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On graphs having uniform size star factors

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Abstract

A *star factor* of a graph with no isolated vertices is a spanning forest in which each component is a star of order at least two. We prove a simple, structural characterization of the class of connected graphs of girth at least five having the property that every star factor has the same number of edges.

1 Introduction

Let $G = (V, E)$ be a finite, undirected, simple graph. An *edge-weighting* of G is a function $w : E(G) \to \{1, 2, \ldots\}$. If H is a subgraph of G then $w(H)$, the *weight of H under w*, is the sum of all the weights of edges that belong to H. That is, $w(H) = \sum_{e \in E(H)} w(e)$. A *leaf* is a vertex of degree one and a *stem* is a vertex that has at least one leaf as a neighbor. In this paper a *star* will mean a tree isomorphic to $K_{1,n}$ for some $n \geq 1$. The vertex of degree n is called the *center* of the star. Either vertex can be used as the center when $n = 1$. If G is not a forest, then the *girth* of G, denoted $g(G)$, is the length of a shortest cycle. We say that a forest has infinite girth. A *star factor* of G is a spanning forest of G in which each component is a star. Note that a graph with isolated vertices has no star factor. On the other hand, a straightforward induction argument shows that any graph without isolated vertices has at least one star factor. In [1] Chen, et al, investigated graphs having the property that each edge belongs to a component of order two or three in a star factor of bounded component size. Yu (see [2]) obtained an upper bound on the number of edges in a graph that has just one star factor.

As with the minimum cost spanning tree and the optimal assignment problems, the most general setting would be in a weighted network. This suggests the following question.

Question 1 *For a given graph $G = (V, E)$ is there an edge-weighting w of G such that every star factor of G has the same weight under w?*

As an example consider the 4-cycle and an edge-weighting where one pair of opposite edges have weights 1 and 4 and the other pair have weights 2 and 3. It is easy to see that the only star factors of this graph are the two distinct perfect matchings, and each has a weight of 5. On the other hand, if an edge-weighting of the 5-cycle has assigned consecutive edges the weights w_1, w_2, w_3, w_4, w_5, then no pair of these weights can be distinct if all star factors must have the same weight. This follows from the observation that C_5 has five star factors, each isomorphic to $\{K_{1,1}, K_{1,2}\}$. This implies, for example, that $w_1 + w_2 + w_4 = w_1 + w_3 + w_4$, and hence adjacent weights are the same. But if the 5-cycle had all edge weights being the same integer k, then every star factor has weight $3k$. At the other extreme a graph G constructed from a connected graph H by adding at least one leaf adjacent to each vertex of H has a unique star factor. Every edge-weighting of G will answer Question 1 in the affirmative.

In the remainder of the paper we restrict our attention to the special case in which all edges of a graph G are given the same weight. In this situation every star factor of G has the same weight if and only if all star factors have the same number of edges, and, in fact, the same number of components. For simplicity we assume this uniform edge weight is one. We denote by $\mathcal{U}$ the set of all graphs G such that if $\mathcal{S}_1$ and $\mathcal{S}_2$ are any two star factors of G, then $\mathcal{S}_1$ and $\mathcal{S}_2$ have the same number of edges. As noted above this is equivalent to requiring that $\mathcal{S}_1$ and $\mathcal{S}_2$ have the same number of components. In Section 2 we derive properties of the class $\mathcal{U}$ and in Sections 3 and 4 we give a structural characterization of the graphs in $\mathcal{U}$ that have girth at least five.

2 Girth at Least Six

We begin with a result that is useful (and often used implicitly) in reducing the problem of determining membership in $\mathcal{U}$ to spanning subgraphs. Note that if H is a spanning subgraph of G, and H has no isolated vertices, then any star factor of H is also a star factor of G. This observation proves the following useful lemma.

Lemma 2 *If F is a subset of $E(G)$ such that $G - F$ has no isolated vertices and $G - F$ is not in $\mathcal{U}$, then G is not in $\mathcal{U}$.*

In showing that some graphs do not belong to $\mathcal{U}$ it is sometimes possible to separate G into several components and show that one of the components is not in $\mathcal{U}$. The following lemma is immediate from the definitions.

Lemma 3 *Suppose that $V(G)$ is partitioned as $V_1 \cup V_2$ such that $G[V_1]$ and $G[V_2]$ have no isolated vertices. If $\mathcal{S}_i$ is a star factor of $G[V_i]$, for $i = 1, 2$, then $\mathcal{S}_1 \cup \mathcal{S}_2$ is a star factor of G.*

In what follows we will often remove a set of edges F from G in such a way that $G - F$ has no isolated vertices and has the path P_6 as a component. But by noting that P_6 has star factors of weights three and four and so is not in $\mathcal{U}$, Lemma 2 shows that G is not in $\mathcal{U}$. For example, any cycle of length eight or more does not belong to $\mathcal{U}$ since F can be chosen to be the first and seventh edges of the cycle.

Lemma 4 *If G is a graph with minimum degree at least two and girth at least eight, then G does not belong to $\mathcal{U}$.*

Proof. Let P be the path $v_1, v_2, v_3, v_4, v_5, v_6$ in a graph G that satisfies the hypothesis of the lemma. Let F be the set of all edges in G, other than those in the path, that are incident with a vertex of P. Since G has girth at least eight, no pair of vertices in P have a common neighbor not belonging to the path. In addition G has minimum degree at least two, and so it follows that $G - F$ has no isolated vertices. But $G - F$ has a component isomorphic to P_6, and so by Lemma 2 it follows that $G \notin \mathcal{U}$. □

Lemma 5 *If G has girth seven and minimum degree at least two, then G belongs to $\mathcal{U}$ if and only if G is a cycle of order seven.*

Proof. First we note that C_7 has a unique star factor up to isomorphism and hence belongs to $\mathcal{U}$. Now consider a graph G that has girth seven and minimum degree at least two but is not isomorphic to C_7, such that $G \in \mathcal{U}$. Let $C : v_1, v_2, \ldots, v_7$ be a cycle in G. Assume without loss of generality that v_7 has a neighbor w not on C. Let F be the set of edges consisting of v_6v_7, v_1v_7 and all edges not in C that are incident with a vertex from the set $\{v_1, \ldots, v_6\}$. Since $g(G) = 7$ and G has no leaves, $\delta(G - F) \geq 1$. But $G - F$ has a component isomorphic to P_6. This contradiction establishes the converse. □

Lemma 6 *If G has girth six and minimum degree at least two, then G is not in $\mathcal{U}$.*

Proof. Let G be a graph of girth six and minimum degree at least two. If $G = C_6$, then G has star factors of weights three and four and so does not belong to $\mathcal{U}$. Otherwise let C be a cycle of order six in G. Let F be the set of edges that are not in the cycle C but that are incident to a vertex of C. Since $g(G) = 6$ and $\delta(G) \geq 2$, the graph $G - F$ has no isolated vertices. But the cycle of order six is a component of $G - F$, and so $G \notin \mathcal{U}$. □

Corollary 7 *If G has minimum degree at least two and girth six or more, then G belongs to $\mathcal{U}$ if and only if G is a cycle of order seven.*

3 Girth Five

In this section we consider graphs of girth five in $\mathcal{U}$. A series of reductions culminates in Theorem 12 which characterizes the graphs in $\mathcal{U}$ that have girth five and no leaves. This in turn is used in the proof of our main structural characterization result, Theorem 14. The first two results of this section do not require any assumptions about the girth of the graph.

Lemma 8 *Let G be a graph with an induced cycle of order five such that four of the vertices are of degree two and the fifth is a stem. Then G does not belong to $\mathcal{U}$.*

Proof. Let C be a 5-cycle in such a graph G and assume the vertices are v_1, v_2, v_3, v_4, v_5 where v_1 is a stem adjacent to a leaf u. Let $\mathcal{S}$ be a star factor of the graph $G - \{v_2, v_3, v_4, v_5\}$. Note that v_1 is the center of some star T in $\mathcal{S}$. Let T' be the star formed from T by adding leaves v_2 and v_5 adjacent to v_1, and let $\mathcal{S}' = (\mathcal{S} - \{T\}) \cup \{T', v_3v_4\}$. Then, $\mathcal{S}'$ as well as $\mathcal{S} \cup \{v_2v_3, v_4v_5\}$ are star factors of G having different weights. Hence $G \notin \mathcal{U}$. □

Lemma 9 *Let G be a graph in $\mathcal{U}$ with an induced 5-cycle. If exactly one of the vertices on this 5-cycle has degree at least three, then all of its neighbors that do not belong to this 5-cycle must be stems.*

Proof. Let v be a vertex on the 5-cycle of degree at least three and assume v has a neighbor x not on the 5-cycle such that x is not a stem. By Lemma 8, x is not a leaf. Let F be the set of edges not including vx that are incident with x. The graph $G - F$ has no isolated vertices, and the vertex v is a stem belonging to an induced 5-cycle of the type that satisfies the hypothesis of Lemma 8. Thus G does not belong to $\mathcal{U}$. □

Lemma 10 *Suppose G is a graph of girth five and contains a cycle of order five in which no vertex is a stem but such that two adjacent vertices of this cycle have degree at least three in G. Then G does not belong to $\mathcal{U}$.*

Proof. Assume G does belong to $\mathcal{U}$ and let C be the 5-cycle v_1, v_2, v_3, v_4, v_5 in G where v_1 and v_2 both have degree at least three. Let F_1 be the set of all edges not on C but incident with either v_3, v_4 or v_5. Since $g(G) = 5$ no pair of vertices on C have a common neighbor not on C, and so $G' = G - F_1$ has no isolated vertices. Furthermore, neither v_1 nor v_2 is a stem in G'. Note that if all the edges incident with v_1 but not on C were deleted from G' to form the graph G'', then no stem created in G'' is a neighbor of v_2 since G has girth five. But by Lemma 2, G'' is in $\mathcal{U}$ and so by Lemma 9 all neighbors of v_2 in G'' must be stems. Thus all neighbors of v_2 in G' must be stems. By a symmetric argument all neighbors of v_1 in G' must be stems. Let w_1 and w_2 be stems adjacent to v_1 and v_2, respectively, in G'. Let F_2 be the set of all edges joining v_1 to stems other than w_1 together with all edges joining v_2 to stems other than w_2. Let G^* denote the graph $G' - F_2$. G^* belongs to $\mathcal{U}$. Let $\mathcal{S}$ be a star factor of the graph $G^* - C$. Suppose $\mathcal{S}$ has k edges. Note that w_1 and w_2 are centers of stars T_1 and T_2, respectively, in $\mathcal{S}$. Let S_1 be the star with center v_1 induced by $\{v_1, v_2, v_5\}$, and let S_2 be the star of order two induced by $\{v_3, v_4\}$. Then, $\mathcal{S} \cup \{S_1, S_2\}$ is a star factor of G with $k+3$ edges. For $i = 1, 2$ let T_i' be the star obtained from T_i by adding the leaf v_i adjacent to the center w_i of T_i, and let S_3 be the star of order three induced by $\{v_3, v_4, v_5\}$. The star factor $(\mathcal{S} - \{T_1, T_2\}) \cup \{T_1', T_2', S_3\}$ is also a star factor of G but has $k+4$ edges. Thus G does not belong to $\mathcal{U}$. □

Lemma 11 *Suppose G is a graph of girth five and contains a cycle C of order five in which no vertex is a stem but such that two nonadjacent vertices v_1 and v_3 of C have degree at least three. If G belongs to $\mathcal{U}$, then all neighbors of v_1 and v_3 not belonging to C must be stems.*

Proof. Assume that G and $C = v_1, v_2, v_3, v_4, v_5$, are as in the statement of the lemma but that v_3 has a neighbor x not on C such that x is not a stem. Note that v_2, v_4 and v_5 must all have degree two in G by Lemma 10. Let F be the set of all edges, other than xv_3, that are incident with x, and let $G' = G - F$. Since x is not a stem in G, $G' \in \mathcal{U}$. We note that v_1 could be a stem in G'. If v_1 is not a stem in G', then let F_1 be the set of edges, not in C, but incident with v_1. Then $G' - F_1$ is in $\mathcal{U}$, but it contains a 5-cycle satisfying the conditions of Lemma 8. Therefore v_1 is a stem in G'. Let $\mathcal{S}$ be a star factor of $G' - \{v_2, v_4, v_5\}$. Suppose $\mathcal{S}$ has k edges. Note that v_1 and v_3 are centers of stars T_1 and T_3, respectively, in $\mathcal{S}$. Let T_3' be the star obtained from T_3 by adding the leaf v_2 adjacent to the center v_3. The star factor $(\mathcal{S} - \{T_3\}) \cup \{T_3', v_4v_5\}$ of G has $k+2$ edges. Add the leaf v_5 adjacent to the center v_1 of T_1 to obtain the star T_1' and construct the star T_3'' by adding the leaf v_4 adjacent to the center v_3 of star T_3'. Then $(\mathcal{S} - \{T_1, T_3\}) \cup \{T_1', T_3''\}$ is also a star factor of G but with $k+3$ edges. This contradiction implies that all neighbors of v_1 and v_3 that are not on C must be stems in G. □

Theorem 12 *Let G be a graph of girth five and minimum degree at least two. Then all star factors of G have the same weight if and only if G is a 5-cycle.*

Proof. The star factors of C_5 are all isomorphic and have weight three. Assume G belongs to $\mathcal{U}$, has girth five and minimum degree at least two but that $G \neq C_5$. Thus G must contain a 5-cycle C with a vertex v of degree at least three. By Lemmas 9, 10 and 11, G must have a leaf. This contradiction establishes the theorem. □

4 Girth at Least Five

We now characterize the graphs in $\mathcal{U}$ of girth at least five. The results of Sections 2 and 3 have essentially reduced the problem to considering those graphs that have at least one leaf. The following lemma is true regardless of girth.

Lemma 13 *If G is a graph in which every vertex is either a leaf or a stem, then G belongs to $\mathcal{U}$.*

Proof. Let L be the set of leaves of G and W be the set of stems. Any star factor of G must contain all edges joining a vertex in L and a vertex in W but cannot contain any edge incident with two vertices of W. Therefore, every star factor of G has weight equal to $|L|$, and so $G \in \mathcal{U}$. □

Theorem 14 *Let G be a connected graph of girth at least five. Then every star factor of G has the same weight if and only if G is a cycle of order five or seven or G has a vertex of degree one and each component of the graph obtained by removing all the leaves and stems from G is*

1. *A 5-cycle in which at most two vertices have degree more than two in G such that if there are two such vertices they are nonadjacent on the 5-cycle; or*
2. *A star $K_{1,m}$ for some $m \geq 2$ such that the center of this $K_{1,m}$ has degree m in G; or*
3. *An isolated vertex.*

Proof. Suppose G has girth at least five and $\delta(G) \geq 2$. The theorem follows from Corollary 7 and Theorem 12.

Now suppose G has at least one leaf. Let L be the set of leaves of G and $W = N(L)$ the set of stems. Assume all star factors of G have the same weight, say k. If $G - (L \cup W)$ has a component that is a cycle C of order five, then, in G, by the results of Section 3 at most two of the vertices of C have degree larger than two (and if there are two such vertices, they are nonadjacent on C) and all of their neighbors not on C must be stems.

Hence we now consider those components of $G - (L \cup W)$ that have no 5-cycles. Let H be such a component. Suppose H has diameter at least three. There exists a path a, b, c, d in H such that a is adjacent to a stem w of G. Since $g(H) \geq 6$, there are no isolated vertices in the graph $G' = G - \{a, b, c, d\}$. Let $\mathcal{S}$ be a star factor of G'. Assume that $T \in \mathcal{S}$ is the star with center w. Let T' be the star constructed from T by adding the leaf a adjacent to w, and let R be the star of order three induced by $\{b, c, d\}$. Then $(\mathcal{S} - \{T\}) \cup \{T', R\}$ and $\mathcal{S} \cup \{ab, cd\}$ are star factors of G having different weights.

Hence we may assume that the diameter of H is at most two. So H is isomorphic to a star, say $K_{1,m}$. Suppose first that $m = 1$ and H is the path a, b. Let w_1 be a stem of G adjacent to a and w_2 be a stem of G adjacent to b. Let $\mathcal{S}$ be a star factor of $G' = G - \{a, b\}$. Suppose T_i is the star with center w_i for $i = 1, 2$. If leaves a and b are added adjacent to the centers w_1 and w_2, in T_1 and T_2, respectively, then it follows that G has a star factor whose weight is two larger than that of $\mathcal{S}$. But $\mathcal{S} \cup \{ab\}$ is also a star factor of G. Thus we may assume that $m \geq 2$. Let the vertices of H be $c, b_1, \ldots, b_m$ where c has degree m in H. For each $1 \leq i \leq m$, let w_i be a stem of G adjacent to b_i. Suppose c has a neighbor w in W. Let $\mathcal{S}$ be a star factor of $G - H$ having k edges. Assume that for each $1 \leq i \leq m$, $T_i \in \mathcal{S}$ is the star with center w_i and that $T \in \mathcal{S}$ is the star with center w. Then $\mathcal{S} \cup \{H\}$ is a star factor of G of weight $k + m$. But by adding one leaf (namely, $c, b_1, \ldots, b_m$) adjacent to the centers (namely, $w, w_1, \ldots, w_m$) of the stars $T, T_1, T_2, \ldots, T_m$, respectively, a star factor of G of weight $k + m + 1$ is constructed. This contradiction shows that the center c of the star $H = K_{1,m}$ has degree m in G.

Therefore, if G has minimum degree one and all its star factors have the same weight, then every component of $G - (L \cup W)$ is one of the three specified types.

Conversely, assume G has the specified structure. Let $\mathcal{S}$ be any star factor of G. $\mathcal{S}$ contains exactly one edge for each leaf of G. Let H be a component of $G-(L\cup W)$. If H is a 5-cycle, then $\mathcal{S}$ either contains three edges of H or two edges of H and exactly one edge joining H to a stem of G. If $H = K_{1,m}$ for some $m \geq 2$, then $\mathcal{S}$ includes precisely m edges incident with a vertex of H. In particular, for each leaf x of H either the edge joining x to the center of H or exactly one edge joining x to a stem of G must be in $\mathcal{S}$. Also note that at least one edge of H must be in $\mathcal{S}$. Finally, if H is an isolated vertex u, then $\mathcal{S}$ contains exactly one edge joining u to a stem of G. □

By using the structure given in Theorem 14 it is straightforward to determine whether a graph of girth at least five belongs to the class $\mathcal{U}$.

References

[1] Ciping Chen, Yoshimi Egawa and Mikio Kano, Star factors with given properties. *Ars Combin.* **28** (1989), 65–70.

[2] Qinglin Yu, Counting the number of star-factors in graphs. *J. Combin. Math. Combin. Comput.* **23** (1997), 65–76.

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