AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **36** (2006), Pages 167–176

Hexagon biquadrangle systems

Lucia Gionfriddo

Dipartimento di Matematica
Università di Catania
Città Universitaria, Viale A. Doria, 6
95125 Catania
ITALY
lucia.gionfriddo@dmi.unict.it

Abstract

A *hexagon biquadrangle* is the graph consisting of two 4-cycles (x_1, x_2, x_3, x_4), (x_1, x_4, x_5, x_6) where $x_1, x_2, x_3, x_4, x_5, x_6$ are distinct vertices such that $(x_1, x_2, x_3, x_4, x_5, x_6)$ is a hexagon. A *hexagon biquadrangle system* of order n and index ρ [HBQS] is a pair (X, H), where X is a finite set of n vertices and H is a collection of edge disjoint hexagon biquadrangles (called *blocks*) which partitions the edge set of ρK_n, with vertex set X. A *hexagon biquadrangle system* is said to be a 4-*nesting* [$N(4)$ or also (4)-HBQS] if the collection of all the 4-cycles contained in the hexagon biquadrangles form a μ-fold 4-cycle system. It is said to be a 6-*nesting* [$N(6)$ or also (6)-HBQS] if the collection of 6-cycles contained in the hexagon biquadrangles is a λ-fold 6-cycle system. It is said to be a (4, 6)-*nesting*, briefly a [$N(4,6)$ or also (4, 6)-HBQS], if it is both 4-*nesting* and a 6-*nesting*. It is said to be a $(4^2, 6)$-*nesting* if it is (4, 6)-nesting and the μ-fold 4-cycle system, nested in it, is decomposable into two $\frac{\mu}{2}$-fold 4-cycle systems.

In this paper we determine completely the spectrum of $(4^2, 6)$-HBQS for $\rho = 7h$, $\lambda = 6h$ and $\mu = 8h$, h a positive integer.

1 Introduction

A λ-fold *m-cycle system* of order n is a pair (X, C), where X is a finite set of n elements, called *vertices*, and C is a collection of edge disjoint m-cycles which partitions the edge set of λK_n, the complete graph with vertex set X and where every pair of vertices is joined by λ edges. In this case, $|C| = \lambda n(n-1)/2m$. When $\lambda = 1$, we will simply say *m-cycle system*. A 3-cycle is also called a *triple* and so a λ-fold 3-*cycle system* will also be called a λ-fold 3-*triple system*. When $\lambda = 1$, we have the well known definition of *Steiner triple system* (or, simply, *triple system*).

Fairly recently the spectrum (i.e., the set of all n such that an *m-cycle systems* of order n exists) has been determined to be [1, 14]:

(1) $n \geq m$, if $n > 1$;

(2) n is odd; and

(3) $\frac{n(n-1)}{2m}$ is an integer.

The spectrum for λ-fold *m-cycle systems* for $\lambda \geq 2$ is still an open problem.

The graph given below is called a *hexagon biquadrangle* and will be also denoted by $[(x_1), x_2, x_3, (x_4), x_5, x_6]$.

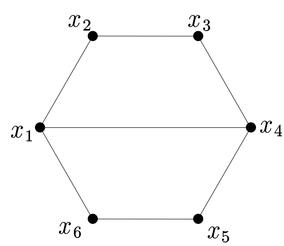

Figure 1:

A *hexagon biquadrangle system* of order n and index ρ [HBQS] is a pair (X, H), where X is a finite set of n vertices and H is a collection of edge disjoint hexagon biquadrangles (called *blocks*) which partitions the edge set of ρK_n, with vertex set X.

A *hexagon biquadrangle system* (X, H) of *order* n and *index* ρ is said to be a 4-*nesting* [(4)-HBQS] if the collection of all the 4-cycles contained in the hexagon biquadrangles form two distinct $\frac{8}{7}\rho$-fold 4-cycle systems. We will say that this $(\mu = \frac{8}{7}\rho)$-fold 4-cycle system is *nested* in the HBQS (X, H).

A *hexagon biquadrangle system* (X, H) of *order* n and *index* ρ is said to be a 6-*nesting* [$N(6)$ or also (6)-HBQS] if the collection of 6-cycles contained in the hexagon biquadrangles is a $(\lambda = \frac{6}{7}\rho)-$fold 6-cycle system. This 6-cycle system is said to be *nested* in (X, H).

A *hexagon biquadrangle system* of *order* n and *index* ϱ is said to be a $(4^2, 6)$-*nesting*, briefly a $N(4^2, 6)$ or also $(4^2, 6)$-HBQS, if it is both a 4-*nesting* and a 6-*nesting* with the additional condition that the 4-*cycle system* of index μ, nested in the HBQS (X, H), is decomposable into two systems of index $\frac{\mu}{2}$. In these cases, we say that the hexagon quadrangle system has indices (ρ, λ, μ).

In the following examples the vertex set is Z_7.

Example 1 *The following HBQS(7) is neither a 4-nesting nor a 6-nesting.*

Base blocks: $[(0), 1, 2, (3), 5, 4]$, $[(0), 2, 4, (3), 1, 6]$, $[(0), 3, 6, (4), 5, 2]$.

Example 2 *The following HBQS(7) is a 6-nesting but not a 4-nesting.*

Base blocks: $[(5), 0, 1, (3), 6, 2]$, $[(6), 0, 1, (3), 2, 5]$,$[(3), 1, 6, (2), 0, 4]$.

Example 3 *The following HBQS(7) is a 4-nesting but not a 6-nesting.*

Base blocks: $[(0), 6, 4, (1), 2, 5]$, $[(0), 2, 4, (1), 3, 6]$, $[(0), 3, 6, (1), 2, 5)]$.

Example 4 *The following HBQS(7) is both a 4^2 and 6 nesting.*

Base blocks: $[(0), 3, 2, (1), 6, 5]$, $[(0), 6, 4, (2), 5, 1]$, $[(0), 2, 6, (3), 5, 4]$.

In this paper we determine completely the spectrum of $(4^2, 6)$-HBQS(n) for $\lambda = 6h$, $\mu = 8h$ and $\rho = 7h$; h a positive integer.

2 Necessary existence conditions

In this section we prove some necessary existence conditions for HBQSs having indices (ρ, λ, μ) and order n.

Theorem 2.1 *Let (X, H) be a $(4^2, 6)$-HBQS. Then:*

(1) $6\rho = 7\lambda$, $4\lambda = 3\mu$, $8\rho = 7\mu$;

(2) $\rho \equiv 0$ *mod* 7, $\mu \equiv 0$ *mod* 8, $\lambda \equiv 0$ *mod* 6; *and*

(3) $\rho = 7h$, $\lambda = 6h$, $\mu = 8h$, *h is a positive integer.*

Proof. Let (X, H) be a $(4^2, 6)$-HBQS and let (X, C') be the 6-cycle system of index λ and (X, C'') the 4-cycle system of index μ, nested in it.

(1) It is immediate that:
$|H| = |C'| = \frac{1}{2}|C''|$,
$|H| = \frac{\binom{n}{2}}{7}\rho$, $|C'| = \frac{\binom{n}{2}}{6}\lambda$, $|C''| = \frac{\binom{n}{2}}{8}\mu$.
It follows that $6\rho = 7\lambda$, $8\rho = 7\mu$ and $4\lambda = 3\mu$.

(2) From (1), the index $\mu = \frac{8}{7}\rho$ must be congruent to 0 (mod 8). So, since $\rho = \frac{7}{6}\lambda$ and $\lambda = \frac{6}{7}\rho$, it follows that $\rho \equiv 0 \pmod 7$ and $\lambda \equiv 0 \pmod 6$.

(3) From (2), directly.

□

Theorem 2.2 *Let (X, H) be a $(4^2, 6)$-HBQS. Then $n \equiv 0, 1 \pmod 2$.*

Proof. This follows from Theorem 2.1. □

3 Existence for p a prime and consequences.

In this section we will examine the existence of $(4^2, 6)$-HBQS of indices (ρ, λ, μ) and order n for $(\rho, \lambda, \mu) = (7, 6, 8)$ and n a prime number or an odd number not divisible by 3 or 5.

Theorem 3.1 *For every prime number p, $p \geq 7$, there exist a $(4^2, 6)$-HBQS with indices $(\rho, \lambda, \mu) = (7, 6, 8)$ and order p.*

Proof. Let $X = \{0, 1, 2, , p-1\} = Z_p$. Observe that, if $x, y \in H$, $x < y$, then: $y - x = \Delta = \{1, 2, ..., \frac{p-1}{2}\}$.

Consider the following families of hexagon biquadrangles, 6-cycles and 4-cycles, respectively:

$H = \{b_{j,i} = [(j), j+n-i, j+n-2i, (i+j), j+2i, j+3i] : i \in \Delta, j \in Z_p\}$;

$C' = \{c_{j,i} = (j, j+n-i, j+n-2i, i+j, j+2i, j+3i) : i \in \Delta, j \in Z_p\}$;

$C'' = \{q_{j,i,1} = (j, j+n-i, j+n-2i, j+i), q_{j,i,2} = (j, j+i, j+2i, j+3i) : i \in \Delta, j \in Z_p\}$.

Observe that $n - i \equiv -i \pmod{n}$ and $n - 2i \equiv -2i \pmod{n}$.

We prove that (Z_p, C'') is a 4-cycle system of index $\mu = 8$. In fact, for every pair $x, y \in Z_p$, $x < y$, if $y - x = i$, then $i \in \Delta$ and the following blocks of C" contain the edge $\{x, y\}$:

$q_{x,i,1} = (x, x-i, x-2i, y = x+i)$,

$q_{x,i,2} = (x, y = x+i, x+2i, x+3i)$,

$q_{y,i,1} = (y = x+i, x, x-i, y+i)$,

$q_{x-i,i,2} = (x-i, x, y = x+i, x+2i)$,

$q_{x+2i,i,1} = (x+2i, x+i = y, x, x+3i)$,

$q_{x-2i,i,2} = (x-2i, x-i, x, y = x+i)$.

Further, since p is a prime number

$\{3i : i \in \Delta\} = \{3, 6, \dots, \frac{3(p-1)}{3}\} = \Delta$.

This implies that there exists an $u \in \Delta$ such that $3u = i$ and

$q_{x+2u,u,1} = (x+2u, x+u, x, x+3u = y) \in C''$

$q_{x,u,2} = (x, x+u, x+2u, x+3u = y) \in C''$.

Since: $|C''| = 2\frac{p(p-1)}{2} = \frac{\binom{p}{2}}{4}8$,

the pair (Z_p, C'') is a 4-cycle system of index $\mu = 8$.

Observe that (Z_p, C'') can be decomposed in two 4-cycle systems, both of index 4, having for blocks $q_{j,i,1}$ and $q_{j,i,2}$ respectively.

We prove that (Z_p, C') is a 6-cycle system of index $\lambda = 6$. In fact, for every pair $x,\ y \in Z_p,\ x < y$, if $y - x = i$, then $i \in \Delta$. Further, there are six blocks of C' containing the edge $\{x, y\}$: precisely, the cycles $c_{x-i,i}, c_{x-2i,i}, c_{x+i,i}, c_{x+2i,i}$, and, since p is a prime number, the two cycles $(x_1, x_2, x_3, x_4, x_5, x_6)$ such that $\{x_1, x_6\} = \{x, y\}$ and $\{x_3, x_4\} = \{x, y\}$, respectively.

Since: $|C'| = \frac{p(p-1)}{2} = (\frac{\binom{p}{2}}{6})6$, the pair (Z_p, C') is a 6-cycle system of index $\lambda = 6$. It follows that (Z_p, H) is a $(4^2, 6)$-HBQS of indices $(7, 6, 8)$ and order p.

Further, if we delete, in every $b_{j,i} \in H$, the edge $\{j, i + j\}$, we obtain the 6-cycle system (Z_p, C') of index $\lambda = 6$. If we separate, in every $b_{j,i} \in H$, the two 4-cycles $(j, j-i, j-2i, i)$ and $(j, j+i, j+2i, j+3i)$, we obtain respectively two 4-cycle systems of index $\mu = 4$, which together give the 4-cycle system (Z_p, C'') of index $\mu = 8$. This completes the proof. □

Theorem 3.2 *For every prime number p, $p \geq 7$, there exist $(4^2, 6)$-HBQS having order $p + 1$ and indices $(\rho, \lambda, \mu) = (7, 6, 8)$.*

Proof. Let $X = \{0, 1, 2, \ldots, p - 1\} = Z_p$, $X^* = X \cup \{\infty\}$, $\Delta = \{1, 2, \ldots, \frac{p-1}{2}\}$, (X, H), where:

$H = \{b_{j,i} = [(j), j + n - i, j + n - 2i, (j + i), j + 2i, j + 3i] : i \in \Delta, j \in Z_p\}$.

From Theorem 3.1, (X, H) is a $(4^2, 6)$-HBQS of indices $(7, 6, 8)$ and order p, p a prime, which defines a 6-cycle system (X, C') and a 4-cycle system (X, C''), where:

$C' = \{c_{j,i} = (j, j + n - i, j + n - 2i, j + i, j + 2i, j + 3i) : i \in \Delta, , j \in Z_p\}$;

$C'' = \{q_{j,i,1} = (j, j+n-i, j+n-2i, j+i), q_{j,i,2} = (j, j+i, j+2i, j+3i) : i \in \Delta, j \in Z_p\}$.

Consider $b_{j,1}$,$b_{j,2} \in H$, for $j \in Z_p$, and define the following blocks:

$b_{j,\infty,1} = [(j), \infty, j + p - 2, (j + 1), j + 2, j + 3]$, for $j \in Z_p$;

$b_{j,\infty,2} = [(j), j + p - 2, j + p - 4, (j + 2), j + 4, \infty]$, for $j \in Z_p$;

$b_{j,\infty} = [(j), j + p - 1, j + p - 2, (\infty), j + 4, j + 6]$ for $j \in Z_p$.

Observe that, if we indicate by $b = [(x_1), x_2, x_3, (x_4), x_5, x_6]$ the blocks of H, then the blocks $b_{j,\infty,1}$, $b_{j,\infty,2}$, $b_{j,\infty}$ are constructed starting from the blocks $b_{j,1}$, $b_{j,2}$ of H, by the same edges, with the same multiplicity and such that the edges $\{\infty, j\}$, for $j \in Z_p$, are repeated 6 times in the cycles $(x_1, x_2, x_3, x_4, x_5, x_6)$ of $b_{j,\infty,1}$, $b_{j,\infty,2}$, $b_{j,\infty}$, 4 times in the cycles (x_1, x_2, x_3, x_4), 4 times in the cycles (x_1, x_4, x_5, x_6) and 7 times in the blocks $b_{j,\infty,1}$, $b_{j,\infty,2}$, $b_{j,\infty}$. So, if $H^* = H \setminus \{b_{j,1}, b_{j,2}\} \cup \{b_{j,\infty,1}, b_{j,\infty,2}, b_{j,\infty}\}$, it is possible to verify that (X^*, H^*) is a $(4^2, 6)$-HBQS of order $p + 1$, completing the proof. □

The results of Theorem 3.1 and Theorem 3.2 can be extended to $(4^2, 6)$-HBQS of indices $(7h, 6h, 8h)$ and order n, by repetition of blocks.

Theorem 3.3 *For every odd number d, not divisible by 3 or 5, there exist $(4^2, 6)$-HBQS having order d and indices $(\rho, \lambda, \mu) = (7, 6, 8)$.*

Proof.

Consider the same families of hexagon biquadrangles defined in Theorem 3.1, where $\Delta = \{1, 2, \ldots, \frac{d-1}{2}\}$:

$H = \{b_{j,i} = [(j), j+n-i, j+n-2i, (i+j), j+2i, j+3i] : i \in \Delta, j \in Z_p\}$;

$C' = \{c_{j,i} = (j, j+n-i, j+n-2i, i+j, j+2i, j+3i) : i \in \Delta, j \in Z_p\}$;

$C'' = \{q_{j,i} = (j, j+n-i, j+n-2i, j+i), q_{j,i,2} = (j, j+i, j+2i, j+3i) : i \in \Delta, j \in Z_p\}$.

These families define a $(4^2, 6)$-HBQS of indices $(7, 6, 8)$ and order n, (Z_d, C), nesting both the 6-cycle system (Z_d, C') and the 4-cycle system (Z_d, C''). Observe that all the edges of the hexagon biquadrangles are obtained by difference methods, starting from the following base blocks:

$b_{0,1}, b_{0,2}, \ldots, b_{0,\frac{d-1}{2}}$

$c_{0,1}, c_{0,2}, \ldots, c_{0,\frac{d-1}{2}}$

$q_{0,1,1}, q_{0,2,1}, \ldots, q_{0,\frac{d-1}{2},1}$

$q_{0,1,2}, q_{0,2,2}, \ldots, q_{0,\frac{d-1}{2},2}$

Since d is not divisible by 3 or 5, there is not any repetition of vertices in all of the previous blocks.

Therefore, the conclusion follows as in Theorem 3.1. □

Theorem 3.4 *For every odd number d, not divisible by 3 or 5, there exist* $(4^2, 6)$-*HBQS of order* $d+1$ *and* $(7, 6, 8)$.

Proof. The statement follows from Theorems 3.1, 3.2, directly. □

4 Construction $v \to 2v$ and Construction $v \to 2v-1$

In this section we give two constructions for $(4^2, 6)$-HBQS. In this case these constructions can be extended to $(4^2, 6)$-HBQS of indices $(7h, 6h, 8h)$ In that follows all $(4^2, 6)$-HBQS have indices $(7, 6, 8)$.

Theorem 4.1 $(4^2, 6)$-*HBQS of order* $2n$ *can be constructed from* $(4^2, 6)$-*HBQS having both indices* $(7, 6, 8)$.

Proof. Let (Z_n, H) be a $(4^2, 6)$-HBQS of order n, $n \geq 6$. Let $X = Z_n \times \{1, 2\}$, and let $(Z_{n,i}, H_i)$ be the HBQS, for $i = 1, 2$ such that $Z_{n,i} = Z_n \times \{i\}$, and $[((a,i)), (b,i), (c,i), ((\alpha,i)), (\beta,i), (\gamma,i)] \in H_i$ if and only if $[(a), b, c, (\alpha), \beta, \gamma] \in H$. Let H^* be the collection of hexagon biquadrangles defined on X by:

$H_1 \subseteq H^*$, $H_2 \subseteq H^*$.

Further, if:

$\Phi = \{[((i,1)),(j+1,2),(i+1,1)),((j,2)),(i+2,1),(j+2,2)]\}$

then: $\Phi \subseteq H^*$.

To begin with (X, H^*) is a $(4^2,6)$-HBQS of order $2n$. It is easy to see that all the edges of type $\{(x,i),(y,i)\}$ are contained in H_i with the correct repetition. In fact, $(Z_{n,i}, H_i)$ is a $(4^2,6)$-HBQS and no edge $\{(x,i),(y,i)\}$ is contained in any of the blocks of Φ, which contains blocks with edges of type $\{(x,1),(y,2)\}$.

Consider an edge of type $\{(x,1),(y,2)\}$.

If $b = [((a,1)),(b,2),(c,1),((\alpha,2)),(\beta,1),(\gamma,2)]$ indicates the blocks of Φ, then an edge $\{(x,1),(y,2)\}$ is contained 6-times in the cycles $((a,1),(b,2),(c,1),$ $(\alpha,2),(\beta,1),(\gamma,2))$, 4-times in the cycles $((a,1),(b,2),(c,1),(\alpha,2))$ and 4-times in the cycles of $((a,1),(\alpha,2),(\beta,1),(\gamma,2))$.

Further, $\{(x,1),(y,2)\}$ is contained 7-times in the blocks of Φ.

Observe that the number of blocks of H^* is:

$|H^*| = |H_1| + |H_2| + |\Phi| = \frac{2\binom{n}{2}}{7}7 + n^2 = n(n-1) + n^2 = 2n^2 - n$

which is exactly the number of blocks of a $(4^2,6)$-HBQS of order $2n$:

$\frac{\binom{2n}{2}}{7}7 = \frac{2n(2n-1)}{2} = 2n^2 - n.$

So, the proof is completed. □

Theorem 4.2 $(4^2,6)$-*HBQSs of order* $(2n-1)$ *can be constructed from* $(4^2,6)$-*HBQS of order* n.

Proof. Let (Z_n, H) be a $(4^2,6)$-HBQS, of order n, and let $x = n-1 \in Z_n$. If $Z_{n-1,i} = Z_{n-1} \times \{i\}$ and $X = (Z_{n-1} \times \{1,2\}) \cup \{x\}$, then $|X| = 2n-1$. Further, let $(x,1) = (x,2) = (x,3) = x$ and let $(Z_{n-1,i} \cup \{x\}, H_i)$ be the HBQS, for $i = 1,2$, such that $[((a,i)),(b,i),(c,i),((\alpha,i)),(\beta,i),(\gamma,i)] \in H_i$ if and only if $[(a),b,c,(\alpha),\beta,\gamma] \in H$.

We define a collection H^* of hexagon biqradrangles on X, as follows:

$H_1 \subseteq H^*$, $H_2 \subseteq H^*$.

Further, let

$\Phi = \{[((i,1)),(j+1,2),(i+1,1),((j,2)),(i+2,1),(j+2,2)] : i,j \in Z_{n-1} \in H^*\}$

Just as in Theorem 4.1, it is possible to verify that the pair (X, H^*) is a $(4^2,6)$-HBQS of order $2n-1$. The number of blocks in H^* is:

$|H^*| = |H_1| + |H_2| + |\Phi| = \frac{2\binom{n}{2}}{7}7 + (n-1)^2 = n(n-1) + (n-1)^2 = 2n^2 - 3n + 1$.

This is exactly the number of blocks of a $(4^2,6)$-HBQS of order $2n-1$:

$$\frac{\binom{2n-1}{2}}{7}7 = (2n-1)(n-1) = 2n^2 - 3n + 1.$$

This completes the proof. □

5 Existence of $(4^2, 6)$-HBQS of orders 6,7,8, 9, 10.

The cases $n = 6, 7, 8, 9, 10$ are necessary to determine the spectrum of $(4^2, 6)$-HBQS completely.

Theorem 5.1 *There exist* $(4^2, 6)$*-HBQS of order* $n = 6, 7, 8, 9, 10$.

Proof. Case $n = 6$. Let H be the family of hexagon biquadrangles defined on $Z_5 \cup \{\infty\}$ as follows:

H= $\{b_{j,1} = [(j), j+4, j+3, (j+1), j+2, \infty)] : j \in Z_5\} \cup \{b_{j,2} = [(j), \infty, j+1, (j+2), j+4, j+3)] : j \in Z_5\} \cup \{b_{j,3} = [(j), j+3, j+1, (\infty), j+2, j+4)] : j \in Z_5\}$.

Observe that the hexagon biquadrangles of H can be obtained by difference methods, starting from the base blocks $b_{0,1}$, $b_{0,2}$, $b_{0,3}$. It is possible to verify that $(Z_5 \cup \{\infty\}, H)$ is a $(4^2, 6)$-HBQS of order 6.

The existence for $n = 7, 8$ follows from Theorem 3.1 and Theorem 3.2.

Case $n = 9$. Let Z_9 and let

$b_{j,1} = [(j), j+3, j+8, (j+1), j+6, j+2)], j \in Z_9;$

$b_{j,2} = [(j), j+4, j+1, (j+2), j+3, j+6], j \in Z_9;$

$b_{j,3} = [(j), j+1, j+8, (j+3), j+5, j+4], j \in Z_9;$

$b_{j,4} = [(j), j+2, j+1, (j+4), j+5, j+3], j \in Z_9.$

If

$H = \{b_{j,1}, b_{j,2}, b_{j,3}, b_{j,4} : j \in Z_9\}$

then it is possible to verify that (Z_9, H) is a $(4^2, 6)$-HBQS of order 9.

Case $n = 10$.

Let $Z_9 \cup \{\infty\}$ and let

$b_{j,1} = [(j), j+3, j+8, (j+1), j+6, j+2)], j \in Z_9;$

$b_{j,2} = [(j), j+4, j+1, (j+2), j+3, j+6], j \in Z_9;$

$b_{j,3,\infty} = [(j), j+1, j+8, (j+3), j+5, \infty], j \in Z_9;$

$b_{j,4,\infty} = [(j), \infty, j+1, (j+4), j+5, j+3], j \in Z_9;$

$b_{j,\infty} = [(j), j+2, j+1, (\infty), j+5, j+4)], j \in Z_9;$

$H = \{b_{j,1}, b_{j,2}, b_{j,3,\infty}, b_{j,4,\infty}, b_{j,\infty} : j \in Z_9\}$.

Observe that the hexagon biquadrangles of H can be obtained by difference methods, starting from the base blocks $b_{0,1}$, $b_{0,2}$, $b_{0,3,\infty,}$, $b_{0,4,\infty,}$, $b_{0,\infty}$. It is possible to verify that $(Z_9 \cup \{\infty\}, H)$ is a $(4^2, 6)$-HBQS of order 10.

6 Conclusion

Collecting together the results of the previous sections, we have the following result:

Theorem 6.1 *There exists a* $(4^2, 6)$*-HBQS of order* v *and indices* $(7h, 6h, 8h)$*, for every* $v \in N$*,* $v \geq 6$.

Proof. The statement follows directly from Theorem 4.1, Theorem 4.2 and Theorem 5.1. □

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(Received 4 June 2005)