_{1,t}$ by $r \geq 1$ operations $\mathcal{O}_3$ and any number of operations $\mathcal{O}_1$, $n(T) \geq n(T_1) + 3r = 3r + t + 1$, $\ell(T) \geq \ell(T_1) + 2r = 2r + t$ and $n(T)+\ell(T) \geq 5r+2t+1 \geq 3r+7$ since $t \geq 2$ and $r \geq 1$. Moreover $n(T)+\ell(T) = 3r+7$ if and only if $t = 2$ and in the construction of T, no operation $\mathcal{O}_1$ and exactly one operation $\mathcal{O}_3$ with $p = 2$ has been used, that is if T is the double star $S_{2,2}$.

If the construction of T uses at least one operation $\mathcal{O}_2$ joining the center x of a star $K_{1,p}$ to a vertex y of T', then y is a nonpendant vertex of $A(T')$. This implies that at least one operation $\mathcal{O}_1$ has been performed before $\mathcal{O}_2$. Hence $n(T)$ is at least equal to $n(T_1)$ plus two vertices added in $\mathcal{O}_1$ plus $2r$ vertices added in the r operations $\mathcal{O}_2$ and $\mathcal{O}_3$ (exactly $2r$ if and only if these r operations are of type $\mathcal{O}_2$ with $p = 1$). Similarly, $\ell(T)$ is at least equal to $\ell(T_1)$ plus r leaves added in the r operations $\mathcal{O}_2$ and $\mathcal{O}_3$ (exactly r if and only if these r operations are of type $\mathcal{O}_2$ with $p = 1$). In this case, $n(T) \geq t + 3 + 2r$, $\ell(T) \geq t + r$ and $n(T) + \ell(T) \geq 3r + 2t + 3 \geq 3r + 7$. Moreover, $n(T) + \ell(T) = 3r + 7$ if and only if T is constructed from $K_{1,2}$ by using one operation $\mathcal{O}_1$ and $r \geq 1$ operations $\mathcal{O}_2$, each of them with $p = 1$. This tree T is a subdivided star SS_q with $q = r + 2 \geq 3$. ■

We are now ready to characterize the trees achieving equality in Theorem 4.

Theorem 6 *Let T be a tree. Then the following statements are equivalent:*

a) $\gamma_2(T) = \beta(T)$,

b) $T = K_1$ *or* $T \in \mathcal{F}$,

c) T *has a unique* $\gamma_2(T)$*-set that also is a unique* $\beta(T)$*-set.*

Proof. Let T be a tree of order n.

(**b**) ⇒ (**c**). The result is obvious if $T = K_1$ and it follows from Lemma 5 if $T \in \mathcal{F}$.

(**c**) ⇒ (**a**). Obvious.

(**a**) ⇒ (**b**). We proceed by induction on the order of T. If $n = 1$ then $T = K_1$. The only tree T of order $n = 2$ does not satisfy $\gamma_2(T) = \beta(G)$. If $n = 3$ then $T = K_{1,2}$ is a tree T_1 of $\mathcal{F}$. Let $n \geqslant 4$ and assume that any tree T' of order $n' < n$ with $\gamma_2(T') = \beta(T')$ is in $\mathcal{F}$.

Let T be a tree of order n such that $\gamma_2(T) = \beta(T)$. If T is a star then $T \in \mathcal{F}$, so assume that T is not a star. Let x be a support vertex of T such that $T - (L_x \cup \{x\})$ is a tree denoted by T' and let y be the unique neighbour of x in T'. If T' has order two, then $\beta(T) = |L_x|+1$ and $\gamma_2(T) = |L_x|+2$, a contradiction. Hence T' has order at least three. Then by Lemma 3, $\beta(T') = \beta(T) - |L_u|$ and $\gamma_2(T') \leq \gamma_2(T) - |L_u|$. Therefore $\gamma_2(T') \leq \beta(T')$ and by Theorem 4, $\gamma_2(T') = \beta(T')$. By induction hypothesis, T' is in $\mathcal{F}$. The set $A(T') \cup L_x$ is then both a $\beta(T)$-set and a $\gamma_2(T)$-set. The addition of the star $\{x\} \cup L_x = K_{1,p}$ to T' in an operation $\mathcal{O}_1$ if y is a leaf of T', $\mathcal{O}_2$ if y is a nonpendant vertex of $A(T')$, $\mathcal{O}_3$ if $y \notin A(T')$ (in this case $|L_x| \geqslant 2$ else $A(T') \cup L_x$ is not a 2-dominating set of T). Therefore $T \in \mathcal{F}$. ■

3 Upper bounds on γ_2 and β

Theorem 7 *If T is a nontrivial tree of order n then $\gamma_2(T) \leq (n + \ell(T))/2$.*

Proof. Let T be a nontrivial tree. If T is a star then the result holds so we assume that T is not a star. The tree T' obtained from T by removing all its leaves is not trivial and admits a bipartition A, B. Every vertex of degree one in T' is a support vertex in T that is adjacent to at least one vertex of $L(T)$. Every vertex of degree at least two of A (resp. B) is dominated twice by B (resp. by A). Thus $L(T) \cup A$ and $L(T) \cup B$ are two 2-dominating sets of T. So $\gamma_2(T) \leqslant \min\{|L(T) \cup A|, |L(T) \cup B|\} \leq \ell(T) + (n - \ell(T))/2 = (n + \ell(T))/2$. ■

In order to characterize the nontrivial trees attaining the upper bound in Theorem 7, we introduce the collection $\mathcal{G}$ of all trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_k$ ($k \geq 1$) of trees, where T_1 is the path P_2, $T = T_k$, and if $k \geq 2$, T_{i+1} is obtained recursively from T_i by one of the two operations listed below.

- Operation $\mathcal{T}_1$: Add a star $K_{1,p}$, $p \geqslant 2$, centered at a vertex x and join x by an edge to a leaf y of T_i.
- Operation $\mathcal{T}_2$: Add a path $P_2 = xz$, join x by an edge to a leaf y of T_i, and add $k \geqslant 0$ new vertices adjacent to y.

For instance an even path P_{2q} is constructed from P_2 by performing $q-1$ times $\mathcal{T}_2$ with $k = 0$. Hence $P_{2q} \in \mathcal{G}$.

Theorem 8 *Let T be a nontrivial tree. Then $\gamma_2(T) = (n + \ell(T))/2$ if and only if $T \in \mathcal{G}$.*

Proof. We first prove the part "if" by induction on the number $k-1$ of operations performed to construct T from $T_1 = P_2$. If $k = 1$, then $T = P_2$, and so $\gamma_2(T) = (n + \ell(T))/2 = 2$. This establishes the basis case.

Assume now that $k \geqslant 2$ and that the result holds for all trees in $\mathcal{G}$ that can be constructed from less than $k-1$ operations. Let $T = T_k \in \mathcal{G}$, and let $T' = T_{k-1}$. By induction hypothesis applied to T' we have $\gamma_2(T') = (n(T') + \ell(T'))/2$. We consider two cases depending on whether T is obtained from T' by using operation $\mathcal{T}_1$ or $\mathcal{T}_2$.

Case 1. T is obtained from T' by using operation $\mathcal{T}_1$.

Let $|L_x| = p \geq 2$. Then $n(T) = n(T') + p + 1$ and $\ell(T) = \ell(T') + p - 1$. By Lemma 3,

$$\gamma_2(T) = \gamma_2(T') + |L_x| = (n(T') + \ell(T'))/2 \; + \; p = (n(T) + \ell(T))/2 \; .$$

Case 2. T is obtained from T' by using operation $\mathcal{T}_2$.

If Y denotes the set (possibly empty) of vertices attached at y by this operation, $n(T) = n(T') + |Y| + 2$ and $\ell(T) = \ell(T') + |Y|$. By Observation 2, let S be a 2-dominating set of T not containing x (but necessarily containing $Y \cup \{z\}$). To 2-dominate x, S also contains y which implies that $S - (Y \cup \{z\})$ is a 2-dominating set of T'. Hence $\gamma_2(T') \leq \gamma_2(T) - |Y| - 1$. On the other hand if S' is a $\gamma_2(T')$-set then it contains the leaf y of T' and thus $S' \cup (Y \cup \{z\})$ is a 2-dominating set of T. Hence $\gamma_2(T) \leq \gamma_2(T') + |Y| + 1$. Therefore

$$\gamma_2(T) = \gamma_2(T') + |Y| + 1 = (n(T') + \ell(T'))/2 + |Y| + 1 = (n(T) + \ell(T))/2 \; .$$

We prove the part "only if" by induction on the order of T. If $n = 2$ then $T = P_2$ which belongs to $\mathcal{G}$. The only tree of order three does not satisfy $\gamma_2(T) = (n(T) + \ell(T))/2$. One of the two trees of order four, namely P_4, satisfies $\gamma_2(T) = (n(T) + \ell(T))/2$ and it belongs to $\mathcal{G}$. For $n \geqslant 5$, suppose that every tree of order less than n and satisfying $\gamma_2(T) = (n(T) + \ell(T))/2$ is in $\mathcal{G}$ and let T be a tree of order n satisfying $\gamma_2(T) = (n(T) + \ell(T))/2$. By Observation 1 on stars and double stars, the only trees of diameter 2 or 3 satisfying $\gamma_2(T) = (n(T) + \ell(T))/2$ are the double stars $S_{1,q}$ with $q \geqslant 1$. These trees are obtained from P_2 by using one operation $\mathcal{T}_2$ with $k = q - 1$ and thus belong to $\mathcal{G}$. So we suppose $diam(T) \geqslant 4$ and consider a $\gamma_2(T)$-set S of T. We root T at a vertex r of maximum eccentricity. Let v be a support vertex at maximum distance from r, u the parent of v and $T' = T - (L_v \cup \{v\})$. Since $diam(T) \geqslant 4$, T' is not trivial and has order $n(T') = n(T) - |L_v| - 1$. We consider two cases.

Case 1. v is a strong support vertex and thus $\gamma_2(T) = \gamma_2(T') + |L_v|$ by Lemma 3. If $\deg_T(u) \geqslant 3$, then $l(T') = l(T) - |L_v|$. By Theorem 7 we have

$$\begin{aligned} \gamma_2(T) &= |L_v| + \gamma_2(T') \leqslant |L_v| + \left(n(T') + l(T')\right)/2 = \left(n(T) + l(T)\right)/2 - 1/2 \\ &< (n(T) + \ell(T))/2, \end{aligned}$$

a contradiction. So u is a leaf of T' and $l(T') = l(T) - |L_v| + 1$. Hence

$$\gamma_2(T') = \gamma_2(T) - |L_v| = (n(T) + \ell(T))/2 - |L_v| = (n(T') + \ell(T'))/2.$$

By induction hypothesis applied to T', we have $T' \in \mathcal{T}$. Since T is obtained from T' by performing $\mathcal{T}_1$, $T \in \mathcal{T}$.

Case 2. From now on we may assume that no child of u is a strong support vertex. Let v' be the unique leaf adjacent to v. We claim that u has no child besides v as a support vertex. Suppose to the contrary that a child w of u is a support vertex with $L_w = \{w'\}$ and let T' be the nontrivial tree $T - (L_v \cup \{v\})$. Let S' be a $\gamma_2(T')$-set not containing w. Then S' contains w' and u. Hence $\{v'\} \cup S'$ is a 2-dominating set of T and so by Theorem 7, $\gamma_2(T) \leqslant 1 + |S'| \leqslant 1 + (n(T') + l(T'))/2$. Since $n(T') = n(T) - 2$ and $l(T') = l(T) - 1$, we get $\gamma_2(T) < (n(T) + l(T))/2$, a contradiction.Thus every child (if any) of u besides v is a leaf.

Now let $T'' = T - (L_u \cup L_v \cup \{v\})$. Then T'' is not trivial. If S is a $\gamma_2(T)$-set not containing v, then S contains u to 2-dominate v, $S - (\{v'\} \cup L_u)$ is a 2-dominating set of T'' and thus $\gamma_2(T'') \leq \gamma_2(T) - 1 - |L_u|$. On the other hand every $\gamma_2(T'')$-set S'' contains the leaf u of T'' and thus $S'' \cup (L_u \cup \{v'\})$ is a 2-dominating set of T, implying $\gamma_2(T) \leq \gamma_2(T'') + 1 + |L_u|$. Since $n(T) = n(T'') + 2 + |L_u|$ and $\ell(T) = \ell(T'') + |L_u|$ we get

$$\gamma_2(T'') = \gamma_2(T) - 1 - |L_u| = (n(T) + \ell(T))/2 - 1 - |L_u| = (n(T'') + \ell(T''))/2 \ .$$

By induction hypothesis applied to T'', we have $T'' \in \mathcal{G}$. Since T is obtained from T'' by using $\mathcal{T}_2$, $T \in \mathcal{G}$. This completes the proof of the theorem. ■

The following observation will be used in Section 5.

Observation 9 *From the part "only if" of the previous proof we deduce that if a tree T satisfying $\gamma_2(T) = (n(T) + \ell(T))/2$ and having no strong support vertex is rooted at a vertex r of maximum eccentricity, then a vertex at distance* $\mathrm{diam}(T) - 2$ *from r has exactly one child as a support vertex.*

Theorems 4 and 7 show that every nontrivial tree satisfies $\beta(T) \leq (n(T)+\ell(T))/2$. The following theorem slightly improves this bound.

Theorem 10 *If T is a nontrivial tree then $\beta(T) \leq (n(T)+\ell(T)-1)/2$ with equality if and only if $T \in \mathcal{F}_1$.*

Proof. In the inequalities chain $\beta(T) \leq \gamma_2(T) \leq (n(T)+\ell(T))/2$, $\beta(T)$ and $\gamma_2(T)$ are integers. Hence each equality $\beta(T) = (n(T) + \ell(T))/2$ or $\beta(T) = (n(T) + \ell(T) - 1)/2$ implies $\beta(T) = \gamma_2(T)$ and thus, by Theorem 6 and Lemma 5, $\beta(T) = \gamma_2(T) = (n(T)+\ell(T)-r-1)/2$ for some $r \geq 0$. Therefore $\beta(T) = (n(T)+\ell(T))/2$ is impossible and $\beta(T) = (n(T)+\ell(T)-1)/2$ implies $T \in \mathcal{F}$ with $r = 0$, that is $T \in \mathcal{F}_1$. Conversely we already observed that each tree T in $\mathcal{F}_1$ satisfies $\beta(T) = (n(T) + \ell(T) - 1)/2$. ■

4 Lower bound on γ_2

From Theorem A and Theorem 4 we deduce that every nontrivial tree satisfies $\gamma_2(T) \geq (n(T) + \ell(T))/3$. In this section we slightly improve this bound.

Theorem 11 *If T is a nontrivial tree then $\gamma_2(T) \geq (n(T)+\ell(T)+2)/3$ with equality if and only if T is a subdivided star SS_q with $q \geq 3$ or the double star $S_{2,2}$.*

Proof. In the inequalities chain $\gamma_2(T) \geq \beta(T) \geq (n(T)+\ell(T))/3$, $\beta(T)$ and $\gamma_2(T)$ are integers. Hence each of the equalities $\gamma_2(T) = (n(T)+\ell(T))/3$, $\gamma_2(T) = (n(T)+\ell(T)+1)/3$, $\gamma_2(T) = (n(T)+\ell(T)+2)/3$ implies $\gamma_2(T) = \beta(T)$ and thus, by Theorem 6 and Lemma 5, $\gamma_2(T) = \beta(T) = (n(T)+\ell(T)-r-1)/2$ and $n(T)+\ell(T) \geq 3r+7$ for some $r \geq 0$. But if $\gamma_2(T) \leq (n(T)+\ell(T)+1)/3$ then $(n(T)+\ell(T)-r-1)/2 \leq (n(T)+\ell(T)+1)/3$, thus implying $n(T)+l(T) \leq 3r+5$ which is impossible. Therefore $\gamma_2(T) \geq (n(T)+\ell(T)+2)/3$ with equality if and only if $n(T)+l(T) = 3r+7$. The result follows from Lemma 5. ■

5 Upper bounds on $\gamma_2(T)/\beta(T)$ and $\gamma_2(T) - \beta(T)$

The upper bound on $\gamma_2(T)$ and the lower bound on $\beta(T)$ show that $\gamma_2(T)$ cannot be arbitrarily large with respect to $\beta(T)$.

We introduce the class $\mathcal{H}$ of all trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_k$ of trees where T_1 is a path $P_4 = x_1^1 y_1^1 y_1^2 x_1^2$, $T = T_k$ and if $k \geq 2$, T_{i+1} is obtained recursively from T_i by the operation defined below.

- **Operation** Ω : Add four vertices forming a path $P_4 = x_{i+1}^1 y_{i+1}^1 y_{i+1}^2 x_{i+1}^2$ and join by an edge an internal vertex y_{i+1}^1 or y_{i+1}^2 of this path to a nonpendant vertex of T_i.

Theorem 12 *If T is a nontrivial tree then $\gamma_2(T)/\beta(T) \leq 3/2$ with equality if and only if T belongs to $\mathcal{H}$.*

Proof. By Theorem A and Theorem 7, $\gamma_2(T)/\beta(T) \leq 3/2$ with equality if and only if $\beta(T) = (n(T)+\ell(T))/3$ and $\gamma_2(T) = (n(T)+\ell(T))/2$.

Let $T = T_k$ be a tree of $\mathcal{H}$.Then every $\gamma_2(T)$-set must contain at least three vertices of each of the k paths P_4. Hence $\gamma_2(T) \geqslant 3k = (4k+2k)/2 = (n(T)+\ell(T))/2$. On the other hand, T is the corona of a tree and thus by Theorem A, $\beta(T) = (n(T)+\ell(T))/3$. Therefore $\gamma_2(T)/\beta(T) = 3/2$.

We prove the part "only if" by induction on n. For $2 \leq n \leq 4$, the only tree T satisfying $\gamma_2(T) = (n(T)+\ell(T))/2$ and $\beta(T) = (n(T)+\ell(T))/3$, namely P_4, is in $\mathcal{H}$.

For $n \geqslant 5$ we suppose this property true for every tree of order less than n and we consider a tree T of order n such that $\beta(T) = (n(T)+\ell(T))/3$ and $\gamma_2(T) = (n(T)+\ell(T))/2$. By Theorem A, T is the corona of a tree. In particular T has no strong support vertex. We root T at a vertex r of maximum eccentricity. Let v be a support vertex at distance $diam(T)-1$ from r, $L_v = \{v'\}$, u the parent of v and $L_u = \{u'\}$. From Observation 9, v is the unique child of u that is a support vertex. The tree $T' = T - \{u, u', v, v'\}$ is not trivial, otherwise u would be a strong support vertex, and $n(T') = n(T) - 4$. Since the parent x of u is a support vertex, the deletion of $\{u, u', v, v'\}$ does not create any new leaf and $\ell(T') = \ell(T) - 2$. Clearly $\beta(T') = \beta(T) - 2$ and thus

$$\beta(T') = (n(T)+\ell(T))/3 - 2 = (n(T')+\ell(T')+6)/3 - 2 = (n(T')+\ell(T'))/3.$$

On the other hand, Let S' be a $\gamma_2(T')$-set. Then $S' \cup \{u, u', v\}$ is a 2-dominating set of T. Hence

$$\gamma_2(T) \leqslant \gamma_2(T') + 3 \leqslant (n(T') + \ell(T'))/2 + 3 \leqslant (n(T) + \ell(T))/2.$$

Since $\gamma_2(T) = (n(T) + \ell(T))/2$, we get $\gamma_2(T') = (n(T') + \ell(T'))/2$. Therefore $\gamma_2(T')/\beta(T') \leq 3/2$. By induction hypothesis, $T' \in \mathcal{H}$. The tree T is obtained from T' by using Ω (the internal vertex u of the path $P_4 = u'uvv'$ is joined by an edge to the nonpendant vertex x of T'). Therefore $T \in \mathcal{H}$ which completes the proof. Note that this means that $\mathcal{H} = \mathcal{WCT} \cap \mathcal{G}$. ■

In a similar way, Theorem A and Theorem 7 show that $\gamma_2(T) - \beta(T) \leq (n(T) + \ell(T))/6$ and that the upper bound is attained if and only if $\beta(T) = (n(T) + \ell(T))/3$ and $\gamma_2(T) = (n(T) + \ell(T))/2$. Therefore we get the following

Corollary 1 *If T is a nontrivial tree then $\gamma_2(T) - \beta(T) \leq (n(T) + \ell(T))/6$ with equality if and only if T belongs to $\mathcal{H}$.*

Concluding remarks For the other domination parameters, ir (the irredundance number), γ and i, which are related in every graph G by the inequalities $ir(G) \leq \gamma(G) \leq i(G)$, the upper bound $i(T) \leq (n(T) + \ell(T))/3$ is known to hold in every tree [3] and is sharp even for ir. But since $ir(K_{1,p}) = \gamma(K_{1,p}) = i(K_{1,p}) = 1$ for arbitrarily large stars, $ir(T)$, $\gamma(T)$ and $i(T)$ cannot admit any lower bound of the form $(n(T) + \ell(T))/c$ where c is a constant. However there exist other lower bounds on these parameters in terms of n and ℓ. Lemańska [8] proved that in every tree, $\gamma(T) \geq (n(T) - \ell(T) + 2)/3$ and characterized the class of extremal trees. Actually, since the connected domination number $\gamma_c(T)$ of a tree T is equal to $n(T) - \ell(T)$, the bound on $\gamma(T)$ given in [8] is the adaptation to trees of a result by Duchet and Meyniel [2] who proved that in every connected graph G, $\gamma(G) \geq (\gamma_c(G) + 2)/3$. This last result was improved to $ir(G) \geq (\gamma_c(G) + 2)/3$ in [4].

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