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The defining numbers for vertex colorings of certain graphs

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Abstract

A c-coloring of a graph G is an assignment of c different colors to the vertices of G such that adjacent vertices receive different colors. In a given graph G a set of vertices S with a specified coloring is called a defining set of the vertex coloring of G, if there exists a unique extension of S to a c-coloring of G. A defining set with minimum cardinality is called a minimum defining set and its cardinality is the defining number. In this paper we give exact values of the defining numbers of vertex colorings of graphs arising from applying Mycielski's construction to paths, cycles, complete graphs and complete bipartite graphs.

1 Introduction

A *c-coloring* of a graph G is an assignment of c different colors to the vertices of G such that adjacent vertices receive different colors. The (vertex) *chromatic number*, $\chi(G)$, of G is the minimum number c, for which there exists a c-coloring of G. A graph with $\chi(G) = c$ is called a *c-chromatic graph.* For a graph G and a number $c \geq \chi(G)$, a set of vertices S with a specified coloring is called a *defining set* of vertex colorings, if there exists a unique extension of the colors of S to a c-coloring of the

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vertices of G. A defining set with minimum cardinality is called a *minimum defining set* (of a vertex coloring). The *defining number*, $d(G, c)$, of G is the cardinality of its smallest defining set [5, 6]. The concept of defining sets has been studied, to some extent, for block designs [8, 9] and also under another name, critical sets, for Latin squares [3] and forcing sets for perfect matchings in graphs [1]. For defining sets in combinatorics the reader may consult [2].

The concept of defining set for vertex colorings is closely related to the concept of a list coloring. In a list coloring, for each vertex v there is a list of colors $L(v)$ available for that vertex. Any defining set S in a graph G naturally induces a list of possible colors for the vertices of the induced subgraph $\langle G - S\rangle$. Furthermore, using this list of colors, $\langle G - S\rangle$ is uniquely list colorable.

A graph G with n vertices is called a uniquely 2-list colorable graph if there exists a list $L(v)$ of at least two colors for each $v \in V(G)$, such that G has a unique list coloring with respect to these lists. We make use of the following theorem.

Theorem A [4] A connected graph is uniquely 2-list colorable if and only if at least one of its blocks is not a cycle, a complete graph or a complete bipartite graph.

For a simple graph G, by graph $M(G)$ we mean the graph arising from applying Mycielski's construction [10]. Mycielski's construction produces a simple graph $M(G)$ containing G as follows. If $V(G) = \{v_1, v_2, \ldots, v_n\}$ then $V(M(G)) = \{v_1, v_2, \ldots, v_n\} \cup \{u_1, u_2, \ldots, u_n, w\}$, where $V(G) \cap \{u_1, u_2, \ldots, u_n, w\} = \emptyset$ and

$$E(M(G)) = E(G) \cup \{u_i v \mid v \in N_G(v_i),\ 1 \leq i \leq n\} \cup \{u_i w \mid 1 \leq i \leq n\}.$$

Theorem B [10] If G is a c-chromatic triangle-free graph then $M(G)$ is a $c+1$-chromatic triangle-free graph.

Let P_n, C_n, K_n and $K_{m,n}$ be the path, the cycle, the complete graph with n vertices and the complete bipartite graph with m vertices in one partite set and n vertices in the other partite set, respectively. In this paper we study the defining numbers for c-colorings of $M(G)$, where $G \in \{P_n, C_n, K_n, K_{m,n}\}$. Throughout this paper $c(v)$ denotes the color of vertex v.

2 Defining numbers for 3-colorings of $M(P_n)$ and $M(C_{2n})$

It is well known that $\chi(P_n) = \chi(C_{2n}) = 2$ for $n \geq 2$. So we have $\chi(M(P_n)) = \chi(M(C_{2n})) = 3$ for $n \geq 2$ by Theorem B. In this section we find the defining numbers for 3-colorings of $M(P_n)$ and $M(C_{2n})$. We always use colors 1, 2 and 3 for a 3-coloring.

2.1 Defining numbers for 3-colorings of $M(P_n)$

Let P_n be a path of length $n \geq 2$ with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$. Figure 1 shows the graph $M(P_{10})$ which arises from applying Mycielski's construction to P_{10}.

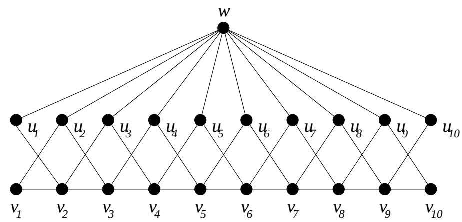

Figure 1: $M(P_{10})$

In this subsection we prove that $d(M(P_n), 3) = 3$ for $n = 2, 3, 4, 5$ and $d(M(P_n), 3) = \lfloor \frac{n+9}{5} \rfloor$ for $n \geq 6$.
Notice that $M(P_n)$ contains 5-cycles for $n \geq 2$. It is now easy to see that any pair of vertices of $M(P_n)$, with any coloring, can be extended to at least two different 3-colorings of $M(P_n)$. Therefore we have:

Lemma 1 $d(M(P_n), 3) \geq 3$ for $n \geq 2$.

Lemma 2 $d(M(P_n), 3) = 3$ *for* $n = 2, 3, 4, 5, 6$.

Proof. For $n = 2$ define $S_2 = \{w, v_1, v_2\}$ with $c(w) = 1$, $c(v_1) = 2$ and $c(v_2) = 3$. For $n = 3, 4$ define $S_3 = S_4 = \{u_1, u_3, v_3\}$ with $c(u_1) = 3$, $c(u_3) = 2$ and $c(v_3) = 3$. For $n = 5$ define $S_5 = \{w, v_1, u_4\}$ with $c(w) = 1$, $c(v_1) = c(u_4) = 2$. Finally, for $n = 6$ define $S_6 = \{u_1, v_1, u_6\}$ with $c(u_1) = 3$, $c(v_1) = c(u_6) = 2$. It is easy to see that S_n is a defining set for a 3-coloring of $M(P_n)$ for $n = 2, 3, 4, 5$ and 6, respectively. Now the result follows by Lemma 1. ■

Now let $n \geq 7$. First we find a lower bound for $d(M(P_n), 3)$.

Lemma 3 Let S be a defining set for a 3-coloring of $M(P_n)$, $n \geq 7$. Then $S \cap \{v_{k+i}, u_{k+i} \mid 0 \leq i \leq 4\} \neq \emptyset$ for each $k = 1, 2, ..., n-4$.

Proof. By assumption there is a unique 3-coloring of $M(P_n)$ containing S. Without loss of generality we may assume $c(w) = 1$ in this 3-coloring. When $1 < k < n-4$ we checked all the different colors for the vertices u_{k-1}, u_{k+5}, v_{k-1} and v_{k+5}, and noticed that if $S \cap \{v_{k+i}, u_{k+i} \mid 0 \leq i \leq 4\} = \emptyset$, then for each case either there was no extension of S to a 3-coloring or there were at least two different 3-colorings of $M(P_n)$. This is a contradiction.
For $k = 1$ (the case $k = n-4$ is similar) we must have $(c(u_6), c(v_6)) \in \{(2,2), (3,3), (2,3), (3,2), (2,1), (3,1)\}$. It is easy to see that for each case there are at least two different 3-colorings for the vertices of $\{v_{i+1}, u_{i+1} \mid 0 \leq i \leq 4\}$. Therefore, $S \cap \{v_{i+1}, u_{i+1} \mid 0 \leq i \leq 4\} \neq \emptyset$. ■

A case-checking similar to that described above leads to the following result.

Lemma 4 If S is a defining set for a 3-coloring of $M(P_n)$, then $S \cap \{v_1, v_2, u_1, u_2\} \neq \emptyset$ and $S \cap \{v_n, v_{n-1}, u_n, u_{n-1}\} \neq \emptyset$.

By Lemmas 3 and 4 we have:

Corollary 5 $d(M(P_n),3) \geq 2 + \lfloor \frac{n-4}{5} \rfloor$ for $n \geq 7$.

Now we find an upper bound for the smallest defining sets in 3-colorings of $M(P_n)$.

Lemma 6 $d(M(P_n),3) \leq \lfloor \frac{n+9}{5} \rfloor$ for $n \geq 7$.

Proof. For $n \equiv 0$ or $4 \pmod 5$ let $S = \{v_1, u_4, u_{5k+4} \mid 1 \leq k \leq \lfloor \frac{n+9}{5} \rfloor - 2\}$ with $c(v_1) = c(u_4) = 2$, and $c(u_{5k+4}) = 3$ if k is odd and 2 otherwise.
For $n \equiv 1 \pmod 5$ let $S = \{u_1, v_1, u_{5k+1} \mid 1 \leq k \leq \lfloor \frac{n+9}{5} \rfloor - 2\}$ with $c(u_1) = 2$, $c(v_1) = 3$, and $c(u_{5k+1}) = 3$ if k is odd and 2 otherwise.
For $n \equiv 2$ or $3 \pmod 5$ let $S = \{u_2, v_2, u_{5k+2} \mid 1 \leq k \leq \lfloor \frac{n+9}{5} \rfloor - 2\}$ with $c(u_2) = 2$, $c(v_2) = 3$, and $c(u_{5k+2}) = 3$ if k is odd and 2 otherwise.
It is easy to see that S is a defining set for a 3-coloring of $M(P_n)$ in each case. ■

By Corollary 5 and Lemma 6 we have:

Corollary 7 If $n \equiv 0$ or $4 \pmod 5$ and $n \geq 9$, then $d(M(P_n),3) = \lfloor \frac{n+9}{5} \rfloor$.

Lemma 8 $d(M(P_{11}),3) = 4$.

Proof. By Lemma 6 we have $d(M(P_{11}),3) \leq 4$. We examined all 3-subsets of $V(M(P_{11}))$ and noticed that each 3-subset, with any coloring, was in at least two different 3-colorings of $M(P_{11})$. So the result follows. ■

A simple case-checking leads to the following result.

Lemma 9 Let $n \equiv 1 \pmod 5$. Consider a 3-coloring of $M(P_n)$ with $c(w) = 1$. Let S be a subset of $V(M(P_n))$ such that $S \cap \{u_1, v_1\} \neq \emptyset$, $S \cap \{u_n, v_n\} \neq \emptyset$ and $|S| = 2 + \lfloor \frac{n-4}{5} \rfloor$.
(1) If $\{u_k, v_k\} \cap S \neq \emptyset$ and $(c(u_k), c(v_k)) \in \{(2,1),(3,1),(2,2),(3,3)\}$ for some k, then S is in at least two different 3-colorings of $M(P_n)$.
(2) If $\{u_k, v_k\} \cap S \neq \emptyset$ and $(c(u_k), c(v_k)) = (2,3)$ or $(3,2)$, then S is a defining set for a 3-coloring of $M(P_n)$ only if $(c(u_{k-5}), c(v_{k-5})) = (c(u_{k+5}), c(v_{k+5})) = (3,2)$ or $(2,3)$, respectively.

Lemma 10 Let $n \equiv 1 \pmod 5$. Let S be a subset of $V(M(P_n))$ with $S \cap \{u_1, v_1\} \neq \emptyset$ and $S \cap \{u_n, v_n\} \neq \emptyset$. If S is a defining set for a 3-coloring of $M(P_n)$ then $|S| > 2 + \lfloor \frac{n-4}{5} \rfloor$.

Proof. Let $n = 5k + 1$. The proof is by induction on k. The statement is true for $k = 1$ and 2 by Lemmas 2 and 8, respectively. Assume $k \geq 3$ and that the statement is true for all $1 \leq k' < k$. We prove that the statement is also true for k. On the contrary, suppose $|S| = 2 + \lfloor \frac{n-4}{5} \rfloor$. Then by assumption and Lemma 3, $S \cap \{u_m, v_m\} \neq \emptyset$ if and only if $m = 5j + 1$ for some $j \in \{0, 1, 2, \ldots, k\}$ and $w \notin S$. As usual we may assume $c(w) = 1$ in the unique 3-coloring arising from S. Now since S is a defining set, by Lemma 3 we must have $(c(u_{5j+1}), c(v_{5j+1})) = (2,3)$ or $(3,2)$ for each j. Furthermore, by Lemma 9 if $(c(u_{5j+1}), c(v_{5j+1})) = (2,3)$, then $(c(u_{5j+6}), c(v_{5j+6})) = (3,2)$ and if $(c(u_{5j+1}), c(v_{5j+1})) = (3,2)$, then $(c(u_{5j+6}), c(v_{5j+6})) = (2,3)$. Now consider the subgraph H of $M(P_n)$ induced by vertices $\{w, u_i, v_i \mid i = 11, 12, \ldots, n\}$. Obviously, H is isomorphic to $M(P_{5(k-2)+1})$. Let $S' = S \cap V(H)$. Then S' is a defining set for a 3-coloring of H. Therefore, by inductive hypothesis we have $|S'| > 2 + \lfloor \frac{5(k-2)+1-4}{5} \rfloor = k-1$. On the other, $|S'| = |S| - 2 = k-1$. This is a contradiction. So $|S| \neq 2 + \lfloor \frac{n-4}{5} \rfloor$. Now the result follows by Corollary 5. ■

Lemma 11 Let $n \equiv 1$, 2 or 3 (mod 5) and $n \geq 6$. Then $d(M(P_n), 3) = 2 + \lceil \frac{n-4}{5} \rceil$.

Proof. Assume that $|S| = 2 + \lfloor \frac{n-4}{5} \rfloor$. By Lemma 4 we have three cases.

Case 1. $S \cap \{u_1, v_1\} \neq \emptyset$ and $S \cap \{u_n, v_n\} \neq \emptyset$.
For $n \equiv 1$ (mod 5) the result follows by Lemmas 6 and 10. For $n \equiv 2$ or 3 (mod 5) the result follows by Lemmas 3 and 6.

Case 2. $S \cap \{u_1, v_1\} \neq \emptyset$ and $S \cap \{u_{n-1}, v_{n-1}\} \neq \emptyset$.
For $n \equiv 3$ (mod 5) the result follows by Lemmas 3 and 6. For $n \equiv 2$ (mod 5) we remove vertices u_n and v_n. The remaining graph is isomorphic to $M(P_{n-1})$. Now we can apply Case 1 since $n-1 \equiv 1$ (mod 5). For $n \equiv 1$ (mod 5), by Lemma 3, we must have $|S \cap \{u_k, v_k\}| = 1$, $|S \cap \{u_{k+4}, u_{k+5}, v_{k+4}, v_{k+5}\}| = 1$ and $|S \cap \{u_{k+9}, v_{k+9}\}| = 1$ for some k (see Figure 2). Now a simple case-checking shows that S cannot be a defining set.

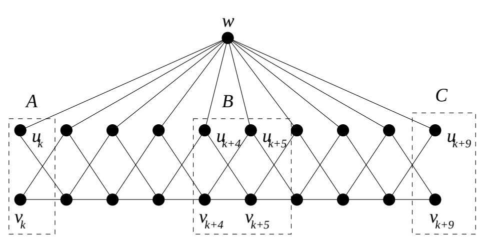

Figure 2: $|S \cap A| = |S \cap B| = |S \cap C| = 1$

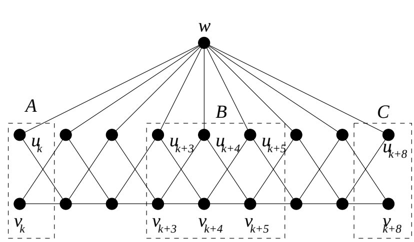

Figure 3: $|S \cap A| = |S \cap B| = |S \cap C| = 1$

Case 3. $S \cap \{u_2, v_2\} \neq \emptyset$ and $S \cap \{u_{n-1}, v_{n-1}\} \neq \emptyset$.
For $n \equiv 3 \pmod 5$ we remove vertices u_1, v_1, u_n and v_n. The remaining graph is isomorphic to $M(P_{n-2})$. So we can apply Case 1 since $n-2 \equiv 1 \pmod 5$. For $n \equiv 2 \pmod 5$ we remove vertices u_n and v_n. The remaining graph is isomorphic to $M(P_{n-1})$. So we can apply Case 2 since $n-1 \equiv 1 \pmod 5$. For $n \equiv 1 \pmod 5$, by Lemma 3, we have either the case shown in Figure 2 or the case shown in Figure 3 for some k. A case-checking shows that both cases are impossible since S is a defining set. ■

Now we are ready to state the main result of this section.

Theorem 12 $d(M(P_n), 3) = 3$ for $n = 2, 3, 4, 5$ and $d(M(P_n), 3) = \lfloor \frac{n+9}{5} \rfloor$ for $n \geq 6$.

2.2 Defining numbers for 3-colorings of $M(C_{2n})$

Let C_{2n} be a $2n$-cycle, $n \geq 2$, with the vertex set $V = \{v_1, v_2, \ldots, v_{2n}\}$. Figure 4 shows the graph $M(C_{10})$ which arises from applying Mycielski's construction to C_{10}. In this subsection we prove that $d(M(C_{2n}), 3) = 3$ for $n = 2, 3, 4$ and $d(M(C_{2n}), 3) = \lceil \frac{2n}{5} \rceil$ if $2n \equiv 1 \pmod 5$ and $d(M(C_{2n}), 3) = \lceil \frac{2n}{5} \rceil + 1$ otherwise, for $n \geq 5$.

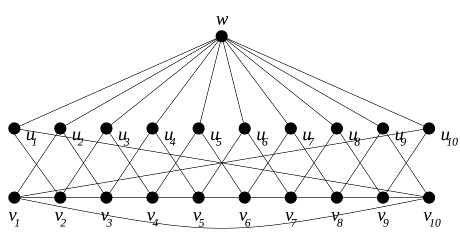

Figure 4: $M(C_{10})$

Similar to Lemma 1 we have:

Lemma 13 $d(M(C_{2n}), 3) \geq 3$ for $n \geq 2$.

Lemma 14 $d(M(C_{2n}), 3) = 3$ for $n = 2, 3, 4$.

Proof. For $n = 2$ define $S_2 = \{u_1, u_3, v_3\}$ with $c(u_1) = 3$, $c(u_3) = 2$, $c(v_3) = 3$. For $n = 3$ define $S_3 = \{w, u_1, v_4\}$ with $c(w) = 1$, $c(u_1) = c(v_4) = 2$. For $n = 4$ define $S_4 = \{u_1, u_3, v_5\}$ with $c(u_1) = 2$, $c(u_3) = 3$ and $c(v_5) = 1$. It is easy to see that S_n is a defining set for a 3-coloring of $M(C_{2n})$ for $n = 2, 3, 4$, respectively. Now the result follows by Lemma 13. ■

The following result gives a lower bound on $d(M(C_{2n}), 3)$.

Lemma 15 $d(M(C_{2n}), 3) \geq \lceil \frac{2n}{5} \rceil$ for $n \geq 4$.

Proof. An argument similar to that described in the proof of Lemma 3 shows that if S is a defining set for a 3-coloring of $M(C_{2n})$, $n \geq 4$, then $S \cap \{v_{k+i}, u_{k+i} \mid 0 \leq i \leq 4\} \neq \emptyset$ for each $k = 1, 2, ..., n$. Now the result follows. ■

For $2n \equiv 0 \pmod 5$ an argument similar to that described in Lemmas 9 and 10 improves the lower bound for $d(M(C_{2n}), 3)$.

Lemma 16 $d(M(C_{2n}), 3) \geq \lceil \frac{2n}{5} \rceil + 1$ for $2n \equiv 0 \pmod 5$ and $n \geq 5$.

Now we are ready to prove the main theorem of this subsection.

Theorem 17 $d(M(C_{2n}), 3) = 3$ for $n = 2, 3, 4$ and

$$d(M(C_{2n}), 3) = \begin{cases} \lceil \frac{2n}{5} \rceil & 2n \equiv 1 (\text{mod } 5), n \geq 8; \\ \lceil \frac{2n}{5} \rceil + 1 & 2n \not\equiv 1 (\text{mod } 5), n \geq 5. \end{cases}$$

Proof. For $n = 2, 3, 4$ we apply Lemma 14. Now let $n \geq 5$.
For $2n \equiv 1 \pmod 5$ let $S = \{u_1, v_4, u_{5k+2} \mid 1 \leq k \leq \lceil \frac{2n}{5} \rceil - 2\}$ with $c(u_1) = c(v_4) = 3$ and $c(u_{5k+2}) = 3$ if k is odd and 2 otherwise.
For $2n \equiv 0$ or $2 \pmod 5$ let $S = \{u_1, v_1, u_{5k+1} \mid 1 \leq k \leq \lceil \frac{2n}{5} \rceil - 1\}$ with $c(u_1) = 2$, $c(v_1) = 3$ and $c(u_{5k+1}) = 3$ if k is odd and 2 otherwise.
For $2n \equiv 4 \pmod 5$ let $S = \{v_1, v_3, u_{5k+1} \mid 1 \leq k \leq \lceil \frac{2n}{5} \rceil - 1\}$ with $c(v_1) = 1$, $c(v_3) = 3$ and $c(u_{5k+1}) = 3$ if k is odd and 2 otherwise.
For $2n \equiv 3 \pmod 5$ let $S = \{u_1, v_1, v_{2n-3}, u_{5k+1} \mid 1 \leq k \leq \lceil \frac{2n}{5} \rceil - 2\}$ with $c(u_1) = 2$, $c(v_1) = 3$, $c(v_{2n-3}) = 1$ and $c(u_{5k+1}) = 3$ if k is odd and 2 otherwise.
In each case it is easy to see that S is a defining set for a 3-coloring of $M(C_{2n})$. So the result follows for $2n \equiv 1 \pmod 5$ by Lemma 15 and for $2n \equiv 0 \pmod 5$ by Lemma 16.
Now let $2n \equiv 2$, 3 or $4 \pmod 5$. If $|S| = \lceil \frac{2n}{5} \rceil$ then we must have (see Lemma 3) one of the cases shown in Figure 2, 3 or 5 for some k. A case-checking shows that these cases are impossible since S is a defining set. ■

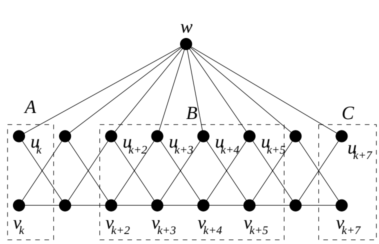

Figure 5: $|S \cap A| = |S \cap B| = |S \cap C| = 1$

3 Defining numbers for 4-colorings of $M(P_n)$ and $M(C_n)$

It is well known that $\chi(C_{2n+1}) = 3$ for $n \geq 1$. So we have $\chi(M(C_{2n+1})) = 4$ by Theorem B. In this section we find defining numbers for 4-colorings of $M(P_n)$ and $M(C_n)$. We use colors 1, 2, 3 and 4 for a 4-coloring. A simple case-checking together with Theorem A shows that:

Lemma 18 Let S be a defining set for a 4-coloring of $M(P_n)$, $n \geq 3$. Then $S \cap \{u_k, v_k, u_{k+1}, v_{k+1}\} \neq \emptyset$ for $1 \leq k \leq n-1$. Furthermore, if $2 \leq k \leq n-1$ and $\{u_k, v_k\} \subseteq G - S$ then $|S \cap \{u_{k-1}, v_{k-1}, u_{k+1}, v_{k+1}\}| \geq 3$.

Lemma 19 Let S be a defining set for a 4-coloring of $M(P_n)$, $n \geq 4$. If $\{u_k, v_k\} \subseteq G - S$ and $\{u_{k-1}, v_{k-1}\} \subseteq S$ for some k, then either $\{u_{k+1}, v_{k+1}\} \subset S$ or there exists a system of distinct representatives $\{x_{k+1}, ..., x_{k+t-1}\}$ of the sets $\{u_{k+1}, v_{k+1}\}, ...,$ $\{u_{k+t-1}, v_{k+t-1}\}$ for some $2 \leq t \leq n-k$, such that $\{x_{k+1}, ..., x_{k+t-1}, u_{k+t}, v_{k+t}\} \subseteq S$.

Proof. Without loss of generality we may assume $c(w) = 1$. Let $\{u_k, v_k\} \subseteq G - S$ and $\{u_{k-1}, v_{k-1}\} \subseteq S$ for some k. Then $|\{u_{k+1}, v_{k+1}\} \cap S| \geq 1$ by Lemma 18. If $\{u_{k+1}, v_{k+1}\} \subset S$ then we are done. Otherwise we have two cases.

Case 1. $u_{k+1} \in S$ and $v_{k+1} \notin S$.
If $|L(v_k)| \neq 1$, then $\{u_k, v_k, v_{k+1}\}$ is not uniquely colorable by Theorem A. So $|L(v_k)| = 1$. If $1 \notin L(v_k)$, then u_k is not uniquely colorable. So $L(v_k) = \{1\}$. If $|L(v_{k+1})| \neq 1$, then $\{u_k, v_k, v_{k+1}\}$ is not uniquely colorable by Theorem A. So $|L(v_{k+1})| = 1$ and hence $u_{k+2} \in S$. Now either $v_{k+2} \in S$ then we are done or $|L(v_{k+2})| = 1$, hence, $u_{k+3} \in S$. Now a simple induction shows that the statement is true since $\{u_n, v_n\} \subset S$.

Case 2. $v_{k+1} \in S$ and $u_{k+1} \notin S$.
If $|L(v_k)| \neq 1$, then $\{v_k, u_{k+1}\}$ is not uniquely colorable by Theorem A. So $|L(v_k)| = 1$. If $L(v_k) = \{1\}$ then u_{k+1} is not uniquely colorable. So $L(v_k) \neq \{1\}$ and, hence, $c(v_{k-1}) = 1$ or $c(v_{k+1}) = 1$. But then u_k is not uniquely colorable. ■

By Lemmas 18 and 19 if S is a defining set for a 4-coloring of $M(P_n)$ and $\{u_k, v_k\} \cap S = \emptyset$ for some k, then there exist r_k, s_k with $r_k < k < s_k$ such that:

1. $\{u_{r_k}, v_{r_k}\} \subseteq S$;
2. $\{u_{s_k}, v_{s_k}\} \subseteq S$;
3. $|\{u_i, v_i\} \cap S| = 1$ for $r_k < i < s_k$, $i \neq k$.

Now we prove our first main result of this section.

Theorem 20 $d(M(P_n), 4) = 5$ for $n = 2, 3$ and $d(M(P_n), 4) = n + 1$ for $n \geq 4$.

Proof. Notice that $\{u_1, v_1, u_n, v_n\} \subseteq S$ since S is a defining set for a 4-coloring. It is now easy to see that the statement is true for $n = 2, 3$. Let $n \geq 4$ and let S

be a defining set. If $\{u_k, v_k\} \cap S \neq \emptyset$ for each $k = 2, ..., n-1$, then $|S| \geq n+2$. Otherwise, let $A = \{\{u_k, v_k\} | \{u_k, v_k\} \cap S = \emptyset\}$. Since for each $\{u_k, v_k\} \in A$ there is a pair $\{u_{s_k}, v_{s_k}\}$ (see above properties) for which $\{u_{s_k}, v_{s_k}\} \subseteq S$ and $|\{u_i, v_i\} \cap S| = 1$, $k < i < s_k$, it follows that $|S| \geq n+1$. Now we find defining sets S of cardinality $n+1$.

Case 1. $n \equiv 1$ or $3 \pmod 4$.
If $n = 5$ we define $S = \{u_1, v_1, u_3, v_3, u_5, v_5\}$ with $c(u_1) = 2$, $c(v_1) = 3$, $c(u_3) = c(v_3) = 4$, $c(u_5) = 3$ and $c(v_5) = 2$. If $n \geq 7$ we define

$$S = \{u_3, u_{4i+1}, v_{4i+1}, v_{4i+3}, u_{4j+3} \mid 0 \leq i \leq \lfloor \frac{n}{4} \rfloor,\ 1 \leq j \leq \lfloor \frac{n}{4} \rfloor\}$$

with $c(u_3) = 3$, $c(u_{4i+1}) = 2$, $c(v_{4i+1}) = 3$ and $c(v_{4i+3}) = 4$, for $0 \leq i \leq \lfloor \frac{n}{4} \rfloor$, and $c(u_{4j+3}) = 4$ for $1 \leq j \leq \lfloor \frac{n}{4} \rfloor$.

Case 2. $n \equiv 2 \pmod 4$.
If $n = 6$ we define $S = \{u_1, v_1, u_3, v_3, u_4, u_6, v_6\}$ with $c(u_1) = 2$, $c(v_1) = 3$, $c(u_3) = 4$, $c(v_3) = 2$, $c(u_4) = 4$, $c(u_6) = 3$ and $c(v_6) = 2$. If $n \geq 10$ we define

$$S = \{u_{n-3}, u_n, v_n, u_{4i+1}, v_{4j+1}, v_{4j+3}, u_{4k+3} \mid 0 \leq i, j, k \leq \lfloor \frac{n}{4} \rfloor,\ 4j \neq n-2,\ 4k \neq n-6\}$$

with $c(u_{n-3}) = 3$, $c(u_n) = 2$, $c(v_n) = 4$, $c(u_{4i+1}) = 2$ for $0 \leq i \leq \lfloor \frac{n}{4} \rfloor$, $c(v_{4j+1}) = 3$ and $c(v_{4j+3}) = 4$ for $0 \leq j < \lfloor \frac{n}{4} \rfloor$, and $c(u_{4k+3}) = 4$ for $0 \leq k < \lfloor \frac{n}{4} \rfloor - 1$.

Case 3. $n \equiv 0 \pmod 4$.
We define

$$S = \{u_n, v_n, u_{4i+1}, v_{4i+1}, u_{4i+3}, v_{4j+3} \mid 0 \leq i, j < \lfloor \frac{n}{4} \rfloor,\ 4j \neq n-4\}$$

with $c(u_n) = 3$, $c(v_n) = 2$, $c(u_{4i+1}) = 2$, $c(v_{4i+1}) = 3$, $c(u_{4i+3}) = 4$ for $0 \leq i < \lfloor \frac{n}{4} \rfloor$ and $c(v_{4j+3}) = 4$ for $0 \leq j < \lfloor \frac{n}{4} \rfloor - 1$. ∎

Similar to Theorem 20 we have the following result for 4-colorings of $M(C_n)$.

Theorem 21 $d(M(C_n), 4) = n+1$ for $n = 3, 4, 5$ and $d(M(C_n), 4) = n$ for $n \geq 6$.

Proof. We leave the cases $n = 3, 4, 5$ for the reader. Let $n \geq 6$. An argument similar to that described above for $M(P_n)$ shows that $d(M(C_n), 4) \geq n$. Now we find defining sets S of cardinality n.

Case 1. $n \equiv 0 \pmod 4$. Define

$$S = \{u_3, u_{4i+1}, v_{4i+1}, v_{4i+3}, u_{4j+3} \mid 0 \leq i, j < \lfloor \frac{n}{4} \rfloor,\ j \neq 0\}$$

with $c(u_3) = 3$, $c(u_{4i+1}) = 2$, $c(v_{4i+1}) = 3$, $c(v_{4i+3}) = 4$ for $0 \leq i < \lfloor \frac{n}{4} \rfloor$ and $c(u_{4j+3}) = 4$ for $1 \leq j < \lfloor \frac{n}{4} \rfloor$.

Case 2. $n \equiv 1 \pmod 4$. Define

$$S = \{u_3, u_{n-4}, u_{n-1}, v_{n-2}, v_{n-4}, u_{4i+1}, v_{4i+1}, v_{4i+3}, u_{4j+3} \mid 0 \le i \le \lfloor \frac{n}{4} \rfloor - 2, \\ 1 \le j \le \lfloor \frac{n}{4} \rfloor - 1\}$$

with $c(u_3) = c(u_{n-4}) = 3$, $c(u_{n-1}) = 4$, $c(v_{n-2}) = 3$, $c(v_{n-4}) = 2$, $c(u_{4i+1}) = 2$, $c(v_{4i+1}) = 3$ and $c(v_{4i+3}) = 4$ for $0 \le i \le \lfloor \frac{n}{4} \rfloor - 2$, and $c(u_{4j+3}) = 4$ for $1 \le j \le \lfloor \frac{n}{4} \rfloor - 1$.

Case 3. $n \equiv 2 \pmod 4$. We leave the cases $n = 6, 10$ for the reader. For $n \ge 14$ define

$$S = \{u_{n-5}, v_{n-5}, v_{n-1}, v_{n-3}, u_{4i+1}, v_{4j+1}, v_{4j+3}, u_{4k+3} \mid 0 \le i \le \lfloor \frac{n}{4} \rfloor,\ 4i \ne n-6, \\ 0 \le j \le \lfloor \frac{n}{4} \rfloor - 2,\ 0 \le k \le \lfloor \frac{n}{4} \rfloor - 1\}$$

with $c(u_{n-5}) = 3$, $c(v_{n-5}) = 2$, $c(v_{n-1}) = 4$, $c(v_{n-3}) = 3$, $c(u_{4i+1}) = 2$ for $0 \le i \le \lfloor \frac{n}{4} \rfloor$ and $4i \ne n-6$, $c(v_{4j+1}) = 3$ and $c(v_{4j+3}) = 4$ for $0 \le j \le \lfloor \frac{n}{4} \rfloor - 2$, and $c(u_{4k+3}) = 4$ for $0 \le k \le \lfloor \frac{n}{4} \rfloor - 1$.

Case 4. $n \equiv 3 \pmod 4$. Define

$$S = \{u_n, u_{4i+1}, v_{4i+1}, u_{4j+3}, v_{4j+3} \mid 0 \le i \le \lfloor \frac{n}{4} \rfloor,\ 0 \le j \le \lfloor \frac{n}{4} \rfloor - 1\}$$

with $c(u_n) = 4$, $c(u_{4i+1}) = 2$ and $c(v_{4i+1}) = 3$ for $0 \le i \le \lfloor \frac{n}{4} \rfloor$, and $c(u_{4j+3}) = 3$ and $c(v_{4j+3}) = 4$ for $0 \le j \le \lfloor \frac{n}{4} \rfloor - 1$. ■

4 Defining numbers for c-colorings of $M(P_n)$ and $M(C_n)$, $c \ge 5$

When $c = 5$ we have $\chi(M(C_{2n+1})) + 1$ colors for $M(C_{2n+1})$ and $\chi(M(C_{2n})) + 2$ colors for $M(C_{2n})$ and $M(P_n)$. Now since $\deg(v_1), \deg(v_n), \deg(u_i) \le 3$ in $M(P_n)$ for $i = 1, 2, \ldots, n$ it follows that any defining set S for a 5-coloring of $M(P_n)$ contains v_1, v_n and u_i for $1 \le i \le n$. Moreover, $\{v_i, v_{i+1}\} \cap S \ne \emptyset$ for $i = 2, 3, \ldots, n-2$. This leads to $d(M(P_{2n}), 5) \ge 3n + 1$ and $d(M(P_{2n+1}), 5) \ge 3n + 2$.

Theorem 22 $d(M(P_2), 5) = 5$, $d(M(P_3), 5) = 6$ and $d(M(P_{2n}), 5) = 3n + 1$ and $d(M(P_{2n+1}), 5) = 3n + 2$ for $n \ge 2$.

Proof. We leave for the reader to check $d(M(P_2), 5) = 5$, $d(M(P_3), 5) = 6$ and $d(M(P_4), 5) = 7$. Let $n \ge 5$. Since $d(M(P_{2n}), 5) \ge 3n + 1$ and $d(M(P_{2n+1}), 5) \ge 3n + 2$ we only need to find defining sets of cardinalities $3n + 1$ and $3n + 2$ for a 5-coloring of $M(P_{2n})$ and $M(P_{2n+1})$, respectively. First consider $M(P_{2n})$. Define

$$S = \{v_{2n}, u_{4i+1}, v_{4i+1}, u_{4j+3}, v_{4j+3}, u_{2k} \mid i, j \ge 0, \\ 4i + 1 < 2n,\ 4j + 3 < 2n,\ 1 \le k \le n\}$$

with $c(v_{2n}) = 1$, $c(u_1) = 3$, $c(v_1) = 2$, $c(u_{4i+1}) = 2$, $c(v_{4i+1}) = 3$ for $i \ge 1$, $4i+1 < 2n$, $c(u_{4j+3}) = 4$, $c(v_{4j+3}) = 5$ for $j \ge 0$, $4j + 3 < 2n$ and $c(u_{2k}) = 1$ for $1 \le k \le n$. Then

S is a defining set for a 5-coloring of $M(P_{2n+1})$ of cardinality $3n+1$.
Now consider $M(P_{2n+1})$. Define

$$S = \{u_{4i+1}, v_{4i+1}, u_{4j+3}, v_{4j+3}, u_{2k} \mid i, j \geq 0, \\ 4i+1 \leq 2n+1,\ 4j+3 \leq 2n+1,\ 1 \leq k \leq n\}$$

with $c(u_1) = 3$, $c(v_1) = 2$, $c(u_{4i+1}) = 2$, $c(v_{4i+1}) = 3$ for $i \geq 1$, $4i+1 \leq 2n+1$, $c(u_{4j+3}) = 4$, $c(v_{4j+3}) = 5$ for $j \geq 0$, $4j+3 \leq 2n+1$ and $c(u_{2k}) = 1$ for $1 \leq k \leq n$. Then S is a defining set for a 5-coloring of $M(P_{2n+1})$ of cardinality $3n+2$. ■

Similar to Theorem 22 we have the following result for defining numbers for 5-colorings of $M(C_n)$.

Theorem 23 $d(M(C_3), 5) = 6$, $d(M(C_6), 5) = 10$, $d(M(C_{2n}), 5) = 3n$ for $n \geq 2$, $n \neq 3$, and $d(M(C_{2n+1}), 5) = 3n+2$ for $n \geq 2$.

Proof. We leave for the reader to check $d(M(C_3), 5) = d(M(C_4), 5) = 6$ and $d(M(C_6), 5) = 10$. As described above for $M(P_n)$ one can see that $d(M(C_{2n}), 5) \geq 3n$ and $d(M(C_{2n+1}), 5) \geq 3n+2$. Now we find defining sets of cardinalities $3n$ and $3n+2$ for a 5-coloring of $M(C_{2n})$ and $M(C_{2n+1})$, respectively.

Case 1. $2n \equiv 0 \pmod 4$.
Define

$$S = \{u_5, v_5, u_{4i+1}, v_{4i+1}, u_{4j+3}, v_{4j+3}, u_{2k} \mid 0 \leq i < n/2, i \neq 1, \\ 0 \leq j < n/2,\ 1 \leq k \leq n/2\}$$

with $c(u_5) = 3$, $c(v_5) = 2$, $c(u_{4i+1}) = 2$, $c(v_{4i+1}) = 3$ for $0 \leq i < n/2$ and $i \neq 1$, $c(u_{4j+3}) = 4$, $c(v_{4j+3}) = 5$ for $0 \leq j < n/2$, and $c(u_{2k}) = 1$ for $1 \leq k \leq n$.

Case 2. $2n \equiv 2 \pmod 4$.
Define

$$S = \{u_{2n-1}, v_{2n-1}, v_{2n-5}, u_{2n-3}, v_{2n-3}, u_{2n-6}, u_{2n-4}, u_{2n-2}, u_{2n}, \\ u_{4i+1}, v_{4j+1}, u_{4j+3}, v_{4j+3}, u_{2k} \mid 0 \leq i \leq (2n-6)/4, \\ 0 \leq j \leq (2n-10)/4,\ 0 \leq k \leq n-4\}$$

with $c(u_{2n-1}) = 1, c(v_{2n-1}) = 4, c(v_{2n-5}) = 1, c(u_{2n-3}) = 5, c(v_{2n-3}) = 3$, $c(u_{2n-6}) = c(u_{2n-4}) = c(u_{2n-2}) = c(u_{2n}) = 2$, $c(u_{4i+1}) = 2$ for $0 \leq i \leq (2n-6)/4$, $c(v_{4j+1}) = 3$, $c(u_{4j+3}) = 4$ and $c(v_{4j+3}) = 5$ for $0 \leq j \leq (2n-10)/4$ and $c(u_{2k}) = 1$ for $1 \leq k \leq n-4$.

Case 3. $2n+1 \equiv 1 \pmod 4$.
Define

$$S = \{u_{2n}, v_{2n}, u_{2n+1}, u_{4i+1}, v_{4i+1}, u_{4i+3}, v_{4i+3}, u_{2j} \mid 0 \leq i \leq \frac{n-2}{2},\ 0 \leq j \leq n-1\}$$

with $c(u_{2n}) = 4, c(v_{2n}) = 1, c(u_{2n+1}) = 5$, $c(u_{4i+1}) = 2$, $c(v_{4i+1}) = 3$, $c(u_{4i+3}) = 4$ and $c(v_{4i+3}) = 5$ for $0 \leq i \leq \frac{n-2}{2}$, and $c(u_{2j}) = 1$ for $1 \leq j \leq n-1$.

Case 4. $2n+1 \equiv 3 \pmod 4$.
Define

$$S = \{u_{2n-1}, v_{2n-1}, u_{2n+1}, u_{4i+1}, v_{4i+1}, u_{4i+3}, v_{4j+3}, u_{2k} \mid 0 \leq i \leq \frac{n-3}{2}, \\ 0 \leq j \leq \frac{n-1}{2},\ 1 \leq k \leq n\}$$

with $c(u_{2n-1}) = 3$, $c(v_{2n-1}) = 2$, $c(u_{2n+1}) = 1$, $c(u_{4i+1}) = 2$, $c(v_{4i+1}) = 3$ and $c(u_{4i+3}) = 4$ for $0 \leq i \leq \frac{n-3}{2}$, $c(v_{4j+3}) = 5$ for $0 \leq j \leq \frac{n-1}{2}$ and $c(u_{2k}) = 1$ for $1 \leq k \leq n$. ■

Finally, when $c \geq 6$ we obviously have:
$$d(M(P_n), c) = d(M(C_n), c) = \begin{cases} 2n & \text{if} \quad 6 \leq c < n+2 \\ 2n+1 & \text{if} \quad c \geq n+2. \end{cases}$$

5 Defining numbers for c-colorings of $M(K_n)$

In this section we study the defining numbers of vertex colorings of $M(K_n)$ with c colors, where $c \geq \chi(M(K_n)) = n+1$. Throughout this section we assume $n \geq 2$.

Lemma 24 Let S be a defining set for a vertex coloring of $M(K_n)$ with $n+1$ colors and $U = \{u_1, u_2, \ldots, u_n\} \not\subseteq S$, then $w \in S$.

Proof. On the contrary, assume that $w \notin S$. Let $u_k \in U \setminus S$. Then u_k has at most $n-1$ colored neighbors in K_n. So $L(u_k)$ has at least two colors. Since for each color for u_k we can find a color for w, the coloring is not unique. This is a contradiction. ■

Theorem 25 $d(M(K_n), n+1) = n+1$.

Proof. Let S be a defining set for an $(n+1)$-coloring of $M(K_n)$. Obviously, $|S| \geq n$. Assume $|S| = n$. By Theorem A, the set U cannot be a defining set for a vertex coloring of $M(K_n)$. So $S \neq U$ and, hence, $w \in S$ by Lemma 24. Without loss of generality we can assume $c(w) = 1$. Since $|S| = n$ we have $\{u_k, v_k\} \cap S = \emptyset$ for some $1 \leq k \leq n$. Without loss of generality we may assume $k = 1$. This forces $v_i \in S$ and $c(v_i) \neq 1$ for $2 \leq i \leq n$, otherwise u_1 is not uniquely colorable. Now since there are $n+1$ colors, vertex v_1 is not uniquely colorable. Hence any defining set S has at least $n+1$ vertices. On the other hand, if $S = \{w, v_1, v_2, \ldots, v_n\}$ with $c(w) = 1$ and $c(v_i) = i+1$ for $1 \leq i \leq n$, then S is a defining set of cardinality $n+1$. ■

Theorem 26 $d(M(K_n), n+i) = n+i+1$ for $i = 2, ..., n$.

Proof. Let S be a defining set for a vertex coloring of $M(K_n)$ with $n+i$ colors. Since $\deg(w) = \deg(u_j) = n$, the vertices w and u_j must be in S for $1 \leq j \leq n$. If $|S| \leq n+i$, then any vertex $x \in V \setminus S$ is not uniquely colorable, a contradiction. So $S \geq n+i+1$. Now define $S = \{w, u_j, v_k \mid 1 \leq j \leq n,\ 1 \leq k \leq i\}$ with $c(w) = 1$, $c(u_j) = j+1$ for $1 \leq j \leq n$, $c(v_1) = 1$ and $c(v_k) = n+k$ for $2 \leq k \leq i$. Then S is a defining set of cardinality $n+i+1$. ■

6 Defining numbers for c-colorings of $M(K_{m,n})$

In this section we study the defining numbers of vertex colorings of $M(K_{m,n})$. Since $\chi(K_{m,n}) = 2$ we have $\chi(M(K_{m,n})) = 3$ by Theorem B. First we settle the case $m = 1$.
Let $V = V(K_{1,n}) = \{v_1, ..., v_n, \theta\}$ and $U = \{u_1, ..., u_n, \theta'\}$, where $\deg(\theta) = 2n$ and $\deg(\theta') = n + 1$.

Theorem 27 $d(M(K_{1,n}), 3) = 3$.

Proof. It is obvious that $d(M(K_{1,n}), 3) \geq 2$. One can also notice that the vertex w and a vertex of V or of U, a vertex of V and a vertex of U, two vertices of V, or two vertices of U cannot uniquely determine a 3-coloring of $M(K_{1,n})$. Hence $d(M(K_{1,n}), 3) \geq 3$. On the other hand, $S = \{w, \theta, \theta'\}$ with $c(w) = 1$, $c(\theta) = 2$ and $c(\theta') = 3$ is a defining set. So the result follows. ■

Let S be a defining set for a vertex coloring of $M(K_{1,n})$ with at least four colors. Then $u_i, v_i \in S$ for $1 \leq i \leq n$ (note that $\deg(u_i), \deg(v_i) = 2$). Now define $S = \{u_i, v_i \mid 1 \leq i \leq n\}$ with $c(u_1) = c(v_1) = 1$, $c(u_2) = c(v_j) = 2$ for $2 \leq j \leq n$ and $c(u_k) = 3$ for $3 \leq k \leq n$. Then S is a defining set. Therefore, $d(M(K_{1,n}), 4) = 2n$. Similarly, $d(M(K_{1,n}), c) = 2n$ for $5 \leq c \leq n + 1$, $d(M(K_{1,n}), n + 2) = 2n + 1$, $d(M(K_{1,n}), c) = 2n + 2$ for $n + 3 \leq c \leq 2n + 1$ and $d(M(K_{1,n}), 2n + 2) = 2n + 3$.

Now we study the defining number for vertex colorings of $M(K_{m,n})$ with 3 colors, where $2 \leq m \leq n$. First we partition the vertices of $M(K_{m,n})$ as follows (see Figure 6). The sets B and D are the partite sets of our original $K_{m,n}$ with $|B| = n$ and $|D| = m$. The set A has m and the set C has n vertices. Moreover, w is adjacent to every vertex of A and C. Notice that $\langle A, B\rangle \cong \langle B, D\rangle \cong \langle D, C\rangle \cong K_{m,n}$.

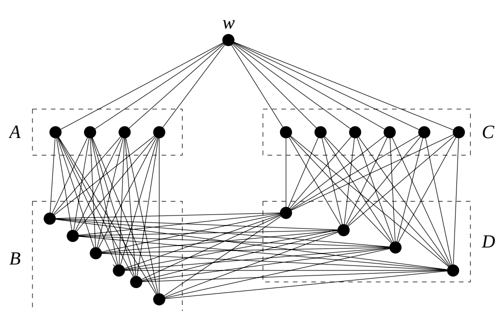

Figure 6: $M(K_{4,6})$

Lemma 28 Let $2 \leq m \leq n$ and let S be a subset of vertices of $M(K_{m,n})$ with $|S| \leq m$. Then S is not a defining set for a 3-coloring of $M(K_{m,n})$.

Proof. On the contrary, let S be a defining set for a 3-coloring of $M(K_{m,n})$. It is easy to see that:

1) $S \setminus \{w\} \not\subseteq X$ for $X \in \{A, B, C, D\}$;
2) $S \cap X \neq \emptyset$ for $X \in \{A, B, C, D\}$;
3) If $x, y \in S \cap X$ then $c(x) = c(y)$.
Now it is straightforward to check that for every coloring of the vertices of S there are at least two different 3-colorings of $M(K_{m,n})$ containing S. This is a contradiction. ■

Theorem 29 *Let $2 \leq m \leq n$. Then $d(M(K_{m,n}), 3) = m + 1$.*

Proof. By Lemma 28 we only need to find a defining set of size $m + 1$. Let $S = D \cup \{w\}$ with $c(w) = c(d) = 1$ for a fixed vertex $d \in D$, and $c(x) = 2$ for every vertex $x \in D \setminus \{d\}$. Then S is a defining set of cardinality $m + 1$. ■

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