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Uniform coverings of 2-paths with 6-cycles in the complete graph

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Abstract

Let $n \geq 6$. There exists a uniform covering of 2-paths with 6-cycles in K_n if and only if $n \equiv 0, 1, 2 \pmod 4$.

1 Introduction

Let K_n be the complete graph on n vertices. A *path of length* l, or an l*-path*, is the graph induced by the edges $\{v_i, v_{i+1}\}$ $(0 \leq i \leq l-1)$, where the vertices v_i $(0 \leq i \leq l)$ are all different. It is denoted by $[v_0, v_1, \ldots, v_l]$.

A uniform covering of the 2-paths in K_n with l-paths [l-cycles] is a set S of l-paths [l-cycles] having the property that each 2-path in K_n lies in exactly one l-path [l-cycle] in S. For a given integer $l \geq 3$, only the following cases of the problem of constructing a uniform covering of the 2-paths in K_n with l-paths or l-cycles have been solved:

1. with 3-cycles,
2. with 3-paths [1],
3. with 4-cycles [2],
4. with 4-paths [3],
5. with 5-paths [4, 5],
6. with 6-paths [6].

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In this paper, we solve the problem in the case of 6-cycles, that is, we prove:

Theorem 1.1 *Let $n \geq 6$. There exists a uniform covering of 2-paths with 6-cycles in K_n if and only if* $n \equiv 0, 1, 2 \pmod 4$.

There are no known cases where the necessary conditions on n are not sufficient for the existence of a uniform covering of 2-paths in K_n.

2 The case n is small

In this section, we construct a uniform covering of 2-paths with 6-cycles in K_n when n is small.

Proposition 2.1 *There exists uniform coverings of 2-paths with 6-cycles in K_n when* $n = 6, 8, 9, 10, 12, 13$.

Proof. Let $V_n = \{\infty, 0, 1, 2, \ldots, n-2\}$ be the vertex set of K_n. We define the vertex permutations α_n, β_n, γ_n in K_n: $\alpha_n = (\infty)(0\ 1\ 2\ \cdots\ n{-}2)$, $\beta_n = (\infty\ 0\ 1\ 2\ \cdots\ n{-}2)$, $\gamma_n = (\infty)(0)(1\ 2\ \cdots\ n{-}2)$.

(1) $n = 6$

Put $C_1 = (\infty, 0, 2, 3, 4, 1)$ and $C_2 = (\infty, 0, 4, 1, 3, 2)$. Then $\{\alpha_n^j C_i | 1 \leq i \leq 2, 0 \leq j \leq 4\}$ is a uniform covering of 2-paths with 6-cycles in K_6.

(2) $n = 8$

Put $C_1 = (\infty, 0, 6, 1, 5, 2)$, $C_2 = (\infty, 0, 5, 2, 4, 3)$, $C_3 = (\infty, 0, 4, 3, 1, 6)$ and $C_4 = (1, 6, 5, 2, 3, 4)$. Then $\{\alpha_n^j C_i | 1 \leq i \leq 4, 0 \leq j \leq 6\}$ is a uniform covering of 2-paths with 6-cycles in K_8.

(3) $n = 9$

Put $C_1 = (2, 0, 3, 6, \infty, 7), C_2 = (2, 0, 7, \infty, 3, 6), C_3 = (2, 0, 5, 4, 6, 3), C_4 = (1, 0, \infty, 6, 5, 3), C_5 = (2, 7, 6, 3, 4, 5)$ and $C_6 = (1, 2, 7, 4, 6, \infty)$. Then $\{\gamma_n^j C_i | 1 \leq i \leq 6, 0 \leq j \leq 6\}$ is a uniform covering of 2-paths with 6-cycles in K_9.

(4) $n = 10$

Put $C_1 = (0, 7, 2, 4, 3, 5), C_2 = (\infty, 1, 4, 3, 6, 8), C_3 = (0, 1, 5, 2, 6, 7)$, $C_4 = (\infty, 4, 3, 8, 2, 5)$, $C_5 = (\infty, 0, 6, 4, 2, 8)$ and $C_6 = (\infty, 2, 0, 4, 8, 6)$. Then $\{\beta_n^j C_i | 1 \leq i \leq 6, 0 \leq j \leq 9\}$ is a uniform covering of 2-paths with 6-cycles in K_{10}.

(5) $n = 12$

Put $C_1 = (0, 4, 3, 10, 9, 2)$, $C_2 = (5, 6, 1, 7, 8, \infty)$, $C_3 = (0, 8, 6, 9, 7, 4)$, $C_4 = (10, 1, 2, 3, 5, \infty)$, $C_5 = (0, 1, 9, 8, 5, 6)$, $C_6 = (4, 7, 3, 10, 2, \infty)$, $C_7 = (0, 5, 1, 7, 3, 8)$, $C_8 = (9, 2, 4, 6, 10, \infty)$, $C_9 = (0, 9, 4, 6, 1, 10)$ and $C_{10} = (3, 8, 5, 2, 7, \infty)$. Then $\{\alpha_n^j C_i | 1 \leq i \leq 10, 0 \leq j \leq 10\}$ is a uniform covering of 2-paths with 6-cycles in K_{12}.

(6) $n = 13$

Put $C_1 = (2, 11, 6, 7, 4, 9)$, $C_2 = (2, 11, 10, 3, 7, 6)$, $C_3 = (1, 9, 11, 2, 5, 8)$, $C_4 = (2, 11, 7, 6, 5, 8)$, $C_5 = (1, 11, 2, 10, 3, 6)$, $C_6 = (2, \infty, 10, 3, 8, 7)$, $C_7 = (1, \infty, 7, 9, 10, 8)$, $C_8 = (2, 4, 9, 0, \infty, 10)$, $C_9 = (1, 2, 5, 0, 4, 9)$, $C_{10} = (3, 10, 0, 8, 4, 9)$, $C_{11} = (1, 11, 0, 3, 5, \infty)$, $C_{12} = (10, \infty, 9, 4, 6, 0)$ and $C_{13} = (1, \infty, 3, 4, 0, 9)$. Then $\{\gamma_n^j C_i |\ 1 \leq i \leq 13, 0 \leq j \leq 10\}$ is a uniform covering of 2-paths with 6-cycles in K_{13}. □

3 Main proposition

Proposition 3.1 *Let $m \geq 6$ and $n = m + 8$. If there exists a uniform covering of 2-paths with 6-cycles in K_m, then there exists a uniform covering of 2-paths with 6-cycles in K_n.*

Proof. Let V_m, V_8 and V_n be the vertex sets of K_m, K_8 and K_n, respectively. Put $V_8 = \{a, b, c, d, e, f, g, h\}$ and $V_n = V_m \cup V_8$.

There exist uniform coverings of 2-paths with 6-cycles in K_m and K_8. Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be the coverings in K_m and K_8, respectively. Then the 2-paths in K_m and the 2-paths in K_8 are covered with $\mathcal{U}_1 \cup \mathcal{U}_2$. The set of 2-paths in K_n which are not covered with $\mathcal{U}_1 \cup \mathcal{U}_2$ is $\Pi_1 \cup \Pi_2 \cup \Pi_3 \cup \Pi_4$, where,

$$\begin{aligned}
\Pi_1 &= \{[u, x, v] \mid u, v \in V_8, u \neq v, x \in V_m\},\\
\Pi_2 &= \{[u, v, x] \mid u, v \in V_8, u \neq v, x \in V_m\},\\
\Pi_3 &= \{[x, u, y] \mid u \in V_8, x, y \in V_m, x \neq y\},\\
\Pi_4 &= \{[u, x, y] \mid u \in V_8, x, y \in V_m, x \neq y\}.
\end{aligned}$$

If $\mathcal{C}$ is a set of cycles in K_n and Γ is a set of vertex permutations in K_n, we define $\Gamma\mathcal{C} = \{\gamma C | \gamma \in \Gamma, C \in \mathcal{C}\}$. A path Q is *contained in* a cycle C if Q is a subgraph of C. More generally, a path Q is *contained in* a set of cycles $\mathcal{C}$ if Q is contained in one of the cycles of $\mathcal{C}$. Define $\pi(\mathcal{C}) = \{[x, y, z] \mid [x, y, z] \text{ is contained in } \mathcal{C}\}$.

(I) Construction of a set of 6-cycles $\mathcal{C}$ in K_n such that $\pi(\mathcal{C}) = \Pi_1 \cup \Pi_2$.

Let $V_m = \{0, 1, 2, \ldots, m-1\}$, where addition in V_m is modulo m. We denote by ρ the vertex permutation $(a)(b\ c\ d\ e\ f\ g\ h)$ of K_8. We can extend ρ to a vertex permutation of K_n by defining $\rho(x) = x$ for $x \in V_m$. Put $P = \{\rho^j | 0 \leq j \leq 6\}$. For $0 \leq i \leq m-1$, define

$$\begin{aligned}
R_1(i) &= (b, i, d, h, i+1, g),\\
R_2(i) &= (a, b, i, f, e, i+1),
\end{aligned}$$

so that $R_1(i)$ and $R_2(i)$ are 6-cycles in K_n. Put $\mathcal{R} = P\{R_1(i), R_2(i) | 0 \leq i \leq m-1\}$.

Claim 3.1 $\pi(\mathcal{R}) = \Pi_1 \cup \Pi_2$.

Proof. It is trivial that $\pi(\mathcal{R}) \subseteq \Pi_1 \cup \Pi_2$, so we will show that $\pi(\mathcal{R}) \supseteq \Pi_1 \cup \Pi_2$.

First we show that $\pi(\mathcal{R}) \supseteq \Pi_1$. Let $Q = [u, x, v]$ $(u, v \in V_8, u \neq v, x \in V_m)$ be any element in Π_1.

When $u = a$ or $v = a$, we can consider $Q = [a, x, v]$ without loss of generality. We have $\rho^j Q = [a, x, e]$ for some j $(0 \leq j \leq 6)$. Then $\rho^j Q$ is contained in $R_2(x-1)$. Hence $\rho^j Q$ is contained in $\{R_2(i) | 0 \leq i \leq m-1\}$. Therefore Q is contained in $P\{R_2(i) | 0 \leq i \leq m-1\}$, so Q is contained in $\mathcal{R}$.

When $u, v \neq a$, we have $\rho^j(\{u, v\}) = \{g, h\}, \{b, d\}$ or $\{b, f\}$ for some j $(0 \leq j \leq 6)$. If $\rho^j(\{u, v\}) = \{g, h\}$, $\rho^j Q$ is contained in $R_1(x-1)$. If $\rho^j(\{u, v\}) = \{b, d\}$, $\rho^j Q$ is contained in $R_1(x)$. If $\rho^j(\{u, v\}) = \{b, f\}$, $\rho^j Q$ is contained in $R_2(x)$. In all cases, Q is contained in $\mathcal{R}$.

Next we show that $\pi(\mathcal{R}) \supseteq \Pi_2$. Let $Q = [u, v, x]$ $(u, v \in V_8, u \neq v, x \in V_m)$ be any element in Π_2.

When $u = a$, we have $\rho^j Q = [a, b, x]$ for some j $(0 \le j \le 6)$. Then $\rho^j Q$ is contained in $R_2(x)$. So Q is contained in $\mathcal{R}$.

When $v = a$, we have $\rho^j Q = [b, a, x]$ for some j $(0 \le j \le 6)$. Then $\rho^j Q$ is contained in $R_2(x-1)$. So Q is contained in $\mathcal{R}$.

When $u, v \ne a$, we have $\rho^j(u, v) = (e, f), (f, e), (b, g), (g, b), (d, h)$ or (h, d) for some j $(0 \le j \le 6)$, where $(\, , \,)$ is an ordered pair. If $\rho^j(u, v) = (e, f)$, $\rho^j Q$ is contained in $R_2(x)$. If $\rho^j(u, v) = (f, e)$, $\rho^j Q$ is contained in $R_2(x-1)$. If $\rho^j(u, v) = (b, g)$, $\rho^j Q$ is contained in $R_1(x-1)$. If $\rho^j(u, v) = (g, b)$, $\rho^j Q$ is contained in $R_1(x)$. If $\rho^j(u, v) = (d, h)$, $\rho^j Q$ is contained in $R_1(x-1)$. If $\rho^j(u, v) = (h, d)$, $\rho^j Q$ is contained in $R_1(x)$. In all cases, Q is contained in $\mathcal{R}$. This completes the proof. □

(II) Construction of a set of 6-cycles $\mathcal{C}$ in K_n such that $\pi(\mathcal{C}) = \Pi_3 \cup \Pi_4$.

Let λ be the vertex permutation $(a\ b\ c\ d)(e\ f\ g\ h)$ in K_8. We can extend λ to a vertex permutation of K_n by defining $\lambda(x) = x$ for $x \in V_m$. Put $\Lambda = \{\lambda^j | 0 \le j \le 3\}$.

(1) The case m is odd.

Assume m is odd and put $r = (m-1)/2$. Let $V_m = \{0, 1, 2, \ldots, m-1\}$, where addition in V_m is modulo m. Let τ be the vertex permutation $(0\ 1\ 2\ \cdots\ m{-}1)$ in K_m. We can extend τ to a vertex permutation of K_n by defining $\tau(u) = u$ for $u \in V_8$. Put $\Gamma = \{\tau^j | 0 \le j \le m-1\}$.

Define 6-cycles S_i $(1 \le i \le r)$ as follows:

$$S_i = \begin{cases} (0, a, -(i+1), -1, e, i+1) & (i\text{: odd, } 1 \le i \le r-2) \\ (0, e, -(i+2), -1, a, i) & (i\text{: even, } 2 \le i \le r-2), \end{cases}$$
$$S_{r-1} = \begin{cases} (0, e, -1, -r, a, 1) & (m \equiv 1 \pmod 4) \\ (0, e, -1, r, a, r-1) & (m \equiv 3 \pmod 4), \end{cases}$$
$$S_r = \begin{cases} (0, a, 1, -(r-1), e, r) & (m \equiv 1 \pmod 4) \\ (0, a, r, r-1, e, -r) & (m \equiv 3 \pmod 4). \end{cases}$$

Put $\mathcal{S} = \Lambda\Gamma\{S_i | 1 \le i \le r\}$.

Claim 3.2 *When m is odd, we have $\pi(\mathcal{S}) = \Pi_3 \cup \Pi_4$.*

Proof. It is trivial that $\pi(\mathcal{S}) \subseteq \Pi_3 \cup \Pi_4$, so we will show that $\pi(\mathcal{S}) \supseteq \Pi_3 \cup \Pi_4$.

Assume $m \equiv 1 \pmod 4$. We show that $\pi(\mathcal{S}) \supseteq \Pi_3$. The 2-path $[x, a, y]$ with $y - x = k$ $(2 \le k \le r)$ is contained in $\Gamma\{S_i | 1 \le i \le r-1\}$. The 2-path $[x, a, y]$ with $y - x = 1$ is contained in ΓS_r. The 2-path $[x, e, y]$ with $y - x = k$ $(3 \le k \le r)$ is contained in $\Gamma\{S_i | 1 \le i \le r-2\}$. The 2-path $[x, e, y]$ with $y - x = 1$ is contained in ΓS_{r-1}. The 2-path $[x, e, y]$ with $y - x = 2$ is contained in ΓS_r. Hence we have $\pi(\mathcal{S}) \supseteq \Pi_3$.

We now show that $\pi(\mathcal{S}) \supseteq \Pi_4$. The 2-path $[a, x, y]$ with $y - x = \pm k$ $(1 \le k \le r-1)$ is contained in $\Gamma\{S_i | 1 \le i \le r-1\}$. The 2-path $[a, x, y]$ with $y - x = \pm r$ is contained in ΓS_r. The 2-path $[e, x, y]$ with $y - x = \pm k$ $(1 \le k \le r-1)$ is contained in $\Gamma\{S_i | 1 \le i \le r-1\}$. The 2-path $[e, x, y]$ with $y - x = \pm r$ is contained in ΓS_r. Hence we have $\pi(\mathcal{S}) \supseteq \Pi_4$. Therefore we have $\pi(\mathcal{S}) \supseteq \Pi_3 \cup \Pi_4$.

When $m \equiv 3 \pmod 4$, we have $\pi(\mathcal{S}) \supseteq \Pi_3 \cup \Pi_4$ in the same way. □

(2) The case m is even.

Assume m is even and put $r = (m-2)/2$. Let $V_m = \{\infty\} \cup \{0, 1, 2, \ldots, m-2\}$, where addition in $V_m \setminus \{\infty\}$ is modulo $m-1$. Let σ be the vertex permutation $(\infty)(0\ 1\ 2\ \cdots\ m{-}2)$ in K_m. We can extend σ to a vertex permutation of K_n by defining $\sigma(u) = u$ for $u \in V_8$. Put $\Sigma = \{\sigma^j | 0 \le j \le m-2\}$.

Define 6-cycles T_i $(1 \le i \le r+1)$ as follows:

$$\begin{aligned}
T_i &= \begin{cases} (0, a, -(i+1), -1, e, i+1) & (i\text{: odd, } 1 \le i \le r-2) \\ (0, e, -(i+2), -1, a, i) & (i\text{: even, } 2 \le i \le r-2), \end{cases} \\
T_{r-1} &= \begin{cases} (0, e, -1, r, a, r-1) & (m \equiv 0 \pmod 4) \\ (0, e, -1, -r, a, r) & (m \equiv 2 \pmod 4), \end{cases} \\
T_r &= \begin{cases} (0, a, r, \infty, e, -r) & (m \equiv 0 \pmod 4) \\ (0, a, -r, \infty, e, r) & (m \equiv 2 \pmod 4), \end{cases} \\
T_{r+1} &= (0, e, 2, \infty, a, 1).
\end{aligned}$$

When $m = 6$, we have $r = 2$ and then we have only T_{r-1}, T_r and T_{r+1}. Put $\mathcal{T} = \Lambda\Sigma\{T_i | 1 \le i \le r+1\}$.

Claim 3.3 *When m is even, we have $\pi(\mathcal{T}) = \Pi_3 \cup \Pi_4$.*

Proof. It is trivial that $\pi(\mathcal{T}) \subseteq \Pi_3 \cup \Pi_4$, so we will show that $\pi(\mathcal{T}) \supseteq \Pi_3 \cup \Pi_4$.

Assume $m \equiv 0 \pmod 4$. We show that $\pi(\mathcal{T}) \supseteq \Pi_3$. The 2-path $[x, a, y]$ with $y - x = k$ $(2 \le k \le r-1)$ is contained in $\Sigma\{T_i | 1 \le i \le r-2\}$. The 2-path $[x, a, y]$ with $y - x = 1$ is contained in ΣT_{r-1}. The 2-path $[x, a, y]$ with $y - x = r$ is contained in ΣT_r. The 2-path $[\infty, a, x]$ is contained in ΣT_{r+1}. The 2-path $[x, e, y]$ with $y - x = k$ $(3 \le k \le r)$ is contained in $\Sigma\{T_i | 1 \le i \le r-2\}$. The 2-path $[x, e, y]$ with $y - x = 1$ is contained in ΣT_{r-1}. The 2-path $[\infty, e, x]$ is contained in ΣT_r. The 2-path $[x, e, y]$ with $y - x = 2$ is contained in ΣT_{r+1}. Hence we have $\pi(\mathcal{T}) \supseteq \Pi_3$.

We now show that $\pi(\mathcal{T}) \supseteq \Pi_4$. The 2-path $[a, x, y]$ with $y - x = \pm k$ $(1 \le k \le r-1)$ is contained in $\Sigma\{T_i | 1 \le i \le r-1, i = r+1\}$. The 2-path $[a, x, y]$ with $y - x = \pm r$ is contained in $\Sigma\{T_i | r-1 \le i \le r\}$. The 2-paths $[a, x, \infty]$ and $[a, \infty, x]$ are contained in $\Sigma\{T_i | r \le i \le r+1\}$. The 2-path $[e, x, y]$ with $y - x = \pm k$ $(1 \le k \le r-1)$ is contained in $\Sigma\{T_i | 1 \le i \le r-1, i = r+1\}$. The 2-path $[e, x, y]$ with $y - x = \pm r$ is contained in $\Sigma\{T_i | r-1 \le i \le r\}$. The 2-paths $[e, x, \infty]$ and $[e, \infty, x]$ are contained in $\Sigma\{T_i | r \le i \le r+1\}$. Hence we have $\pi(\mathcal{T}) \supseteq \Pi_4$. Therefore we have $\pi(\mathcal{T}) \supseteq \Pi_3 \cup \Pi_4$.

When $m \equiv 2 \pmod 4$, we have $\pi(\mathcal{T}) \supseteq \Pi_3 \cup \Pi_4$ in the same way.

Thus we complete the proof of Claim 3.3. □

When m is odd, put $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{R} \cup \mathcal{S}$, and when m is even, put $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{R} \cup \mathcal{T}$. Then we have the claim.

Claim 3.4 *$\mathcal{U}$ is a uniform covering of 2-paths with 6-cycles in K_n.*

Proof. $\mathcal{U}$ is a set of 6-cycles in K_n. By Claims 3.1, 3.2 and 3.3, each 2-path in K_n is contained in $\mathcal{U}$. By counting the number of 6-cycles in $\mathcal{U}$, we find that each 2-path in K_n appears exactly once in $\mathcal{U}$. Therefore the claim holds. □

We have completed the proof of Prop. 3.1. □

4 A proof of Theorem 1.1

We prove Theorem 1.1. Let $n \geq 6$. Assume that there is a uniform covering $\mathcal{C}$ of 2-paths with 6-cycles in K_n. Since there are $n(n-1)(n-2)/2$ 2-paths in K_n and 6 2-paths in a 6-cycle, $n(n-1)(n-2)$ is divisible by 12. Therefore we have $n \equiv 0, 1, 2 \pmod 4$.

To show the converse, we denote by A_n the following statement for an integer $n (\geq 6)$, A_n: There exists a uniform covering of 2-paths with 6-cycles in K_n. Put $N = \{n | n \equiv 0, 1, 2 \pmod 4, n \geq 6\}$. By Prop. 2.1, for $n \in \{6, 8, 9, 10, 12, 13\}$, A_n holds. By Prop. 3.1, for $n \in \{m + 8 | m = 6, 8, 9, 10, 12, 13\}$, A_n holds. Put $M = \{m + 8i | m = 6, 8, 9, 10, 12, 13,\ i \geq 1\}$. By applying Prop. 3.1 repeatedly, A_n holds for all $n \in M$. Since $M = N$, A_n holds for all $n \equiv 0, 1, 2 \pmod 4$, $n \geq 6$.

This completes the proof of Theorem 1.1.

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