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On the critical group of the Möbius ladder graph*

Pingge Chen   Yaoping Hou

Department of Mathematics
Hunan Normal University
Changsha, Hunan 410081
China

Chingwah Woo

Department of Mathematics
City University of Hong Kong
Hong Kong
China

Abstract

The critical group of a connected graph is a finite abelian group whose order is the number of spanning trees and whose structure is a subtle isomorphism invariant of the graph. In this paper we study the structure of the critical group on the Möbius ladder, and we prove that the Smith normal form of the critical group is not cyclic but is always the direct sum of two or three cyclic groups.

1 Introduction

The critical group of a connected graph is a finite abelian group whose structure is a subtle isomorphism invariant of the graph. It is closely connected with the graph Laplacian as follows:

Let G be a finite simple graph with n vertices. Then its Laplacian matrix $L(G) = D(G) - A(G)$, where $D(G) = \operatorname{diag}(d_1, d_2, \cdots, d_n)$ is the degree matrix and $A(G)$ is the adjacency matrix of G. Thinking of $L(G)$ as a map $\mathbf{Z}^n \to \mathbf{Z}^n$, its *cokernel* has the form

$$\operatorname{coker} L(G) = \mathbf{Z}^n / L(G)\mathbf{Z}^n \cong \mathbf{Z} \oplus \mathcal{C}(G)$$

where $\mathcal{C}(G)$ is defined to be the *critical group* of G.

It follows from the Matrix-tree Theorem that the order $|\mathcal{C}(G)|$ is the number $\tau(G)$ of spanning trees in G. The critical group $\mathcal{C}(G)$ has also been called the *Picard group*,

* Corresponding author: yphou@hunnu.edu.cn (Y. Hou). This work was supported by NSFC(10471037) and SRP from the Education Committee of Hunan Province.

the *Jacobian group*, the *tree group*, the *sandpile group*, and has a close connection with the critical configurations in a certain chip-firing game on G, and known as *abelian sandpile* in the physics literature, see e.g. [1, 2, 5, 6, 8, 9].

In general, it is difficult to say much more about the critical group structure of a graph. There are relatively few results describing the critical group structure of $\mathcal{C}(G)$ in terms of the structure of G. There are interesting infinite families of graphs, such as wheel graph, complete multipartite graphs, Cartesian products of complete graphs, threshold graphs, lattice graphs $P_n \times P_3$, and ladder graphs $P_2 \times C_n$, for which the critical group structure has recently been completely determined [2, 3, 7, 10].

The aim of this paper is to describe the critical group of the Möbius ladder M_n (see Fig. 1). We give an explicit expression for the Smith normal form of the sandpile group of M_n which is always the direct sum of two or three cyclic groups. That is, if $n = 2m + 1$, then

$$\mathcal{C}(M_n) \cong Z_{(n,h_m)} \oplus Z_{h_m} \oplus Z_{\frac{3nh_m}{(n,h_m)}},$$

where the sequence h_m is defined as $h_0 = 1, h_1 = 3, h_m = 4h_{m-1} - h_{m-2}$ for $m \geq 2$ and if $n = 2m$ and m is odd then

$$\mathcal{C}(M_n) \cong Z_{\frac{(n,k_m)}{2}} \oplus Z_{2k_m} \oplus Z_{\frac{2nk_m}{(n,k_m)}}$$

and if $n = 2m$ is even and m is even then

$$\mathcal{C}(M_n) \cong Z_{(n,k_m)} \oplus Z_{k_m} \oplus Z_{\frac{2nk_m}{(n,k_m)}},$$

where the sequence k_m is defined as $k_0 = 1, k_1 = 2, k_m = 4k_{m-1} - k_{m-2}$ for $m \geq 2$.

Our main tools will be the use of the Smith normal form for an integer matrix, which can be achieved by row and column operations that are invertible over the ring Z of integers. Given a square integer matrix A, its *Smith normal form* is the unique diagonal matrix $S(A) = \operatorname{diag}(S_{11}, S_{22}, ..., S_{nn})$ whose entries are nonnegative integers and S_{ii} divides $S_{i+1,i+1}$. Note that, for each i, the product $S_{11}S_{22}\cdots S_{ii}$ is the greatest common divisor of all $i \times i$ minor determinants of A, and we will use this fact to determine the Smith normal form of an integer matrix. Say that two matrices $A, B \in Z^{m\times n}$ are *unimodular equivalent* [12] (written $A \sim B$) if there exist matrices $P \in GL(m, \mathbf{Z}), Q \in GL(n, \mathbf{Z})$ such that $B = PAQ$. Equivalently, B is obtainable from A by a sequence of row and column operations mentioned above.

It is easy to see that $A \sim B$ implies $\operatorname{coker} A \cong \operatorname{coker} B$, and if $A = \operatorname{diag}(a_1, a_2, \cdots, a_n)$ then

$$\operatorname{coker} A \cong Z_{a_1} \oplus Z_{a_2} \oplus \cdots \oplus Z_{a_n},$$

where $Z_a = \mathbf{Z}/a\mathbf{Z}$. (Of course, Z_1 is the trivial group and $Z_0 = \mathbf{Z}$.)

2 A system of relations for the critical group on M_n

Let M_n be the Möbius ladder and its vertex set be $V = \{x_1, x_2, ..., x_n; y_1, y_2, ..., y_n\}$ (see Fig. 1). Graph M_n is the Cayley graph $Cay(Z_{2n}, \{1, -1, n\})$, it is different from ladder $P_2 \times C_n$ only two edges. However, $P_2 \times C_n$ is planar, but M_n non-planar. In

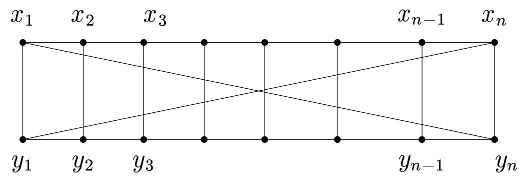

Fig 1. Möbius ladder M_n

this paper we will prove that the critical group of M_n has a similar group structure as $P_2 \times C_n$[7].

Let the image of x_i and y_i in the cokernel $\mathbf{Z}^{|V|}/imL(M_n)$ of $L(M_n)$ are $\overline{x}_i, \overline{y}_i$, respectively. For $3 \le i \le n$, we have

$$\begin{cases} \overline{x}_i &= 3\overline{x}_{i-1} - \overline{x}_{i-2} - \overline{y}_{i-1} \\ \overline{y}_i &= 3\overline{y}_{i-1} - \overline{y}_{i-2} - \overline{x}_{i-1}. \end{cases} \tag{1}$$

Thus the cokernel $\mathbf{Z}^{|V|}/imL(M_n) \cong \mathbf{Z} \oplus \mathcal{C}(M_n)$ of $L(M_n)$ can be generated by $\overline{x}_1, \overline{x}_2, \overline{y}_1, \overline{y}_2$.

For $i \ge 3$, let

$$\overline{x}_i \quad = \quad a_i\overline{x}_2 - b_i\overline{x}_1 - c_i\overline{y}_2 + d_i\overline{y}_1. \tag{2}$$

By induction, we have

$$\overline{y}_i = a_i\overline{y}_2 - b_i\overline{y}_1 - c_i\overline{x}_2 + d_i\overline{x}_1, \text{ for } i \ge 3. \tag{3}$$

and a_i, b_i, c_i, d_i obey the following recurrence relations:

$$\begin{cases} a_i &= 3a_{i-1} - a_{i-2} + c_{i-1}, \\ b_i &= 3b_{i-1} - b_{i-2} + d_{i-1}, \\ c_i &= 3c_{i-1} - c_{i-2} + a_{i-1}, \\ d_i &= 3d_{i-1} - d_{i-2} + b_{i-1}, \end{cases} \tag{4}$$

for $i \ge 3$, and $a_1 = 0, a_2 = 1, b_1 = -1, b_2 = 0, c_1 = 0, c_2 = 0, d_1 = 0, d_2 = 0$.

By solving the Eq. (4), we get

$$b_i = a_{i-1}, d_i = c_{i-1}, a_i - c_i = i - 1, b_i - d_i = i - 2. \tag{5}$$

Lemma 1 *Let $n \ge 3$. Then $\overline{x}_1, \overline{x}_2, \overline{y}_1, \overline{y}_2$ is a generating set of $\mathcal{C}(M_n)$ and the relation matrix A_n of the generating set $\overline{x}_1, \overline{x}_2, \overline{y}_1, \overline{y}_2$ is*

$$A_n = \begin{pmatrix} b_{n+1} & -a_{n+1} & -(d_{n+1}-1) & c_{n+1} \\ -(d_{n+1}-1) & c_{n+1} & b_{n+1} & -a_{n+1} \\ d_n - 3 & -(c_n - 1) & -(b_n - 1) & a_n \\ -(b_n - 1) & a_n & d_n - 3 & -(c_n - 1) \end{pmatrix}.$$

Proof. Since $\overline{y}_n = 3\overline{x}_1 - \overline{y}_1 - \overline{x}_2$, $\overline{x}_n = 3\overline{y}_1 - \overline{x}_1 - \overline{y}_2$, we have

$$\begin{cases} \overline{y}_1 = a_{n+1}\overline{x}_2 - b_{n+1}\overline{x}_1 - c_{n+1}\overline{y}_2 + d_{n+1}\overline{y}_1, \\ \overline{x}_1 = a_{n+1}\overline{y}_2 - b_{n+1}\overline{y}_1 - c_{n+1}\overline{x}_2 + d_{n+1}\overline{x}_1, \\ 3\overline{x}_1 - \overline{y}_1 - \overline{x}_2 = a_n\overline{y}_2 - b_n\overline{y}_1 - c_n\overline{x}_2 + d_n\overline{x}_1, \\ 3\overline{y}_1 - \overline{x}_1 - \overline{y}_2 = a_n\overline{x}_2 - b_n\overline{x}_1 - c_n\overline{y}_2 + d_n\overline{y}_1. \end{cases}$$

Hence, we have

$$\begin{cases} b_{n+1}\overline{x}_1 - a_{n+1}\overline{x}_2 - (d_{n+1}-1)\overline{y}_1 + c_{n+1}\overline{y}_2 &= 0, \\ -(d_{n+1}-1)\overline{x}_1 + c_{n+1}\overline{x}_2 + b_{n+1}\overline{y}_1 - a_{n+1}\overline{y}_2 &= 0, \\ (d_n-3)\overline{x}_1 - (c_n-1)\overline{x}_2 - (b_n-1)\overline{y}_1 + a_n\overline{y}_2 &= 0, \\ -(b_n-1)\overline{x}_1 + a_n\overline{x}_2 + (d_n-3)\overline{y}_1 - (c_n-1)\overline{y}_2 &= 0. \end{cases}$$

Thus lemma 1 follows. □

In order to obtain the structure of group $\mathcal{C}(M_n)$, it suffices to give the Smith normal form of A_n. In order to do this, we give

Lemma 2 *Let $s_n = a_n - a_{n-1}$. We have*

$$s_n = 4s_{n-1} - s_{n-2} - 1, s_0 = 2, s_1 = 1, \tag{6}$$

and

$$a_n = \frac{s_{n+2} - 3s_{n+1} + n}{2} = \frac{s_{n+1} - s_n + n - 1}{2}. \tag{7}$$

Proof. Since $a_n = 3a_{n-1} - a_{n-2} + c_{n-1} = 3a_{n-1} - a_{n-2} + a_{n-1} - n + 2 = 4a_{n-1} - a_{n-2} - n + 2$, we have $a_n - a_{n-1} = 4a_{n-1} - 4a_{n-2} - a_{n-2} - a_{n-3} - 1$. Thus $s_n = 4s_{n-1} - s_{n-2} - 1$, $s_0 = 2$, $s_1 = 1$. Since $a_n = 4a_{n-1} - a_{n-2} - n + 2$ and $a_n - a_{n-1} = 3(a_{n-1} - a_{n-2}) + 2a_{n-2} - n + 2$, we have $s_n = 3s_{n-1} + 2a_{n-2} - n + 2$ and $a_{n-2} = \dfrac{s_n - 3s_{n-1} + n - 2}{2}$. Therefore $a_n = \dfrac{s_{n+2} - 3s_{n+1} + n}{2} = \dfrac{s_{n+1} - s_n + n - 1}{2}$. □

Theorem 3 *Let the sequence s_n be defined as Lemma 2. Then the relation matrix A_n of $\mathcal{C}(M_n)$ is equivalent to $\begin{pmatrix} 0 & B_n \\ 0 & 0 \end{pmatrix}$, where B_n is equivalent to*

$$A'_n = \begin{pmatrix} n & \dfrac{s_{n+2} - 3s_{n+1} + n}{2} & \dfrac{s_{n+2} - s_{n+1} + n}{2} \\ 0 & s_{n+1} & s_{n+2} \\ 0 & s_n + 1 & s_{n+1} \end{pmatrix}.$$

Proof. By Eq.(4), we have the row sums and column sums of A_n are zero and $A_n \sim \begin{pmatrix} 0 & B_n \\ 0 & 0 \end{pmatrix}$, where

$$B_n = \begin{pmatrix} -a_{n+1} & -(d_{n+1}-1) & c_{n+1} \\ c_{n+1} & b_{n+1} & -a_{n+1} \\ -(c_n - 1) & -(b_n - 1) & a_n \end{pmatrix} \sim \begin{pmatrix} -a_{n+1} + c_{n+1} & -(d_{n+1}-1) & c_{n+1} \\ -a_{n+1} + c_{n+1} & b_{n+1} & -a_{n+1} \\ a_n - c_n + 1 & -(b_n - 1) & a_n \end{pmatrix}$$

$$\sim \begin{pmatrix} n & -b_{n+1} & a_{n+1} \\ 0 & b_{n+1}+d_{n+1}-1 & -a_{n+1}-c_{n+1} \\ 0 & b_{n+1}-b_n+1 & a_n-a_{n+1} \end{pmatrix} \sim \begin{pmatrix} n & a_n & a_{n+1} \\ 0 & 2a_n-n & 2a_{n+1}-n \\ 0 & a_n-a_{n-1}+1 & a_{n+1}-a_n \end{pmatrix}$$

$$= \begin{pmatrix} n & a_n & a_{n+1} \\ 0 & s_{n+1}-s_n-1 & s_{n+2}-s_{n+1} \\ 0 & s_n+1 & s_{n+1} \end{pmatrix} \sim \begin{pmatrix} n & \frac{s_{n+2}-3s_{n+1}+n}{2} & \frac{s_{n+2}-s_{n+1}+n}{2} \\ 0 & s_{n+1} & s_{n+2} \\ 0 & s_n+1 & s_{n+1} \end{pmatrix}.$$

□

We conclude this section with an explicit formula for the sequence s_n. By solving the Eq. (6), we get

$$s_n = \frac{9-5\sqrt{3}}{12}(2+\sqrt{3})^n + \frac{9+5\sqrt{3}}{12}(2-\sqrt{3})^n + \frac{1}{2}. \tag{8}$$

3 The critical group of the Möbius ladder M_n

In this section, we give an explicit expression of the Smith normal form $S = S_n$ of the matrix A'_n. In order to do this we give some properties of the sequence s_n.

Lemma 4 *For each n, we have $s_n s_{n+2} = s_{n+1}^2 + s_{n+1}$*

Proof. This can be proved by induction. □

By Lemma 4, the determinant of the minor $\begin{pmatrix} s_{n+1} & s_{n+2} \\ s_n+1 & s_{n+1} \end{pmatrix}$ of A'_n is $-w_n = -(s_{n+1}+s_{n+2})$, and

$$w_n = \frac{(2+\sqrt{3})^n + (2-\sqrt{3})^n}{2} + 1. \tag{9}$$

Lemma 5 *For $n = 2m+1$ odd, we have $w_n = s_{2m+2} + s_{2m+3} = 3h_m^2$, where the sequence h_m is defined as*

$$h_0 = 1, h_1 = 3, h_m = 4h_{m-1} - h_{m-2}. \tag{10}$$

For $n = 2m$ even, we have $w_n = s_{2m+1} + s_{2m+2} = 2k_m^2$, where the sequence k_m is defined as

$$k_0 = 1, k_1 = 2, k_m = 4k_{m-1} - k_{m-2}. \tag{11}$$

Proof. We prove that $3h_{m-1}h_m = w_{2m}+1, w_{2m+1} = 3h_m^2, 2k_{m-1}k_m = w_{2m-1}+1$ $w_{2m} = 2k_m^2$ for $m \geq 1$. Since

$$\begin{aligned} (w_{2m}+1)^2 - 3h_{m-1}^2 3h_m^2 &= (s_{2m}+s_{2m+1})(s_{2m+2}+s_{2m+3}) - (s_{2m+1}+s_{2m+2}+1)^2 \\ &= 6s_{2m+1}^2 - 24s_{2m+1}s_{2m+2} + 6s_{2m+2}^2 + 6s_{2m+1} + 6s_{2m+2} \\ &= [s_{2m+1}(s_{2m+1} - 4s_{2m+2} + 1) + s_{2m+2}^2 + s_{2m+2}] \\ &= 6(-s_{2m+1}s_{2m+3} + s_{2m+2}^2 + s_{2m+2}) \\ &= 0, \end{aligned}$$

we have $3h_{m-1}h_m = w_{2m} + 1$.

We can prove $w_{2m+1} = 3h_m^2$ by induction as follows.

$$\begin{aligned}
3(4h_{m-1} - h_{m-2})^2 &= 3(16h_{m-1}^2 - 8h_{m-1}h_{m-2} + h_{m-2}^2) \\
&= 16(s_{2m} + s_{2m+1}) - 8(s_{2m-1} + s_{2m+1} + 1) + s_{2m-2} + s_{2m-1} \\
&= 16s_{2m+1} + s_{2m} - 7s_{2m-1} + s_{2m-2} - 8 \\
&= s_{2m+2} + 12s_{2m+1} + 9s_{2m} - 7s_{2m-1} + s_{2m-2} - 7 \\
&= s_{2m+2} + 12s_{2m+1} + 8s_{2m} - 3s_{2m-1} - 8 \\
&= s_{2m+2} + 4(4s_{2m+1} - s_{2m} - 1) - 4s_{2m+1} + 12s_{2m} - 3s_{2m-1} - 4 \\
&= s_{2m+2} + 4s_{2m+2} - s_{2m+1} - 1 - 3s_{2m+1} + 3(4s_{2m} - s_{2m-1} - 1) \\
&= s_{2m+2} + s_{2m+3}.
\end{aligned}$$

Hence, $3h_m^2 = 3(4h_{m-1} - h_{m-2})^2$, that is, $h_m = 4h_{m-1} - h_{m-2}$ and $h_0 = 1, h_1 = 3$. The other two equations can be proved similarly. □

We list two relations linking h_n, k_n and s_n in the next lemma, they can be proved by induction.

Lemma 6 $h_m k_m = s_{2m+2}, \quad h_m k_{m+1} = s_{2m+3}$.

Lemma 7 *If* 3^t *divides* $2m + 1$ *then* 3^t *divides* h_m.

Proof. By solving the recurrence relation (10), we can obtain an explicit formula for h_m :

$$\begin{aligned}
h_m &= \frac{3+\sqrt{3}}{6}(2+\sqrt{3})^m + \frac{3-\sqrt{3}}{6}(2-\sqrt{3})^m \\
&= \frac{\sqrt{3}(\sqrt{3}+1)}{6}\frac{(\sqrt{3}+1)^{2m}}{2^m} + \frac{\sqrt{3}(\sqrt{3}-1)}{6}\frac{(\sqrt{3}-1)^{2m}}{2^m} \\
&= \frac{1}{2^m} \cdot \frac{1}{2\sqrt{3}}[(\sqrt{3}+1)^{2m+1} + (\sqrt{3}-1)^{2m+1}] \\
&= \frac{1}{2^m} \cdot \frac{1}{\sqrt{3}} \sum_{1 \le 2j+1 \le 2m+1} \binom{2m+1}{2j+1} \sqrt{3}^{2j+1} \\
&= \frac{1}{2^m} \sum_{1 \le 2j+1 \le 2m+1} \binom{2m+1}{2j+1} 3^j.
\end{aligned}$$

From this we have that 3^t divides h_m if 3^t divides $2m + 1$. □

3.1 Computation of S_{11}

Notice that for each m we have $(h_m, h_{m+1}) = (k_m, k_{m+1}) = 1$. This implies that

- If $n = 2m + 1$ then $(s_{n+1}, s_{n+2}) = (h_m k_m, h_m k_{m+1}) = h_m$.

- If $n = 2m$ then $(s_{n+1}, s_{n+2}) = (h_{m-1}k_m, h_m k_m) = k_m$.

  Since $s_n + 1 = 4s_{n+1} - s_{n+2}$ and $\frac{s_{n+2} - 3s_{n+1} + n}{2} = \frac{s_{n+2} - s_{n+1} + n}{2} - s_{n+1}$, so $S_{11} = (n, s_{n+1}, s_{n+2}, (s_{n+2} - s_{n+1} + n)/2)$.

- If $n = 2m + 1$ then $(n, s_{n+1}, s_{n+2}, (s_{n+2} - s_{n+1} + n)/2) = (n, s_{n+1}, s_{n+2})$. In fact, if d divides n, s_{n+1}, s_{n+2} then d is odd because n is odd and d divides $(s_{n+2} - s_{n+1} + n)/2$. Hence $S_{11} = (n, h_m)$.

- If $n = 2m$ then $S_{11} = (n, k_m, \frac{h_m k_m - h_{m-1}k_m + n}{2}) = (n, k_m, k_m \frac{h_m - h_{m-1}}{2} - m) = (n, k_m, m) = (m, k_m)$.

  If m is odd then k_m is even and then $S_{11} = (m, k_m) = \frac{(n, k_m)}{2}$.

  If m is even then k_m is odd and then $S_{11} = (m, k_m) = (n, k_m)$.

3.2 Computation of S_{22}

$$S_{11}S_{22} = (s_{n+1} + s_{n+2}, ns_{n+1}, ns_{n+2}, \frac{(s_{n+1} + s_{n+2})^2 + ns_{n+2} - ns_{n+1}}{2} - 3s_{n+1}s_{n+2}).$$

- If $n = 2m + 1$ then $S_{11}S_{22} = (s_{n+1} + s_{n+2}, ns_{n+1}, ns_{n+2}, 3s_{n+1}s_{n+2})$. In fact, it is easy to see that either s_{n+1} or s_{n+2} is odd and if d divides $s_{n+1} + s_{n+2}, ns_{n+1}$ and ns_{n+2}, hence d is odd and d divides also $\frac{(s_{n+1} + s_{n+2})^2 + ns_{n+2} - ns_{n+1}}{2}$. Thus,

$$\begin{aligned} S_{11}S_{22} &= (3h_m^2, nh_m k_m, nh_m k_{m+1}, 3h_m^2 k_m k_{m-1}) \\ &= h_m(3h_m, n, 3h_m k_m k_{m+1}) \\ &= h_m(3h_m, n) = h_m(n, h_m), \end{aligned}$$

  the last equality follows from Lemma 7. Hence $S_{22} = h_m$.

- If $n = 2m$ then

$$\begin{aligned} S_{11}S_{22} &= (2k_m^2, nk_m, \frac{4k_m^4 + nh_m k_m - nh_{m-1}k_m}{2} - 3h_{m-1}k_m^2 h_m) \\ &= k_m(2k_m, n, 2k_m^3 + n\frac{h_m - h_{m-1}}{2} - 3h_{m-1}k_m h_m) \\ &= k_m(2k_m, n, 3h_{m-1}k_m h_m) \\ &= k_m(n, k_m(2, 3h_{m-1}h_m)) \\ &= k_m(2m, k_m) = k_m(n, k_m). \end{aligned}$$

  Hence

  If m is odd then $S_{22} = 2k_m$.

  If m is even then $S_{22} = k_m$.

Since $\det(S_n) = |\det(A'_n)| = nw_n = n(s_{n+1}+s_{n+2})$, we have $S_{33} = \dfrac{n(s_{n+1}+s_{n+2})}{S_{11}S_{22}}$. Hence the Smith normal S_n of the matrix A'_n is:

- For $n = 2m+1$

$$\begin{pmatrix} (n,h_m) & 0 & 0 \\ 0 & h_m & 0 \\ 0 & 0 & \frac{3nh_m}{(n,h_m)} \end{pmatrix}.$$

- For $n = 2m$ with m odd

$$\begin{pmatrix} \dfrac{(n,k_m)}{2} & 0 & 0 \\ 0 & 2k_m & 0 \\ 0 & 0 & \dfrac{2nk_m}{(n,k_m)} \end{pmatrix}.$$

- For $n = 2m$ with m even

$$\begin{pmatrix} (n,k_m) & 0 & 0 \\ 0 & k_m & 0 \\ 0 & 0 & \dfrac{2nk_m}{(n,k_m)} \end{pmatrix}.$$

Thus, we obtain main result of this paper:

Theorem 8 *Let sequences h_m and k_m be defined by Eq. (10) and Eq. (11), respectively.*

(1). If $n = 2m+1$ then

$$\mathcal{C}(M_n) \cong Z_{(n,h_m)} \otimes Z_{h_m} \oplus Z_{\frac{3nh_m}{(n,h_m)}}.$$

(2a). If $n = 2m$ and m is odd then

$$\mathcal{C}(M_n) \cong Z_{\frac{(n,k_m)}{2}} \oplus Z_{2k_m} \oplus Z_{\frac{2nk_m}{(n,k_m)}}.$$

(2b). If $n = 2m$ and m is even then

$$\mathcal{C}(M_n) \cong Z_{(n,k_m)} \oplus Z_{k_m} \oplus Z_{\frac{2nk_m}{(n,k_m)}}.$$

Corollary 9 *The number of spanning trees of M_n is $\kappa(M_n) = 3nh_m^2$ if $n = 2m+1$ and $\kappa(M_n) = 2nk_m^2$ if $n = 2m$.*

4 Some particular cases

It is interesting to investigate the case in which the critical group of M_n is the direct sum of exactly three cyclic groups, that is, $S_{11} \neq 1$. By Lemma 7 we have that if $t \geq 1$ and 3^t divides $2m+1$ then 3^t divides h_m. Thus if 3^t divides $2m+1$ then $(2m+1, h_m) > 1$. For even n less than 200, $S_{11} > 1$ holds only for $n = 28, 78, 84, 140, 196$ by computer calculation.

4.1 The case $n = p^t$ where p is a prime

- If $p > 2$, we have the following equalities mod p:

$$\begin{aligned} w_{p^t} &= \frac{(2+\sqrt{3})^{p^t} + (2-\sqrt{3})^{p^t}}{2} + 1 \\ &= \frac{2^{p^t} + \sqrt{3}^{p^t} + 2^{p^t} - \sqrt{3}^{p^t}}{2} + 1 \\ &= 2^{p^t} + 1 \\ &= 3 \mod p. \end{aligned}$$

  This argument implies that $(p, w_{p^t}) = 1$ for $p \neq 3$ odd prime, thus the Smith normal form is

$$S_{p^t} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & h_{\frac{p^t-1}{2}} & 0 \\ 0 & 0 & 3p^t h_{\frac{p^t-1}{2}} \end{pmatrix}.$$

- For $p = 3$, we have

$$S_{3^t} = \begin{pmatrix} 3^t & 0 & 0 \\ 0 & h_{\frac{3^t-1}{2}} & 0 \\ 0 & 0 & 3h_{\frac{3^t-1}{2}} \end{pmatrix}.$$

- For $p = 2$, we have

$$S_{2^t} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & k_{2^{t-1}} & 0 \\ 0 & 0 & 2^{t+1} k_{2^{t-1}} \end{pmatrix}.$$

4.2 The case $n = 2p^t$, where p is an odd prime

If $p > 2$, we have the following calculation mod p:

$$\begin{aligned} w_{2p^t} &= \frac{(2+\sqrt{3})^{2p^t} + (2-\sqrt{3})^{2p^t}}{2} + 1 \\ &= \frac{(7+4\sqrt{3})^{p^t} + (7-4\sqrt{3})^{p^t}}{2} + 1 \\ &= 7^{p^t} + 1 \\ &= 8 \mod p. \end{aligned}$$

Thus $(p, w_{p^t}) = 1$ and $(2p^t, k_{p^t}) = 2$ and the Smith normal form is

$$S_{2p^t} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2k_{p^t} & 0 \\ 0 & 0 & 2p^t k_{p^t} \end{pmatrix}.$$

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