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Edge-reconstruction of the decay number of a connected graph*

Han-yuan Deng

Department of Mathematics
Hunan Normal University
Changsha, Hunan 410081
P. R. China

Abstract

The decay number $\zeta(G)$ of a connected graph G is the smallest number of components a cotree of G can have. In this paper we show that the decay number $\zeta(G)$ can be determined from the values $\zeta(G-e)$ on the edge-deleted subgraphs of G. In particular, the decay number is edge-reconstructible.

1 Introduction

Graphs in this paper are finite and can have multiple edges and loops; they are multigraphs in the sense of [1]. For a connected graph G, its *decay number* $\zeta(G)$ is defined by setting

$$\zeta(G) = \min\{c(G - E(T));\ T \textit{ is a spanning tree of } G\}.$$

where $c(H)$ denotes the number of components of a graph H. This invariant was defined by Škoviera in [2] and was used for studying the maximum genus of a graph. Nebeský [3] found the following characterization of the decay number of a graph.

Theorem A (Nebeský [3]). *Let G be a connected graph. Then*

$$\zeta(G) = \max\{2c(G - A) - |A| - 1; A \subseteq E(G)\}.$$

Later Škoviera [4] established a different but related characterization:

$$\zeta(G) = \max\{2l(G - A) - |A|; A \subseteq E(G)\}$$

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where $l(G-A)$ denotes the number of leaves of $G-A$. (A leaf of a graph G is any 2-edge-connected subgraph of G, trivial or not, maximal with respect to inclusion.)

The purpose of this note is to show that for every connected graph G the value $\zeta(G)$ can be determined from the values $\zeta(G-e)$ of all edge-deleted subgraphs $G-e$ of G.

The famous Reconstruction Conjecture [5,7] claims that a graph with at least three vertices can be reconstructed up to isomorphism if we know all of its vertex-deleted subgraphs up to isomorphism. The edge-analogue of this conjecture is the Edge-Reconstruction Conjecture which claims that a graph with at least four edges can be reconstructed up to isomorphism if we know all of its edge-deleted subgraphs up to isomorphism. Many results concerning both conjectures have been obtained but the conjectures themselves remain elusive [5-8]. One possible approach to these conjectures is therefore to determine which graph invariants or properties can be determined from the set of all vertex-deleted subgraphs or edge-deleted subgraphs. Such properties are called vertex-reconstructible or edge-reconstructible, respectively. More precisely, a graph invariant $\pi(G)$ is *edge-reconstructible* if it can be determined by the set $\{\pi(G-e); e \in E(G)\}$.

Our main result shows that the decay number of a graph is edge-reconstructible.

Theorem B *Let G be a connected and bridgeless graph. Then*

$$\zeta(G) = \begin{cases} 1, & \text{if there is } e \in E(G) \text{ such that } \zeta(G-e) = 1 \\ \max\{\zeta(G-e); e \in E\} - 1, & \text{otherwise.} \end{cases}$$

Theorem C *The decay number $\zeta(G)$ of a connected graph G is edge-reconstructible.*

2 Proofs

In this section, we prove the main results. First, from the definition of decay number, we note that $\zeta(G) = \zeta(G_1) + \zeta(G_2)$ if e is a bridge of G and G_1 and G_2 are the components of $G-e$.

Next, we prove two properties of $\zeta(G)$ to be used later.

Lemma 1 *Let G be a connected and bridgeless graph, then*

$$\zeta(G) \leq \zeta(G-e) \leq \zeta(G) + 1$$

for every edge e of G.

Proof. The left inequality is obvious from the definition. We prove the right inequality. Choose a spanning tree T of G such that $c(G-E(T)) = \zeta(G)$. If e does not belong to T, then T is a spanning tree of $G-e$, and the claim follows immediately because the removal of e can disconnect at most one component of $G-E(T)$. If

e belongs to T, then it suffices to find a spanning tree T' of G such that e is not contained in T' and $c(G - E(T')) \leq c(G - E(T))$.

Since e is not a bridge, there exists an edge f of $G - E(T)$ whose end-vertices belong to different components of $T - e$. If the end-vertices of e belong to the same component C of $G - E(T)$, we choose f from C. It follows that e lies on the cycle of $T + f$, so $T' = T + f - e$ is a spanning tree of G not containing e. Moreover, in both cases we have $c(G - E(T')) \leq c(G - E(T))$, and the result follows.

Lemma 2 *Let G be a connected and bridgeless graph. If $\zeta(G) \geq 2$, then G contains an edge $f \in E(G)$ such that $\zeta(G) = \zeta(G - f) - 1$.*

Proof. By Theorem A, there is a set $A \subseteq E(G)$ such that $\zeta(G) = 2c(G-A)-|A|-1$. Note that A is nonempty since G is connected and $\zeta(G) \geq 2$. Take any $f \in A$ and set $A' = A - \{f\}$. Then $G - f$ is connected and $c(G - f - A') = c(G - A)$. Moreover,

$$\zeta(G - f) \geq 2c(G - f - A') - |A'| - 1 = 2c(G - A) - |A| = \zeta(G) + 1$$

By Lemma 1, $\zeta(G - f) \leq \zeta(G) + 1$. So $\zeta(G - f) = \zeta(G) + 1$, i.e., $\zeta(G) = \zeta(G - f) - 1$.

Finally, we prove Theorem B and Theorem C.

Proof of Theorem B. If G contains an edge e such that $\zeta(G - e) = 1$, then by Lemma 1 we have that $1 \leq \zeta(G) \leq \zeta(G - e) = 1$. So $\zeta(G) = 1$, as claimed. Otherwise, $\zeta(G - e) \geq 2$ for each edge e. We distinguish two cases.

Case 1. There is an edge $e \in E(G)$ such that $\zeta(G - e) \geq 3$. Then $\zeta(G) \geq \zeta(G - e) - 1 \geq 2$ by Lemma 1, and there is an edge $f \in E(G)$ such that $\zeta(G) = \zeta(G - f) - 1$ by Lemma 2. Again, for any $e \in E(G)$, $\zeta(G) \geq \zeta(G - e) - 1$ by Lemma 1. So, $\zeta(G) = \max\{\zeta(G - e); e \in E(G)\} - 1$.

Case 2. Assume that $\zeta(G - e) = 2$ for each edge e of G. Then $\zeta(G) = 1$, for otherwise Lemma 2 would provide an edge f such that $\zeta(G) = \zeta(G - f) - 1 = 1$, which is a contradiction.

Proof of Theorem C. If G is bridgeless, then $G - e$ is connected for each edge $e \in E(G)$, and $\zeta(G)$ is edge-reconstructible by Theorem B. If G contains a bridge $e \in E(G)$, then $\zeta(G) = \zeta(G_1) + \zeta(G_2)$, where G_1 and G_2 are the components of $G - e$. $\zeta(G)$ is also edge-reconstructible.

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