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A method for constructing symmetric Hamiltonian double Latin squares

M. A. Ollis

Marlboro College, P. O. Box A, South Road,
Marlboro, VT 05344
U.S.A.
matt@marlboro.edu

Abstract

We give a method for constructing symmetric Hamiltonian double Latin squares from groups that have a symmetric sequencing. This allows us to find orthogonal pairs of symmetric Hamiltonian double Latin squares of order $2p$ for all primes p congruent to 3 modulo 4. We also show that there is a triple of mutually orthogonal symmetric Hamiltonian double Latin squares of order 18.

1 Introduction

A *Latin square* of order n is an $n \times n$ array of n symbols with the property that each symbol occurs once in each row and once in each column. A *double Latin square* of order $2n$ is a $2n \times 2n$ array of n symbols with the property that each symbol occurs twice in each row and twice in each column.

Let A be a double Latin square of order $2n$ on symbol set Ω. Consider a symbol $k \in \Omega$ and suppose that $A(i_1, j_1) = k$. We can uniquely produce a cycle $(i_1, j_1), (i_1, j_2), (i_2, j_2), (i_2, j_3), \ldots, (i_l, j_l), (i_l, j_1)$ of cells of A where each of these cells contains the symbol k. By repeating this procedure for all the cells containing k we get a set of disjoint cycles of total length $4n$. If this set contains just one cycle then that cycle is called *Hamiltonian*. If each symbol of Ω gives rise to a Hamiltonian cycle then then the double Latin square A is also called *Hamiltonian*.

Example 1.1 *The double Latin square A of order* 4 *in Figure 1 is Hamiltonian.*
The Hamiltonian cycle associated with the symbol 1 *is*

$$(1,1),(1,2),(4,2),(4,3),(3,3),(3,4),(2,4),(2,1).$$

A Hamiltonian double Latin square A is *symmetric* if $A(i,j) = A(j,i)$ for all pairs (i,j). The Hamiltonian double Latin square in Figure 1 is symmetric. Following

1	1	2	2
1	2	2	1
2	2	1	1
2	1	1	2

Figure 1: A

Hilton et al. [13] we use SHLS($2n$) for "symmetric Hamiltonian double Latin square of order $2n$."

Let K_{2n} denote the complete graph on $2n$ vertices. A *Hamiltonian path* of K_{2n} is a path which visits each vertex exactly once. Symmetric Hamiltonian double Latin squares are connected to Hamiltonian paths in complete graphs by the following result.

Theorem 1.1 [13] *An* SHLS($2n$) *is equivalent to a decomposition of* K_{2n} *into Hamiltonian paths.* □

The equivalence arises as follows. We first obtain a Hamiltonian path decomposition of K_{2n} from an SHLS($2n$) A on symbol set Ω. Label the vertices of the graph with the symbols $1, 2, \ldots, 2n$. For each $\alpha \in \Omega$ define a subgraph H_α of K_{2n} by the following rule:

$$(i, j) \text{ is an edge of } H_\alpha \text{ if and only if } A(i, j) = \alpha.$$

Each H_α is a Hamiltonian path and, as H_α and H_β have no edges in common when $\alpha \neq \beta$, the set $\{H_\alpha : \alpha \in \Omega\}$ is a Hamiltonian path decomposition of K_{2n}.

For the converse suppose that $\{H_\alpha : \alpha \in \Omega\}$ is a Hamiltonian decomposition of K_{2n}. For each H_α put symbol α in cells (i, j) and (j, i) whenever (i, j) is an edge in H_α. Also put α in cell (i, i) whenever i is a vertex of degree one in H_α. This gives an SHLS($2n$).

In [13] a method is given for constructing SHLS($2n$)s using cyclic groups of even order. In the next section we see that the cyclic group of this construction can be replaced by an arbitrary group that has a single involution. We also note that symmetric Hamiltonian double Latin squares are related to particular types of Tuscan square and row-complete Latin square.

Consider two double Latin squares, A and B, of order $2n$ on the same symbol set Ω. Then A and B are *orthogonal* if for each ordered pair (α, β) of symbols from Ω there are precisely four ordered pairs (i, j) such that $A(i, j) = \alpha$ and $B(i, j) = \beta$.

Figure 2 gives a pair of orthogonal symmetric Hamiltonian double Latin squares of order 6.

Hilton et al. [13] show that there exists a pair of mutually orthogonal SHLS($2^k n$)s for any $k \geq 1$ and $n \in \{1, 3, 5, 7, 9, 11, 13\}$. In Section 3 we extend the set of possible values of n to include 15, 17 and all primes congruent to 3 modulo 4. We also demonstrate the existence of a triple of mutually orthogonal SHLS(18)s.

1	1	2	2	3	3		1	3	3	2	1	2
1	2	2	3	3	1		3	2	1	1	3	2
2	2	3	3	1	1		3	1	3	2	2	1
2	3	3	1	1	2		2	1	2	1	3	3
3	3	1	1	2	2		1	3	2	3	2	1
3	1	1	2	2	3		2	2	1	3	1	3

Figure 2: A pair of orthogonal SHLS(6)s

2 Group-based SHLS($2n$)s

As we saw in the previous section, a Hamiltonian path decomposition of K_{2n} is equivalent to an SHLS($2n$). Here we mention some other combinatorial objects which are related to Hamiltonian path decompositions and note one well-known construction that provides many examples of symmetric Hamiltonian double Latin squares.

An *Italian square* of order n is an $n \times n$ array of n symbols with the property that each symbol occurs exactly once in each row. A *Tuscan square* is an Italian square in which each pair of distinct symbols occur as horizontal neighbours twice, once in each order. A Tuscan square which is also a Latin square is called a *row-complete* Latin square. We write RCLS(n) for "row-complete Latin square of order n". An Italian square is called *row-reversible* if each row is the reverse of some other row. Necessarily, row-reversible Italian squares must have even order.

Golomb and Taylor [11] introduced Tuscan squares as a way of representing Hamiltonian path decompositions of complete directed graphs. To see the equivalence between these objects label the vertices of the graph and with the symbols of the Tuscan square and obtain a set of Hamiltonian paths, one for each row, by taking the entries of that row in order. As the square is Tuscan each edge will appear exactly once. Clearly, this process can be reversed to get a Tuscan square from a Hamiltonian path decomposition of the complete directed graph. The Hamiltonian paths in the decomposition will come in pairs, one the reverse of the other, if and only if the associated Tuscan square is row-reversible. So we have:

Lemma 2.1 [11] *A Hamiltonian path decomposition of K_{2n} is equivalent to a row-reversible Tuscan square.* □

Theorem 2.3.1 of [9] gives a construction of a RCLS(2n) for each n. It is noted [9, pp. 300–301] that this gives a Hamiltonian path decomposition for K_{2n} for all n.

We now introduce some concepts that allow us to construct row-reversible Tuscan squares. In fact, all of these squares are Latin.

Let G be a group of order n. Let $\mathbf{a}$ be an arrangement $(a_1, a_2, \ldots, a_n)$ of the elements of G, and set $\mathbf{b} = (b_1, b_2, \ldots, b_{n-1})$ where $b_i = a_i^{-1}a_{i+1}$ for $1 \leq i \leq n-1$. If $\mathbf{b}$ contains all of the non-identity elements of G then $\mathbf{a}$ is a *directed terrace* for G and

$\mathbf{b}$ is a *sequencing* of G. If $b_{n-i} = b_i^{-1}$ for each i then the sequencing is *symmetric*. In this case we will also call the directed terrace associated with the sequencing *symmetric*. For a group to have a symmetric directed terrace it must be of even order with a single involution. Following [15] we call such groups *binary* groups.

Note that if $(a_1, a_2, \ldots, a_n)$ is a directed terrace for G then so is the sequence obtained by premultiplying each element by any $g \in G$, denoted $g(a_1, a_2, \ldots, a_n)$. If a_1 is the identity of the group then the directed terrace is called *basic*. Given any directed terrace we can produce a basic directed terrace by premultiplying by a_1^{-1}.

Theorem 2.1 [12] *Let G be a group of order n and $\mathbf{a}$ be a directed terrace for G. Then the Latin square with rows $\{g\mathbf{a} : g \in G\}$ is an* RCLS(n).

Proof: A square with these rows is the Cayley table of G, and so is Latin. Let x and y be distinct elements of G. Then there is exactly one value of i with $a_i^{-1}a_{i+1} = x^{-1}y$. Set $h = xa_i^{-1}$ and consider the row $h\mathbf{a}$. The ith entry of this row is $xa_i^{-1}a_i = x$ and the $(i+1)$st entry is $xa_i^{-1}a_{i+1} = xx^{-1}y = y$. So we have found an occurrence of the ordered pair (x, y) in the square. As x and y were arbitrary choices we must have that every ordered pair of distinct elements appears in the square. Therefore the square is row-complete. □

If the directed terrace is symmetric then we get the desired row-reversibility property.

Theorem 2.2 [8] *Let G be a binary group of order $2n$ and $\mathbf{a}$ be a symmetric directed terrace for G. Then the Latin square with rows $\{g\mathbf{a} : g \in G\}$ is a row-reversible* RCLS(n).

Proof: Let A be a square whose rows are $\{g\mathbf{a} : g \in G\}$. Theorem 2.1 says that A is an RCLS$(2n)$.

Let $\mathbf{a} = (a_1, a_2, \ldots, a_{2n})$ and let z be the involution of G. Then z must be in the centre of G as it is the only element of order 2. As $\mathbf{a}$ is symmetric we have $a_{2n-i} = a_i z$ for $1 \leq i \leq n$.

Consider the rows $h\mathbf{a}$ and $hz\mathbf{a}$. The ith entry of $h\mathbf{a}$ is ha_i and the $(2n-i)$th entry of $hz\mathbf{a}$ is $hza_{2n-i} = hza_i z = ha_i$. So $hz\mathbf{a}$ is the reverse of $h\mathbf{a}$. □

Example 2.1 *The following symmetric directed terrace for the (additively written) cyclic group $\mathbb{Z}_{2n}$ was used implicitly by Lucas (who gave credit to Walecki) in* [14] *and given explicitly by Williams in* [18].

$$\mathbf{a} = (0, 1, 2n-1, 2, 2n-2, 3, \ldots, n+1, n)$$

The associated symmetric sequencing is

$$\mathbf{b} = (1, 2n-2, 3, 2n-4, 5, \ldots, 2, 2n-1).$$

0	1	5	2	4	3
1	2	0	3	5	4
2	3	1	4	0	5
3	4	2	5	1	0
4	5	3	0	2	1
5	0	4	1	3	2

Figure 3: A row-reversible RCLS(6)

Following [8] *we call* **a** *the* LWW directed terrace*.*

For $\mathbb{Z}_6$ *we have* $\mathbf{a} = (0, 1, 5, 2, 4, 3)$ *and* $\mathbf{b} = (1, 4, 3, 2, 5)$*. Theorem 2.1 gives the row-reversible* RCLS(6) *in Figure 3.*

Example 2.2 [8] *Let* Q_{12} *be the dicyclic group of order* 12*:*

$$Q_{12} = \langle u, w : u^6 = e,\ w^2 = u^3,\ wu = u^5 w \rangle.$$

A symmetric directed terrace for Q_{12} *is*

$$(e, u, u^3w, u^5, u^5w, uw, u^4w, u^2w, u^2, w, u^4, u^3)$$

whose associated symmetric sequencing is

$$(u, u^2w, uw, w, u^4, u^3, u^2, u^3w, u^4w, u^5w, u^5).$$

Figure 4 gives half of the rows of the row-reversible RCLS(12) *obtained by applying Theorem 2.1. The remaining rows may be found by reversing those given.*

e	u	u^3w	u^5	u^5w	uw	u^4w	u^2w	u^2	w	u^4	u^3
u	u^2	u^4w	e	w	u^2w	u^5w	u^3w	u^3	uw	u^5	u^4
u^2	u^3	u^5w	u	uw	u^3w	w	u^4w	u^4	u^2w	e	u^5
w	u^5w	e	uw	u^4	u^2	u^5	u	u^4w	u^3	u^2w	u^3w
uw	w	u	u^2w	u^5	u^3	e	u^2	u^5w	u^4	u^3w	u^4w
u^2w	uw	u^2	u^3w	e	u^4	u	u^3	w	u^5	u^4w	u^5w

Figure 4: Half of a row-reversible RCLS(12)

Let G be a binary group of order $2n$. Given a symmetric directed terrace for G we can construct a row-reversible RCLS($2n$) and hence a Hamiltonian path decompostion K_{2n} and finally an SHLS($2n$). We call an SHLS($2n$) obtained in this way *based* on G. More generally, if it possible to obtain an SHLS($2n$) from a symmetric directed terrace of some group we call it *group-based*. Figure 5 gives the SHLS(12) obtained from the symmetric directed terrace for Q_{12} given in Example 2.2. The

	e	u	u^2	u^3	u^4	u^5	w	uw	u^2w	u^3w	u^4w	u^5w
e	1	1	5	5	6	3	2	4	3	6	2	4
u	1	2	2	6	6	4	5	3	5	1	4	3
u^2	5	2	3	3	4	4	1	6	1	6	2	5
u^3	5	6	3	1	1	5	6	2	4	2	4	3
u^4	6	6	4	1	2	2	1	4	3	5	3	5
u^5	3	4	4	5	2	3	6	2	5	1	6	1
w	2	5	1	6	1	6	4	5	2	3	3	4
uw	4	3	6	2	4	2	5	5	6	3	1	1
u^2w	3	5	1	4	3	5	2	6	6	4	1	2
u^3w	6	1	6	2	5	1	3	3	4	4	5	2
u^4w	2	4	2	4	3	6	3	1	1	5	5	6
u^5w	4	3	5	3	5	1	4	1	2	2	6	6

Figure 5: An SHLS(12) based on Q_{12}

square has been bordered with the elements of Q_{12} to make the correspondence to the row-reversible RCLS(12) clearer.

The method for constructing symmetric Hamiltonian double Latin squares given in [13] is exactly equivalent to constructing SHLS($2n$)s based on $\mathbb{Z}_{2n}$. They use the terminology "n-procession" and "n-gradation." In the vocabulary we are using, an *n-procession* is a list of the first n elements of a symmetric directed terrace for $\mathbb{Z}_{2n}$; an *n-gradation* is a list of the first $n-1$ elements of a symmetric sequencing. Having either the n-procession or the n-gradation is sufficient to reconstruct the symmetric sequencing or symmetric directed terrace.

Much is known about symmetric sequencings for binary groups. It is known that the quaternion group Q_8 does not have a symmetric directed terrace, but that every other soluble binary group does [5]. Some insoluble binary groups are also known to have symmetric directed terraces [6]. It is conjectured that Q_8 is the only binary group which does not have a symmetric directed terrace [5].

Let G be a binary group with involution z. We use $\Lambda(G)$ for $\langle z \rangle$, which is a normal subgroup of G. There is a connection between "terraces" in $G/\Lambda(G)$ and symmetric directed terraces in G which we now outline.

Let H be a group of order n. Let $\mathbf{a}$ be the arrangement $(a_1, a_2, \ldots, a_n)$ of the elements of H and let $\mathbf{b} = (b_1, b_2, \ldots, b_{n-1})$, where $b_i = a_i^{-1}a_{i+1}$ for $1 \leq i \leq n-1$. Suppose that $\mathbf{b}$ contains each involution of H exactly once and for each element h of H of order greater that 2 one of the following applies:

- h occurs twice in $\mathbf{b}$ and h^{-1} does not occur,
- h and h^{-1} both occur once in $\mathbf{b}$,
- h^{-1} occurs twice in $\mathbf{b}$ and h does not occur.

Then $\mathbf{a}$ is called a *terrace* for H and $\mathbf{b}$ is called a *2-sequencing* for H.

Example 2.3 *The following is a terrace for* $\mathbb{Z}_{2m+1}$*:*

$$(0, 1, 2m, 2, 2m-1, 3, \ldots, m, m+1).$$

Its associated 2*-sequencing is* $(1, 2m-1, 3, 2m-3, \ldots, 2m-1, 1)$*. This terrace was first given by Williams in* [18]*. Due to the similarity with the LWW directed terrace for cyclic groups of even order we call this the LWW terrace for* $\mathbb{Z}_{2m+1}$*.*

Terraces were introduced by Bailey [7] to construct quasi-complete Latin squares. They were used by Anderson [3] to construct symmetric directed terraces:

Theorem 2.3 [3] *Let* G *be a binary group. Then* G *has a symmetric sequencing if and only if* $G/\Lambda(G)$ *has a* 2*-sequencing.*

Proof: We outline the construction. Details of its correctness may be found in [3] or [15].

Suppose that G has order $2n$ and involution z. Define $\phi : G \to G/\Lambda(G)$ to be the natural projection. If $(b_1, b_2, \ldots, b_{n-1}, z, b_{n-1}^{-1}, \ldots, b_2^{-1}, b_1^{-1})$ is a symmetric sequencing of G then apply ϕ to the first $n-1$ elements to get the sequence $(\phi(b_1), \phi(b_2), \ldots, \phi(b_{n-1}))$ of elements of $G/\Lambda(G)$. This sequence is a 2-sequencing of $G/\Lambda(G)$.

Suppose now that we have a 2-sequencing $(d_1, d_2, \ldots, d_{n-1})$ of $G/\Lambda(G)$. Construct a sequence $(b_1, b_2, \ldots, b_{n-1})$ of elements of G as follows, where $d_i = \{x_i, x_i z\}$:

- if d_i has order 2 then either set $b_i = x_i$ or set $b_i = x_i z$,
- if d_i has order greater than 2 and $d_j = d_i$ for some $j \neq i$ then either set $b_i = x_i$ and $b_j = x_i z$ or set $b_i = x_i z$ and $b_j = x_i$,
- if d_i has order greater than 2 and $d_j = d_i^{-1}$ then either set $b_i = x_i$ and $b_j = x_i^{-1} z$ or set $b_i = x_i z$ and $b_j = x_i^{-1}$.

Extend $(b_1, b_2, \ldots, b_{n-1})$ to a sequence $(b_1, b_2, \ldots, b_{2n-1})$ by setting $b_n = z$ and $b_{2n-i} = b_i^{-1}$ for $1 \leq i \leq n-1$. Then $(b_1, b_2, \ldots, b_{2n-1})$ is a symmetric sequencing of G. □

Note that this result gives many symmetric sequencings for G for each 2-sequencing of $G/\Lambda(G)$. If we obtain a symmetric sequencing $\mathbf{b}$ for G in this way from a 2-sequencing $\mathbf{d}$ of $G/\Lambda(G)$, then we say that $\mathbf{b}$ is a *lift* of $\mathbf{d}$, and that $\mathbf{d}$ is the *half-projection* of $\mathbf{b}$. If $\mathbf{a}$ is a symmetric directed terrace associated with $\mathbf{b}$ and $\mathbf{c}$ is a terrace associated with $\mathbf{d}$ then we also call $\mathbf{a}$ a *lift* of $\mathbf{c}$ and $\mathbf{c}$ the *half-projection* of $\mathbf{a}$.

Example 2.4 *The LWW directed terrace for* $\mathbb{Z}_{2n}$ *is a lift of the LWW (directed) terrace for* $\mathbb{Z}_n$*.*

Example 2.5 [8] *The dihedral group of order* 6*, denoted* D_6*, is given by*

$$D_6 = \langle u, w : u^3 = e = w^2, wu = u^2w \rangle.$$

A terrace for D_6 *is* (e, u, w, u^2, u^2w, uw)*, which has associated* 2*-sequencing* (u, u^2w, uw, w, u)*. We have* $Q_{12}/\Lambda(Q_{12}) \cong D_6$*. The symmetric sequencing for* Q_{12} *given in Example 2.2 is a lift of the above* 2*-sequencing for* D_6*.*

For each group of order at most 11, and for $\mathbb{Z}_{13}$, the total number of terraces is known. This can be used to calculate the number of symmetric directed terraces for binary groups of order at most 22, and for $\mathbb{Z}_{26}$ [8, 16]. Large numbers of terraces for other small groups (order up to 20) are known [16] and hence many symmetric directed terraces for binary groups (order up to 40) can be constructed.

As we have seen, row-reversible Tuscan squares are equivalent to symmetric Hamiltonian double Latin squares. Several alternative constructions for row-reversible Tuscan squares (not all of them Latin) are given in [8].

3 Constructing orthogonal SHLS($2n$)s

Let G be a binary group of order $2n$. Define two symmetric directed terraces $(a_1, a_2, \ldots, a_{2n})$ and $(c_1, c_2, \ldots, c_{2n})$ to be *orthogonal* if

- $a_1 = c_1$ and $a_n = c_n$,
- for each $g \in G$ there is a pair (i, j) with $1 \leq i \leq 2n-1$ and $1 \leq j \leq n$ such that $\{a_i, a_{i+1}\} = \{gc_j, gc_{j+1}\}$.

When $\{a_i, a_{i+1}\} = \{gc_j, gc_{j+1}\}$ we have $a_i^{-1}a_{i+1} = (c_j^{-1}c_{j+1})^{\pm 1}$. So, as $j \leq n$, each i has a unique value of j associated with it, and different values of i in the range $1 \leq i \leq n$ give different values of j.

Example 3.1 *In* $\mathbb{Z}_6$*, the two symmetric directed terraces* $(0, 1, 5, 2, 4, 3)$ *and* $(0, 4, 5, 2, 1, 3)$ *are orthogonal.*

Our first aim is to show that symmetric Hamiltonian double Latin squares constructed from mutually orthogonal symmetric directed terraces are mutually orthogonal (Theorem 3.2).

Let H be a subgraph of K_{2n}. Let $E(H)$ denote the set of edges of H and $V_1(H)$ denote the set of vertices of degree one of H. Two Hamiltonian paths H_1 and H_2 of K_{2n} are *orthogonal* if

$$2\,|E(H_1 \cap H_2)| + |V_1(H_1) \cap V_1(H_2)| = 4.$$

Two Hamiltonian path decompositions $\mathcal{H}_1$ and $\mathcal{H}_2$ are *orthogonal* if each path of $\mathcal{H}_1$ is orthogonal to every path of $\mathcal{H}_2$. More on the orthogonality of graphs may be found in [1].

Theorem 3.1 [13] *Two* SHLS($2n$)*s are orthogonal if and only if their underlying Hamiltonian path decompositions of* K_{2n} *are orthogonal.* □

Theorem 3.2 *Let* **a** *and* **c** *be orthogonal symmetric directed terraces for a binary group* G. *Let* A *and* C *be the symmetric Hamiltonian double Latin squares constructed from* **a** *and* **c** *respectively. Then* A *and* C *are orthogonal.*

Proof: By Theorem 3.1 it is sufficient to show that the Hamiltonian path decompositions given by **a** and **c** are orthogonal.

Let $|G| = 2n$, let e be the identity of G and let z be the involution of G. Let $\mathbf{a} = (a_1, a_2, \ldots, a_{2n})$ and $\mathbf{c} = (c_1, c_2, \ldots, c_{2n})$. Choose a subset $\tilde{G}$ of G with $e \in \tilde{G}$ and for each $g \in G$ exactly one of g and gz is in $\tilde{G}$ (note: $|\tilde{G}| = n$ and $z \notin \tilde{G}$). The Hamiltonian path decompositions of K_{2n} associated with **a** and **c** are $\{g\mathbf{a} : g \in \tilde{G}\}$ and $\{h\mathbf{c} : h \in \tilde{G}\}$ respectively. So we need to show that $g\mathbf{a}$ and $h\mathbf{a}$ are orthogonal (as Hamiltonian paths) for each pair $g, h \in \tilde{G}$. It is sufficient to show that $2\,|E(g\mathbf{a} \cap h\mathbf{c})| + |V_1(g\mathbf{a}) \cap V_1(h\mathbf{c})| \geq 4$ for each pair g, h.

First suppose that $g = h$. Then

- $ga_1 = ha_1$,
- $ga_{2n-1} = ga_1 z = hc_1 z = hc_{2n-1}$,
- $ga_n = hc_n$,
- $ga_{n+1} = ga_n z = hc_n z = hc_{n+1}$.

This gives $2\,|E(g\mathbf{a} \cap h\mathbf{c})| + |V_1(g\mathbf{a}) \cap V_1(h\mathbf{c})| \geq 2 + 2 = 4$.

Now suppose that $g \neq h$. As **a** and **c** are orthogonal, for each $f \in G$ there are i and j such that $\{a_i, a_{i+1}\} = \{fc_j, fc_{j+1}\}$ for some $1 \leq i \leq 2n-1$ and $1 \leq j \leq n$. Set $f = g^{-1}h$. We cannot have $i = n$. If we did then $a_i^{-1}a_{i+1} = z$ (as **a** is symmetric) and so $(fc_j)^{-1}fc_{j+1} = z$ and so j must be n. But $a_n = c_n$ and so $f = e$ and hence $g = h$. For similar reasons we cannot have $j = n$ either.

So we have $\{a_i, a_{i+1}\} = \{g^{-1}hc_j, g^{-1}hc_{j+1}\}$ and hence $\{ga_i, ga_{i+1}\} = \{hc_j, hc_{j+1}\}$ for some $1 \leq i \leq 2n-1$, with $i \neq n$, and $1 \leq j \leq n-1$. Set $i' = 2n - i$ and $j' = 2n - j$. Then

$$\begin{aligned} \{ga_{i'}, ga_{i'+1}\} &= \{ga_i z, ga_{i+1} z\} \\ &= \{hc_j z, hc_{j+1} z\} \\ &= \{hc_{j'}, hc_{j'+1}\}. \end{aligned}$$

So $2\,|E(g\mathbf{a} \cap h\mathbf{c})| + |V_1(g\mathbf{a}) \cap V_1(h\mathbf{c})| \geq 4 + 0 = 4$.

The Hamiltonian path decompostions given by **a** and **c** are orthogonal and hence the symmmetric Hamiltonian double Latin squares A and C are orthogonal. □

The two orthogonal SHLS(6)s in Figure 2 are based on $\mathbb{Z}_6$ using the pair of orthogonal symmetric directed terraces in Example 3.1.

Let G be a binary group. The study of terraces for $G/\Lambda(G)$ has often led to fruitful results concerning symmetric directed terraces for G, see, for example, [4, 5, 8, 17]. In our case, it also seems that there is something to be gained from the study of terraces.

Let H be a group of order n, and let $\mathbf{a}$ and $\mathbf{c}$ be the basic terraces $(a_1, a_2, \dots, a_n)$ and $(c_1, c_2, \dots, c_n)$ respectively, with $a_n = c_n$. For a fixed value of i satisfying $1 \leq i \leq n-1$, consider the equation

$$\{a_i, a_{i+1}\} = \{hc_j, hc_{j+1}\},$$

where $1 \leq j \leq n-1$ and $h \in H$. For the equation to hold we must have $a_i^{-1}a_{i+1} = (c_j^{-1}c_{j+1})^{\pm 1}$. If $a_i^{-1}a_{i+1}$ is an involution, then there is one value of j for which this holds and there are two possible values for h which give solutions to the equation. If $a_i^{-1}a_{i+1}$ has order greater than 2, then there are two values of j for which this holds and there is one value of h corresponding to each of these values of j which gives a solution to the equation. So $\{a_i, a_{i+1}\} = \{hc_j, hc_{j+1}\}$ has two solutions for any fixed i. Let S_i be the pair of solutions $\{(j_{i_1}, h_{i_1}), (j_{i_2}, h_{i_2})\}$ for a fixed i. If it is possible to choose a sequence

$$(j_1, h_1), (j_2, h_2), \dots, (j_{n-1}, h_{n-1}),$$

where $(j_i, h_i) \in S_i$ for each i and $1 \leq i \leq n-1$, with the additional properties that $\{j_1, j_2, \dots, j_{n-1}\} = \{1, 2, \dots, n-1\}$ and $\{h_1, h_2, \dots, h_{n-1}\} = H \setminus \{e\}$ then we say that $\mathbf{a}$ and $\mathbf{c}$ are *crossing.*

Example 3.2 *Let* $\mathbf{a}$ *be the LWW terrace* $(0, 1, 6, 2, 5, 3, 4)$ *for* $\mathbb{Z}_7$ *and let* $\mathbf{c}$ *be the terrace* $(0, 5, 1, 2, 3, 6, 4)$. *Then* $\mathbf{a}$ *and* $\mathbf{c}$ *are crossing; we can choose the sequence*

$$(4, 5), (1, 1), (5, 3), (2, 4), (6, 6), (3, 2).$$

Theorem 3.3 *Let* G *be a binary group with orthogonal symmetric basic directed terraces* $\mathbf{a}$ *and* $\mathbf{c}$ *whose half-projections onto* $G/\Lambda(G)$ *are* $\bar{\mathbf{a}}$ *and* $\bar{\mathbf{c}}$. *Then* $\bar{\mathbf{a}}$ *and* $\bar{\mathbf{c}}$ *are crossing.*

Proof: Let G have order $2n$ and involution z, let $\mathbf{a} = (a_1, a_2, \dots, a_{2n})$ and $\mathbf{c} = (c_1, c_2, \dots, c_{2n})$, and let $\bar{\mathbf{a}} = (\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n)$ and $\bar{\mathbf{c}} = (\bar{c}_1, \bar{c}_2, \dots, \bar{c}_n)$. As $\mathbf{a}$ and $\mathbf{c}$ are orthogonal we have that $a_n = c_n$ and hence $\bar{a}_n = \bar{c}_n$.

Set $\tilde{G}$ to be as in Theorem 3.2. As $\mathbf{a}$ and $\mathbf{c}$ are orthogonal we have that for each $g \in \tilde{G}$ there is a pair (i, j) with $1 \leq i \leq 2n-1$ and $1 \leq j \leq n$ such that $\{a_i, a_{i+1}\} = \{gc_j, gc_{j+1}\}$.

As we observed immediately after the definition of orthogonality for symmetric directed terrraces, if we choose a value of i there is only one possibility for the value of j; call this value j_i. Different values of i in the range $1 \leq i \leq n$ give different values of j_i and if $i \geq n$ then $j_i = j_{2n-i}$, so $\{j_1, j_2, \dots, j_n\} = \{1, 2, \dots, n\}$. Also,

$j_n = n$, as the involution must occur at position n in the symmetric sequencings associated with $\mathbf{a}$ and $\mathbf{c}$, and so $\{j_1, j_2, \ldots, j_{n-1}\} = \{1, 2, \ldots, n-1\}$.

Suppose that two distinct elements $g, h \in \tilde{G}$ both use the same value of i. Then $\{gc_{j_i}, gc_{j_i+1}\}$ and $\{hc_{j_i}, hc_{j_i+1}\}$ both occur as adjacent elements in the first half of $\mathbf{a}$, contradicting the fact that $\mathbf{a}$ is a symmetric directed terrace.

For each i, let g_i be the element of $\tilde{G}$ which uses the value i. Let h_i be the projection of g_i onto $G/\Lambda(G)$. If $h_i = h_{i'}$ for any pair i, i' then we must have $g_i = g_{i'}$ as $\tilde{G}$ contains just one of g and gz for each $g \in G$. Hence $|\{h_1, h_2, \ldots, h_n\}| = n$. Moreover, $g_n = e$ as $a_n = c_n$, so $h_n = e$ and $\{h_1, h_2, \ldots, h_{n-1}\} = H \setminus \{e\}$.

To show that $\bar{\mathbf{a}}$ and $\bar{\mathbf{c}}$ are crossing it only remains to show that, for each i, we have $\{\bar{a}_i, \bar{a}_{i+1}\} = \{h_i\bar{c}_{j_i}, h_i\bar{c}_{j_i+1}\}$. This follows immediately as $\{a_i, a_{i+1}\} = \{g_i c_{j_i}, g_i c_{j_i+1}\}$. □

Theorem 3.3 gives a necessary condition that a pair of terraces must satisfy if it is possible to lift them to a pair of orthogonal symmetric directed terraces. As each terrace may be lifted to a large number of different symmetric directed terraces this helps considerably in our search.

Example 3.3 *Let $p = 2m+1$ be a prime congruent to 3 modulo 4. Williams* [18] *shows that*

$$(1, 2, \ldots, m, m, m-1, \ldots, 1)$$

is a 2-sequencing for $\mathbb{Z}_p$. The associated basic terrace is

$$(0, 1, 3, \ldots \frac{m(m+1)}{2}, \frac{m(m+1)}{2} + m, \ldots, m(m+1)).$$

This is a triangular numbers terrace, of which there are m for each prime p. The general family appeared in [4] *and was named in* [17].

If we apply the automorphism of multiplying by -2 modulo p then we get the 2-sequencing $(2m-1, 2m-3, \ldots, 1, 1, 3, \ldots, 2m-1)$ which has associated basic terrace

$$(0, 2m-1, 2m-5, \ldots \frac{(m+1)}{2}, \frac{(m+1)}{2} + 1, \ldots, (m+1)).$$

The LWW terrace and this terrace are crossing.

Theorem 3.4 *If p is be a prime congruent to 3 modulo 4 then there is a pair of orthogonal symmetric directed terraces for $\mathbb{Z}_{2p}$.*

Proof: Let $p = 2m+1$ and take $\mathbf{a}$ to be the LWW directed terrace for $\mathbb{Z}_{2p}$. Let $\mathbf{c}$ be the lift of the last terrace in Example 3.3 obtained by choosing the symmetric sequencing

$$(4m, 4m-2, \ldots, 2m+2, 1, 3, \ldots, 4m+1, 2m, 2m-2, \ldots, 2).$$

As $p \equiv 3 \pmod 4$, we have that -1 is non-square and that each element of $\mathbb{Z}_{2p}$ is of the form 0, p, k^2 or $-k^2$ for some $1 \leq k \leq 2m$. For each element g of $\mathbb{Z}_{2p}$ we need a pair (i,j) as described in the definition of orthogonality. Calculations reveal that

$$(i,j) = \begin{cases} (p,p) & \text{if } g = 0 \text{ or } g = p \\ (2k,k) & \text{if } g = k^2 \text{ and } 1 \leq k \leq m \\ (2k, p-k) & \text{if } g = k^2 \text{ and } m+1 \leq k \leq 2m \\ (p + (-1)^{k+\delta}2k, p-k) & \text{if } g = -k^2 \text{ and } 1 \leq k \leq m \\ (p - (-1)^{k+\delta}2k, k) & \text{if } g = -k^2 \text{ and } m+1 \leq k \leq 2m \end{cases}$$

where $\delta = 0$ if $p \equiv 3 \pmod 8$ and $\delta = 1$ otherwise. □

Example 3.4 *Theorem 3.4 gives the following pair of orthogonal symmetric directed terraces for $\mathbb{Z}_{14}$:*

$$\begin{aligned} \mathbf{a} &= (0,1,13,2,12,3,11,4,10,5,9,6,8,7) \\ \mathbf{c} &= (0,12,8,2,3,6,11,4,13,10,9,1,5,7) \end{aligned}$$

The (i,j) pairs are

$(7,7),(2,1),(8,3),(11,5),(4,2),(13,4),(5,6),$

$(7,7),(12,1),(6,3),(3,5),(10,2),(1,4),(9,6)$

for $g = 0,1,2,\ldots,13$ respectively. These symmetric directed terraces are equivalent to the 14-processions of [13].

Corollary 3.1 *There is a pair of mutually orthogonal* SHLS$(2p)$*s for all primes p congruent to 3 modulo 4.*

Proof: Apply Theorems 3.4 and 3.2. □

An inflation theorem for symmetric Hamiltonian double Latin squares is known:

Theorem 3.5 [13] *The existence of a pair of mutually orthogonal* SHLS$(2n)$*s implies the existence of a pair of mutually orthogonal* SHLS$(4n)$*s.* □

In [2, 16] heuristic algorithms for finding terraces are described. Programs using ideas from these algorithms (written in GAP v4.2 [10]) have been used to find pairs of crossing terraces and to test whether they can be lifted to pairs of orthogonal symmetric directed terraces. Collecting together the theoretical results and the output of these programs, we have:

Theorem 3.6 *Let k be a positive integer. There is a pair of* SHLS$(2^k n)$*s whenever n is a prime congrent to 3 modulo 4 or $n \in \{1,5,9,13,15,17\}$.*

Proof: The primes congruent to 3 modulo 4 are covered by Corollary 3.1 and Theorem 3.5. The cases $n \in \{1, 5, 9, 13\}$ are covered in [13]. We give pairs of orthogonal symmetric directed terraces for $\mathbb{Z}_{30}$ and $\mathbb{Z}_{34}$; applying Theorem 3.5 completes the proof. For brevity, we give just the first half of the symmetric directed terraces (that is, the "n-procession" in the terminology of [13]).

$$\begin{array}{rl} \mathbb{Z}_{30}: & 0,16,29,17,28,18,27,19,26,20,25,21,24,22,23 \\ & 0,21,18,1,5,28,9,4,14,12,11,17,25,7,23 \\ \mathbb{Z}_{34}: & 0,1,33,2,32,3,31,4,30,5,29,6,28,7,27,8,26 \\ & 0,10,31,33,4,18,15,23,30,12,8,7,19,28,22,3,26 \end{array}$$

In both cases the first of the symmetric directed terraces is a lift of the LWW terrace for the cyclic group of half the order. □

The programs turned up two further items of note. Firstly, they found a pair of orthogonal symmetric directed terraces for $\mathbb{Z}_3 \times \mathbb{Z}_6$, a non-cyclic group. Secondly, they found three mutually orthogonal symmetric directed terraces for $\mathbb{Z}_{18}$, allowing us to construct a set of three mutually orthogonal SHLS(18)s. As in the proof of Theorem 3.6, we give the first half of the symmetric directed terraces.

$$\begin{array}{rl} \mathbb{Z}_3 \times \mathbb{Z}_6: & (0,0),(1,3),(2,4),(2,5),(1,5),(0,1),(0,5),(2,3),(1,4) \\ & (0,0),(1,2),(0,2),(1,3),(2,1),(2,0),(2,2),(0,1),(1,4) \\ \mathbb{Z}_{18}: & 0,2,6,1,4,3,14,8,16 \\ & 0,6,14,1,8,12,11,13,16 \\ & 0,12,1,17,13,5,2,15,16 \end{array}$$

References

[1] B. Alspach, K. Heinrich, and G. Liu, Orthogonal factorizations of graphs, In J. H. Dinitz and D. R. Stinson, editors, *Contemporary Design Theory, A Collection of Surveys*, pp. 13–40, Wiley, New York, 1992.

[2] B. A. Anderson, A fast method for sequencing low order non-abelian groups, *Ann. Discrete Math.* 34 (1987), 27–41.

[3] B. A. Anderson, Sequencings of dicyclic groups, *Ars. Combin.* 23 (1987), 131–142.

[4] B. A. Anderson, Some quasi-2-complete latin squares, *Congr. Numer.* 70 (1990), 65–79.

[5] B. A. Anderson and E. C. Ihrig, Every finite solvable group with a unique element of order two, except the quaternion group, has a symmetric sequencing. *J. Combin. Des.* 1 (1993), 3–14.

[6] B. A. Anderson and E. C. Ihrig, Symmetric sequencings of non-solvable groups, *Congr. Numer.* 93 (1993), 73–82.

[7] R. A. Bailey, Quasi-complete latin squares: construction and randomization, *J. Royal Statist. Soc. Ser. B* 46 (1984), 323–334.

[8] R. A. Bailey, M. A. Ollis, and D. A. Preece, Round-dance neighbour designs from terraces, *Discrete Math.* 266 (2003), 69–86.

[9] J. Dénes and A. D. Keedwell. *Latin Squares and their Applications*, Akadémiai Kiadó, Budapest/English Universities Press, London/Academic Press, 1974.

[10] GAP group, GAP—Groups, Algorithms, and Programming, Version 4, 1999.

[11] S. W. Golomb and H. Taylor, Tuscan squares—a new family of combinatorial designs, *Ars Combin.* 20B (1985), 115–132.

[12] B. Gordon, Sequences in groups with distinct partial products, *Pacific J. Math.* 11 (1961), 1309–1313.

[13] A. J. W. Hilton, M. Mays, C. A. Rodger, and C. St. J. A. Nash-Williams, Hamiltonian double latin squares, *J. Combin. Theory Ser. B* 87 (2003), 81–129.

[14] E. Lucas, *Récréations Mathématiques, Tôme II*, Albert Blanchard, Paris, 1892, reprinted 1975.

[15] M. A. Ollis, Sequenceable groups and related topics, *Electron. J. Combin.* Dynamic Surveys (10) (2002), 34pp.

[16] M. A. Ollis, *Terraces for groups: some constructions and applications (PhD thesis)*, Queen Mary, University of London, 2003.

[17] M. A. Ollis and D. A. Preece, Sectionable terraces and the (generalised) Oberwolfach problem, *Discrete Math.* 266 (2003), 399–416.

[18] E. J. Williams, Experimental designs balanced for the estimation of residual effects of treatments, *Aust. J. Scient. Res. A* 2 (1949), 149–168.

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