AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **36** (2006), Pages 83–90

Finite and infinite hypercubes as direct products

Wilfried Imrich

Montanuniversität Leoben
A-8700 Leoben
Austria
wilfried.imrich@uni-leoben.at

Douglas Rall

Furman University
Greenville, SC 29613
U.S.A.
drall@herky.furman.edu

Abstract

We characterize the factorizations of finite or infinite hypercubes with respect to the direct product in the class of unoriented simple graphs with loops. The paper extends the corresponding result for finite graphs (B. Brešar, W. Imrich, S. Klavžar and B. Zmazek, Hypercubes as direct products, *SIAM J. Discrete Math.* 18 (2005), 779–786). It is based on a new approach that yields a simple, unified proof for both the finite and the infinite case. The main theorem also characterizes the graphs whose Kronecker coverings are hypercubes.

1 Introduction

This paper is concerned with the decomposition of finite and infinite hypercubes with respect to the direct product. It generalizes the results of [1] to infinite graphs. It is shown that every decomposition of the k-dimensional hypercube Q_k with respect to the direct product, where k is a finite or infinite cardinal, is of the form $K_2 \times G$ where K_2 is the complete graph on two vertices and G a non-bipartite graph that contains a spanning subgraph S that is isomorphic to Q_{k-1}. Furthermore, G is characterized by the condition that the endpoints of the edges in $E(G) \setminus E(S)$ have even distance in S and induce an involution of S.

The main result in [1] is the existence of a spanning hypercube in every graph G with the property that $K_2 \times G$ is a hypercube. It is effected by an application of Graham's density lemma [2] for (the number of edges of) subgraphs of the hypercube, a method that does not extend to the infinite case. Here Graham's density lemma is replaced by application of metric properties of the sets of parallel edges in hypercubes and symmetries induced by these sets.

To prove that every factorization has exactly two factors, where one of them is always a K_2, the Cartesian skeleton is invoked in [1]. This argument is replaced by a short direct proof.

The paper is organized as follows. The next section contains the main definitions and basic facts about the direct product and the connection with coverings. Then comes the proof of the main result, that is Lemma 2. Formally this lemma is identical to the Lemma 2 in [1], except that the new proof is also valid for the infinite case as will be displayed in the section on infinite hypercubes. The main difference to the proof in [1] is that it does not refer to Graham's density lemma.

Then follow the definition of the Cartesian product of infinitely many graphs, of the weak Cartesian product and of the infinite hypercube. It will then be clear that Lemma 2 also holds in the infinite case. The paper ends with a proof of the fact that all factorizations of the hypercube with respect to the direct product have exactly two factors and a complete statement of the main result.

2 Preliminaries

All graphs considered here are undirected graphs that may contain loops but not multiple edges. In this section we shall define the direct and the Cartesian product of graphs and collect the main properties needed in the sequel. For both products there exists an extensive literature. Let us only mention that R. McKenzie [5] and Sabidussi [6] were the first to investigate these products for infinite graphs. For a survey of graph products in general the interested reader is referred to [4].

The *direct product* $G \times H$ of two graphs G and H is defined on the Cartesian product $V(G) \times V(H)$ of the vertex sets of the factors. Its edge set is the set of all pairs of vertices $(a, x), (b, y) \in V(G) \times V(H)$ where $ab \in E(G)$ and $xy \in E(H)$. It is commutative, associative and the one-vertex graph with a loop is a unit.

Figure 1 depicts the direct products $K_2 \times P_3^*$, where P_3^* is a path of length 2 with loops added to its endpoints, and $K_2 \times K_3$. In both cases the product is a cycle of length 6.

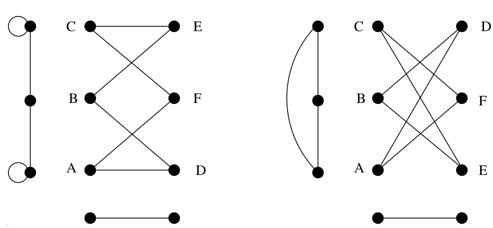

Figure 1: Two decompositions of C_6

The *Cartesian product* $G \square H$ has the same vertex set as the direct product. Its edge set consists of all pairs (a, x), (b, y) with $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$. It is also commutative and associative with K_1 as a unit.

The subgraph of $G \Box H$ induced by the vertices $(a, x), x \in V(H)$, is called an H-*layer* of $G \Box H$ and is denoted by $H^{(a,x)}$. Note that any H-layer is isomorphic to H. Analogously one defines G-layers. The d-dimensional *hypercube* or d-*cube* Q_d is the Cartesian product of d copies of the complete graph K_2 on two vertices. So $Q_1 = K_2$ and we also set $Q_0 = K_1$. The vertices of Q_d can be considered to be all binary vectors of length d. Two such vertices $x = (x_1, x_2, \ldots, x_d)$ and $y = (y_1, y_2, \ldots, y_d)$ are adjacent in Q_d if and only if there exists an index i such that $x_i = 1 - y_i$ and $x_j = y_j$ for $j \neq i$.

Let $Q_{d-1} \Box K_2$ be an arbitrary factorization of Q_d. The edges between the two Q_{d-1}-layers are said to be of the same *color* or are *parallel* in Q_d. The set of all edges between two Q_{d-1}- layers will be referred to as a *color class* or a *parallel class* of the edge set of the factorization. Such classes are also equivalence classes with respect to the Djoković-Winkler relation Θ as defined in [4, p. 48]. We will therefore denote such classes containing the edge e by Θ_e. The main property (cf. [4, Lemma 2.3]) of this relation that we shall use is the fact that in a bipartite graph two edges $e = [u, v]$, $f = [x, y]$ are in the relation Θ if and only if the notation can be chosen such that

$$d(u, x) = d(v, y) = d(u, y) - 1 = d(v, x) - 1.$$

Layers of direct products are defined analogously to those of the Cartesian product. In the case of the direct product the layer $H^{(a,x)}$ is isomorphic to H only if a carries a loop (in G), otherwise the edge-set of $H^{(a,x)}$ is empty, cf. Figure 1.

For a given graph G it follows directly from the definition of the direct product that interchanging vertices between the G-layers is an automorphism of $K_2 \times G$. In fact, direct products by K_2 can be characterized by *involutions*, that is by automorphisms of order two.

Lemma 1 *Suppose that H is bipartite. Then there exists a graph G such that H is isomorphic to $K_2 \times G$ if and only if H has an involution that interchanges the color classes of $V(H)$.*

Proof. If H can be factored as $K_2 \times G$ for some graph G, then the involution φ defined by $\varphi(i, x) = (1 - i, x)$ for $i = 0, 1$ is such an automorphism of H. Conversely, suppose H has color classes $C_0 = \{u_1, u_2, \ldots, u_n\}$ and $C_1 = \{v_1, v_2, \ldots, v_n\}$ together with an automorphism φ such that $\varphi(u_i) = v_i$ and $\varphi(v_i) = u_i$ for each i. We construct a graph G on the vertex set $\{w_1, w_2, \ldots, w_n\}$ by including the edge $w_i w_j$ in G if and only if $u_i v_j \in E(H)$. It follows immediately from the definition of the direct product and the assumptions about φ that H is isomorphic to $K_2 \times G$. □

We note that if the involution φ in the above lemma has the property that $\varphi(x)$ is not adjacent to x for any vertex x of H, then the resulting factor G has no loops. However, each pair of vertices x and $\varphi(x)$ that are adjacent in H gives rise to a loop in G. The former case is illustrated in the second factorization of C_6 and the latter by the first factorization in Figure 1.

To establish the connection with covering spaces we recall that a graph $\tilde{G}$ is said to be a covering graph of a graph G if there exists a surjective homomorphism (called a covering) $f\colon \tilde{G} \to G$ such that for every vertex v of $\tilde{G}$ the set of edges incident with v is mapped bijectively onto the set of edges incident with $f(v)$. A covering f is k-fold if the preimage of every vertex of G consists of k vertices. Clearly $K_2 \times G$ is a 2-fold covering of G.

To simplify the description of large graphs, the concept of *voltage graphs* is generally used, see for example [3] or [7]. Then the $\mathbb{Z}_2$-covering graph of G with voltages 1 on all edges is known as the *Kronecker cover* or the *canonical double cover* of G. Is is identical with $K_2 \times G$.

3 The main result

For a bipartite graph G with bipartition $V(G) = X + Y$ we call an involution α *bipartite* if $\alpha(X) = X$. For such an involution we let G^α denote the graph obtained from G by addition of the edges $\{uv \mid u = \alpha(v), v \in V(G)\}$.

It is not hard to see that $K_2 \times Q_{k-1}^\alpha$ is isomorphic to Q_k for any bipartite involution. We wish to show that every graph G with $Q_k \cong K_2 \times G$ is of that form. The main step in the proof is the following lemma.

Lemma 2 *If G is a connected, nonbipartite graph such that $K_2 \times G$ is a k-dimensional hypercube, then G has a spanning subgraph isomorphic to a $(k-1)$-dimensional hypercube.*

Proof. Assume that G is connected and nonbipartite such that $H = K_2 \times G$ is isomorphic to Q_k. We denote the vertex set of K_2 by $\{0, 1\}$ and the projection map from H onto G by p_G. The idea of the proof is to find a parallel class Θ of edges in the hypercube Q_k such that for each $e \in \Theta$ either $p_G(e)$ is a loop or there is another edge $f \in \Theta$ with $p_G(e) = p_G(f)$.

Choose a odd cycle $C = v_1, v_2, \ldots, v_{2\ell+1}, v_1$ of shortest length in G. If C is a loop, then there is an edge e in H projecting onto it, so assume that $\ell \geq 1$. Consider the subgraph $K_2 \times C \simeq C_{4\ell+2}$ of H. The edges $e = [a, d]$ and $f = [b, c]$ belong to $K_2 \times C$ where $a = (0, v_1), b = (0, v_{2\ell+1}), c = (1, v_1)$ and $d = (1, v_{2\ell+1})$. Denote by Θ_e the parallel class of H containing the edge e.

Suppose $d_H(a, b) < 2\ell$. Since a and b belong to the same color class of H, $d_H(a, b) = 2r < 2\ell$. If P is a shortest path in H from a to b, then P projects under the homomorphism p_G to a walk Q of length $2r$ joining v_1 and $v_{2\ell+1}$. If P does not contain a pair $(0, x)$ and $(1, x)$ for some vertex x of G, then adding the edge $v_1 v_{2\ell+1}$ to Q yields an odd cycle of length $2r + 1$. Hence, there exist vertices $(0, v)$ and $(1, v)$ in P. From among all such pairs choose $(0, w)$ and $(1, w)$ that are closest together along P. The segment of P from $(0, w)$ to $(1, w)$ projects onto an odd cycle in G whose length is less than $2r$. This contradicts the choice of C, and so $d_H(a, b) = 2\ell = d_H(c, d)$. It follows that $f \in \Theta_e$.

Let R be a shortest path in H from a to c. Since a and c are from different color classes of H, the length of R is odd. The image $p_G(R)$ is a closed walk beginning and ending at v_1 that must contain an odd cycle. For if not, then its edges induce a bipartite subgraph of G that has a closed walk of odd length. Therefore, the length of $p_G(R)$, and hence also of R, is at least $2\ell+1$. We conclude that $d_H(c,d) < d_H(c,a)$, and so c and d belong to the same component of $H \setminus \Theta_e$.

Let $g = [x, y]$ be an arbitrary edge of H where $x = (0, u)$ and $y = (1, v)$, and suppose that $g \in \Theta_e$. The involution φ that interchanges vertices $(0, t)$ and $(1, t)$ for all $t \in V(G)$ leaves the parallel classes of H invariant and $\varphi(e) = f$. Thus, $\varphi(g) = [(1, u), (0, v)]$ is also in Θ_e. That is, for all $u, v \in V(G)$, $[(0, u), (1, v)] \in \Theta_e$ implies that $[(1, u), (0, v)] \in \Theta_e$.

The graph $H \setminus \Theta_e$ consists of two components, say S_1 and S_2 each of which is a hypercube of dimension $k-1$. To complete the proof we will now show that p_G is injective when restricted to S_1 or to S_2. We assume without loss of generality that $a, b \in S_1$ and that $c, d \in S_2$. Consider the following partition $V(H) = A \cup B \cup C \cup D$ depicted in Figure 2 where

$$A = \bigcup_{(0,x)\in S_1,\,(1,x)\in S_1} \{(0,x),(1,x)\} \qquad B = \bigcup_{(0,x)\in S_2,\,(1,x)\in S_1} \{(0,x),(1,x)\}$$

$$C = \bigcup_{(0,x)\in S_1,\,(1,x)\in S_2} \{(0,x),(1,x)\} \qquad D = \bigcup_{(0,x)\in S_2,\,(1,x)\in S_2} \{(0,x),(1,x)\}.$$

Since $a \in S_1$ and $c \in S_2$ and $p_G(a) = p_G(c)$, it follows that $a, c \in C$ and hence $C \neq \emptyset$. Suppose $A \neq \emptyset$. Let $u = (0, x) \in A$. By definition of A, $v = (1, x) \in A$ and both of u and v belong to S_1. Each vertex of S_1 and each vertex of S_2 is incident with exactly one edge that belongs to Θ_e. Let $w, z \in S_2$ such that $[u, w] \in \Theta_e$ and $[v, z] \in \Theta_e$. Since H is bipartite it follows that $w \in \{1\} \times G$ and that $z \in \{0\} \times G$, and therefore $w, z \in D$. The hypercube S_1 is connected so there must be an edge h in S_1 from A to $B \cap S_1$ or $C \cap S_1$. Without loss of generality we can assume that h connects A with $C \cap S_1$ and that it is of the form $h = [(1, t), (0, s)]$, where $(1, t) \in A \subset S_1$ and $(0, s) \in S_1 \cap C$.

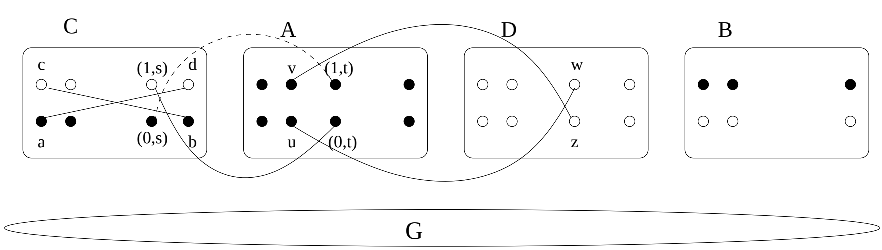

Figure 2: The partition $V(H) = A \cup B \cup C \cup D$

This implies that s and t are adjacent in G, and hence $g = [(1, s), (0, t)]$ is an edge of H. But $(0, t) \in S_1$ and $(1, s) \in S_2$. Hence g is the unique edge of H incident with $(0, t)$ that belongs to the parallel class Θ_e. By the above argument it follows that h must also belong to Θ_e, but this is a contradiction since h is incident with two vertices of S_1.

Therefore, $A = \emptyset$, and we conclude that p_G is injective on S_1. That is, G has a spanning hypercube. □

Corollary 3 *The hypercube Q_k is representable as a product of the form $G \times K_2$ if and only if G is isomorphic to Q_{k-1}^{α} for some bipartite involution α of Q_{k-1}.*

Proof. Suppose $G \times K_2 = Q_k$. Then the projections of endpoints of the edges in Θ_e induce α. In other words, let $[u, v] \in \Theta_e$. Then $\alpha(p_G u) = p_G v$.

The converse is just as easy. □

4 Infinite hypercubes

We have introduced the hypercube as the Cartesian product of finitely many complete graphs on two vertices. So we begin with the generalization of the Cartesian product of finitely many factors to that of infinitely many. Clearly the vertex set of $G_1 \square \cdots \square G_n$ consists of the coordinate vectors $v = (v_1, v_2, \ldots, v_n)$ where $v_i \in V(G_i)$, and the edge set of $G_1 \square \cdots \square G_n$ is the set of all unordered pairs $[u, v] = [(u_1, u_2, \ldots, u_n), ((v_1, v_2, \ldots, v_n)]$ for which there exists a $k \in \{1, 2, \ldots n\}$ such that $[u_k, v_k] \in E(G_k)$ and $u_i = v_i$ for $i \neq k, i \in \{1, 2, \ldots n\}$.

For infinitely many factors, we replace the coordinate vector by a function from an index set into the sets of vertices of the factors. Thus, let I be an index set and G_ι, $\iota \in I$, be a family of graphs. Then the Cartesian product

$$G = \prod_{\iota \in I} G_\iota$$

is defined on the set x of all functions $x : \iota \mapsto x_\iota$, $x_\iota \in V(G_\iota)$, where two vertices x, y are adjacent if there exist a $\kappa \in I$ such that $[x_\kappa, y_\kappa] \in E(G_\kappa)$ and $x_\iota = y_\iota$ for $\iota \in I \setminus \{\kappa\}$. We call the x_ι the coordinates of x.

For finite I this clearly coincides with the usual definition, and in this case the product is connected if and only if the factors are. However, if we have infinitely many nontrivial factors, there are vertices that differ in infinitely many coordinates. They cannot be connected by paths of finite length, since the endpoints of every edge differ in just one coordinate, so the product is disconnected.

The connected components of G are called *weak Cartesian products*. To identify a component, it suffices to specify an arbitrary vertex of it. Thus the weak Cartesian product $G = \prod_{\iota \in I}^{a} G_\iota$ is the connected component of $\prod_{\iota \in I} G_\iota$ containing a.

It is easy to see that the components of the Cartesian product of infinitely many factors K_2 are pairwise isomorphic. We can thus define the $\mathfrak{n}$-dimensional hypercube $Q_{\mathfrak{n}}$ as the weak Cartesian product $\prod_{\iota \in I}^{a} G_\iota$, where all G_ι are isomorphic to K_2, $a_\iota = 0$ for all $\iota \in I$, and $|I| = \mathfrak{n}$.

One can show that every connected graph is uniquely representable as a weak Cartesian product of connected prime graphs. This result is due to Imrich and Miller, for a proof cf. [4]. For us this implies that there are no other representations of $Q_{\mathfrak{n}}$ as a weak Cartesian product.

More importantly, we note that $K_2 \Box Q_{\mathfrak{n}} = Q_{\mathfrak{n}}$ and that we can define parallel edges as in the finite case. Removal of a set of parallel edges from $Q_{\mathfrak{n}}$ yields two isomorphic copies of $Q_{\mathfrak{n}}$ and the set of parallel edges induces an isomorphism between these copies in a natural way, just as in the case of finite graphs. Moreover, the characterization of parallel edges by the distances between their endpoint is the same as in the finite case.

Therefore, both Lemma 2 and Corollary 3 also hold for the infinite hypercube.

5 Two factors only

We begin with a direct product $G = G_1 \times G_2$ and observe that the neighborhood $N_G(u)$ of $u = (u_1, u_2)$ in G is the Cartesian product, $N_{G_1}(u_1) \times N_{G_2}(u_2)$, of the neighborhood $N_{G_1}(u_1)$ of u_1 in G_1 by the neighborhood $N_{G_2}(u_2)$ of u_2 in G_2. For $v = (v_1, v_2)$ it is clear that

$$N_G(u) \cap N_G(v) = (N_{G_1}(u_1) \cap N_{G_1}(v_1)) \times (N_{G_2}(u_2) \cap N_{G_2}(v_2)).$$

In a hypercube Q, whether finite or infinite, the neighborhoods of two different vertices are either disjoint or have exactly two vertices in common. Thus, in our case where $Q = G_1 \times G_2$, $N_Q(u) \cap N_Q(v)$ is empty or has exactly two elements if u and v are distinct. Also, $N_Q(u) \cap N_Q(v) = N_Q(u) \cap N_Q(w) \neq \emptyset$ and $u \neq v$ implies that $v = w$, otherwise Q would contain a $K_{2,3}$. Of course this generalizes in the obvious way to any number - finite or infinite - of factors.

Theorem 4 *Let Q be a finite or infinite hypercube and assume that Q has a direct product decomposition $Q = G_1 \times G_2 \times \cdots \times G_k$. Then $k = 2$ and one of the two factor graphs is K_2.*

Proof. Since Q is connected and bipartite, exactly one of $G_1, G_2, \ldots, G_k$ is bipartite. Assume first that there exists an i such that some vertex $x \in G_i$ has $r \geq 3$ neighbors. Suppose also that there is a $j \neq i$ and a vertex $y \in G_j$ such that $N_{G_j}(y)$ contains two distinct vertices y_1 and y_2. We may assume that $i = 1$ and $j = 2$. Choose a vertex $w_n \in G_n$ for each $n \geq 3$. If we let $u = (x, y_1, w_3, \ldots, w_k)$ and $v = (x, y_2, w_3, \ldots, w_k)$, then we see that $|N_Q(u) \cap N_Q(v)| \geq r$. This contradiction implies that for each $j \neq i$, every vertex in G_j is adjacent to exactly one vertex, and so $k = 2$ and $G_2 = K_2$.

Therefore, if $k \geq 3$, then it must be the case that for every $1 \leq i \leq k$ and for every vertex x in G_i, $|N_{G_i}(x)| \leq 2$. Suppose that there exist $x \in G_1$, $y \in G_2$ and $z \in G_3$ such that x has distinct neighbors x_1 and x_2, y has distinct neighbors y_1 and y_2, and z has distinct neighbors z_1 and z_2. With w_n chosen as above, $u = (x, y, z_1, w_4, \ldots, w_k)$ and $v = (x, y, z_2, w_4, \ldots, w_k)$ we arrive at the contradiction $|N_Q(u) \cap N_Q(v)| \geq 4$. That is, if $k \geq 3$, then at most two of the factors have a vertex with more than one neighbor and all vertices in each remaining factor have degree one.

Since exactly one of the factor graphs is bipartite it follows that $k \leq 3$. If $k = 3$, then one of the factors is K_2. In this case $Q = G_1 \times G_2 \times K_2$, and no vertex of Q has degree more than 4. So, Q must be one of Q_2, Q_3 or Q_4. It is straightforward to check that none of these hypercubes has a direct product factorization unless $k = 2$ and one of the factors is K_2. □

We have thus shown

Theorem 5 *Every decomposition of a nontrivial hypercube Q into a direct product has exactly two factors. One factor is always K_2 and the other one any of the graphs Q^{α}_{k-1} for a bipartite involution α of Q_{k-1} if Q has finite dimension k, or $Q^{\alpha}_{\mathfrak{n}}$ for a bipartite involution α of $Q_{\mathfrak{n}}$ if Q has infinite dimension $\mathfrak{n}$.*

Corollary 6 *The Kronecker cover of a graph G is a hypercube if and only if G is isomorphic to a graph Q^{α}_{k} or $Q^{\alpha}_{\mathfrak{n}}$.*

References

[1] B. Brešar, W. Imrich, S. Klavžar and B. Zmazek, Hypercubes as direct products, *SIAM J. Discrete Math.* 18 (2005), 779–786.

[2] R. L. Graham, On primitive graphs and optimal vertex assignments, *Ann. New York Acad. Sci.* 175 (1970), 170–186.

[3] J. L. Gross and T. W. Tucker, *Topological Graph Theory*, Wiley Interscience, 1987.

[4] W. Imrich and S. Klavžar, *Product Graphs*, Wiley-Interscience, New York, NY, 2000.

[5] R. McKenzie, Cardinal multiplication of structures with a reflexive relation, *Fund. Math.* 70 (1971), 59–101.

[6] G. Sabidussi, Graph multiplication, *Math. Z.* 72 (1960), 446–457.

[7] A. T. White, *Graphs of Groups on Surfaces*, North-Holland Mathematics Studies 188, North-Holland Publishing Co., Amsterdam 2001.

(Received 13 Apr 2005)