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On certain group embeddings in cross-characteristic

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Abstract

We explore some geometry of three group embeddings in cross- characteristic. The first embedding is $Sz(8) < P\Omega_8^+(5)$. We found that this embedding is associated with a cap in $PG(7,5)$. The corresponding cap-code is also determined. The second embedding is $PSL_2(13) < G_2(4)$ and we explore its action on quadrics of $PG(5,4)$ invariant under $PSp_6(4)$. The last embedding is $PSL_2(7) < A_7 < PSU_3(25)$ which is associated with a spread of the Hermitian curve $\mathcal{H}(2,25)$.

1 Introduction

Let G be one of the finite classical groups acting on a vector space V over a finite field F of characteristic r. It has been shown by Aschbacher [1] that every subgroup of G either lies in one of a family $\mathcal{C}_i$, $i = 1, \ldots, 8$ of natural geometric subgroups of G or has the form $N_G(X)$, where X is a quasisimple subgroup of G and V is an absolutely irreducible X-module realisable over no proper subfield of F. In the latter case, the problem is that of deciding when $N_G(X)$ is maximal. If $N_G(X)$ lies in no subgroup in $\mathcal{C}_i$ but is not maximal in G then X must lie in a larger quasisimple subgroup Y of G. This leads to the problem of determining all triples (X, Y, V) of this kind. Let p be the characteristic of X. Much is known about the case $p = r$. Here, it is assumed that $p \neq r$, so that we are in the cross-characteristic case.

In this paper, we do not prove any maximality result, but we are mainly interested in the geometry of two group embeddings in cross-characteristic. The first embedding is $Sz(8) < P\Omega_8^+(5)$. We found that this embedding is associated with a cap in $PG(7,5)$. The corresponding cap-code is also determined. The second embedding is $PSL_2(13) < G_2(4)$ and we explore its action on quadrics of $PG(5,4)$ invariant under $PSp_6(4)$.

2 The embedding $Sz(8) < P\Omega_8^+(5)$

The embedding $Sz(8) < P\Omega_8^+(5)$ arises from the embedding of $2 \cdot Sz(8)$ in $P\Omega_8^+(5)$. Notice that the double cover $2Sz(8)$ has a $(p,8)$-representation only when $p=5$ and it is absolutely irreducible. The group $Sz(8)$ is maximal in $P\Omega_8^+(5)$, [4].

It follows from the ATLAS that the group $2.Sz(8)$ may be generated by the maps A and B which take $(X_\infty, X_0, \ldots, X_6)$ (subscripts mod 7) to $(X_4, -X_5, X_2, X_1, -X_6, X_\infty, X_0, X_3)$ and $(X_\infty, X_1, X_2, X_3, X_4, x_5, X_6, X_0)$, respectively, together with the map $C : X_\infty \mapsto 2X_\infty + X_0 + X_1 + X_2 + X_3 + X_4 + X_5 + X_6$, $X_t \mapsto X_\infty - X_{-t} - X_{3-t} + X_{4-t} + 3X_{1-t} + 3X_{2-t} + 3X_{6-t}$.

Generating matrices for $Sz(8)$ in its orthogonal representation are:

$$M1 = \begin{pmatrix} 1 & 4 & 3 & 2 & 1 & 3 & 3 & 0 \\ 3 & 3 & 0 & 1 & 4 & 1 & 2 & 2 \\ 2 & 3 & 2 & 3 & 2 & 3 & 1 & 2 \\ 0 & 2 & 4 & 3 & 4 & 3 & 2 & 4 \\ 1 & 2 & 1 & 4 & 0 & 2 & 3 & 2 \\ 3 & 4 & 2 & 4 & 3 & 0 & 3 & 1 \\ 2 & 1 & 4 & 0 & 3 & 1 & 2 & 2 \\ 4 & 0 & 2 & 2 & 2 & 1 & 2 & 4 \end{pmatrix}$$

and

$$M2 = \begin{pmatrix} 3 & 3 & 4 & 1 & 0 & 2 & 4 & 3 \\ 3 & 1 & 4 & 2 & 4 & 3 & 0 & 3 \\ 4 & 4 & 0 & 2 & 2 & 2 & 1 & 2 \\ 1 & 2 & 2 & 1 & 4 & 0 & 2 & 3 \\ 0 & 4 & 2 & 4 & 3 & 4 & 3 & 2 \\ 2 & 3 & 2 & 0 & 4 & 1 & 4 & 3 \\ 4 & 0 & 1 & 2 & 3 & 4 & 2 & 2 \\ 3 & 3 & 2 & 3 & 2 & 3 & 2 & 4 \end{pmatrix}.$$

The absolutely irreducible subgroup $2.2^{3+3} : 7$ of $2.Sz(8)$ (of index 65) is represented monomially. Thus there is a configuration of 65 coordinate frames preserved by $2.Sz(8)$ (and also by $Sz(8)$), but not by $P\Omega_8^+(5)$. These frames correspond to the points of the Suzuki-Tits ovoid in $PG(3,8)$. The frames of course are, the canonical frame $\{< e_1 >, < e_2 >, < e_3 >, < e_4 >, < e_5 >, < e_6 >, < e_7 >, < e_8 >\}$ and its images. By direct computations, the $Sz(8)$-invariant quadric of $PG(7,5)$ is $\mathcal{Q} : X_0^2 + X_\infty^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2 + X_6^2 = 0$.

From our previous discussion, $Sz(8)$ has an orbit $\mathcal{O}$ of size $520 = 8 \times 65$ consisting of 65 non-singular frames. A point of $\mathcal{O}$ corresponds to a secant section of the Suzuki-Tits ovoid of $PG(3,8)$.

The map $(X, Y, Z) \to (x, y, z)$ where,

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$Y = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$Z = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 4 & 3 & 3 & 4 & 1 & 0 & 3 \\ 1 & 3 & 3 & 4 & 1 & 0 & 3 & 4 \\ 1 & 3 & 4 & 1 & 0 & 3 & 4 & 3 \\ 1 & 4 & 1 & 0 & 3 & 4 & 3 & 3 \\ 1 & 1 & 0 & 3 & 4 & 3 & 3 & 4 \\ 1 & 0 & 3 & 4 & 3 & 3 & 4 & 1 \\ 1 & 3 & 4 & 3 & 3 & 4 & 1 & 0 \end{pmatrix},$$

and

$$x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix},$$

$$y = \begin{pmatrix} w^3 & 0 & 0 & 0 \\ 0 & w^2 & 0 & 0 \\ 0 & 0 & w^{-2} & 0 \\ 0 & 0 & 0 & w^{-3} \end{pmatrix},$$

$$z = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

gives an isomorphism between the orthogonal representation of $Sz(8)$ and the natural $GF(8)$-representation. It turns out that $\langle X^2, Y\rangle$ is the group $2^3 : 7$, which fixes the point $(1,0,0,0,0,0,0,0)$ in the $GF(5)$-representation and the point $(0,1,0,0)$ in the natural representation.

It should be noted that the group $2 \cdot Sz(8)$ is an exceptional cover, and may be treated sporadically. Its 8-dimensional representation is exceptional in the sense that it is not a 5-modular reduction of a characteristic zero representation of $2 \cdot Sz(8)$.

We have the following proposition.

Proposition 2.1 *The group $Sz(8)$ in its orthogonal representation is the full stabilizer in $P\Omega_8^+(5)$ of a set of 65 non-singular frames. The points of these frames are permuted in a single orbit $\mathcal{O}$ and form a 520-cap of $PG(7,5)$.*

Proof. We need to prove the second part of the Proposition. Suppose there are three collinear points X_1, X_2, X_3 on $\mathcal{O}$. Of course, they cannot be points of a frame and they can lie on at most three frames. Notice that, the group $Sz(8)$ is 2-transitive on the 65 frames of $\mathcal{O}$ and transitive on $\mathcal{O}$. It follows that we can choose a frame F_1 as the canonical frame $\{< e_1 >, < e_2 >, < e_3 >, < e_4 >, < e_5 >, < e_6 >, < e_7 >, < e_8 >\}$ and a frame F_2 as the image of F_1 under M_1. Direct and tedious calculations show that in any case we get a contradiction. □

2.1 The code associated to $\mathcal{O}$

A $GF(q)$-linear code C of length n, dimension k and minimum distance d is said to be a q-ary $[n,k,d]$-code.

For given q, n and k the largest value of d for which a q-ary $[n,k,d]$-code exists is denoted by $d_q(n,k)$. An extensive table of bounds on $d_2(n,k)$ is provided in [2, Table I].

An n-cap of the $(k-1)$-dimensional projective space $\mathrm{PG}(k-1,q)$ over the finite field $GF(q)$ is a set of n points no three of which are collinear. A cap is *complete* if it cannot be extended to a larger cap with addition of points.

In [12] a construction of linear codes arising from caps is described. More precisely, let K be an n-cap of $\mathrm{PG}(k-1,q)$ generating the whole space and let A be the matrix whose columns are the homogeneous coordinate vectors of points of K. The code C arising from K is the $[n,k]$-code with generator matrix A. Codes arising from caps are called *cap-codes*.

For a cap-code given by an n-cap K of $\mathrm{PG}(k-1,q)$, any subspace of rank 1 of the code corresponds to a hyperplane of $\mathrm{PG}(k-1,q)$. The non-zero words of a rank 1 subspace have weight $n-t$ if and only if the corresponding hyperplane intersects the cap K in exactly t points.

Using the software package MAGMA [3], we found that the cap-code associated with C has length 520, dimension 8 and minimum distance 387 and information rate 1/65. The automorphism group of C is $2 \cdot Sz(8) \cdot 2$.

3 The embedding $PSL_2(13) < G_2(4)$

Let $PG(5,4)$ be the 5-dimensional projective space over $GF(4)$ equipped with a symplectic polarity Λ. Let G be the group $PSp_6(4)$ of Λ. Although $PSp_6(4)$ contains a number of subgroups isomorphic to the Cartan-Dickson-Chevalley exceptional group $G_2(4)$, we write $G_2(4)$ for a fixed representative. The group $G_2(4)$ is maximal in $PSp_6(4)$, [5], [6].

When q is even, an orthogonal polarity is a symplectic one. The group G acts on the set Φ of all quadrics of $PG(5,4)$ inducing the polarity Λ and contains the corresponding orthogonal groups.

Let V be the 6-dimensional vector space underlying $PG(5,4)$, and let $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ be a basis of V so that the basic nonsingular alternating form B has the canonical coordinate form

$$B(x,y) = \sum_{i=1}^{3}(x_i y_{3+i} + y_i x_{3+i}). \tag{1}$$

A quadratic form Q on V has B for its corresponding polar form if and only if

$$B(x,y) = Q(x+y) + Q(x) + Q(y). \tag{2}$$

From (1) it follows that Q is a quadratic form for a quadric in Φ, with associated alternating form B, if and only if

$$Q(x) = x_1x_4 + x_2x_5 + x_3x_6 + \sum_{i=1}^{6}\alpha_i x_i^2,$$

for some $\alpha_i \in GF(4)$. Dye shows in [10] that two members $\mathcal{Q}_1, \mathcal{Q}_2$ of Φ (with corresponding quadratic forms Q_1, Q_2 associated with B) are equivalent, that is, have the same Witt index, if and only if there is a transvection of the group $Sp_6(4)$ of B transforming Q_1 into Q_2. The choices for the α_i's lead to 4^6 distinct quadratic forms associated with B. The group G has two orbits on Φ, [8, Theorem 14]: an orbit of size $4^3(4^3+1)/2$ consisting of maximal Witt index quadratic forms and denoted by $\mathcal{H}_G$, and an orbit of size $4^3(4^3-1)/2$ consisting of non-maximal Witt index quadratic forms and denoted by $\mathcal{E}_G$. From a projective point of view, members of $\mathcal{H}_G$ give rise to hyperbolic quadrics $\mathcal{Q}^+(5,4)$ inducing the polarity Λ, and members of $\mathcal{E}_G$ give rise to elliptic quadrics $\mathcal{Q}^-(5,4)$ inducing the polarity Λ.

The *Arf invariant* of Q with respect to the basis $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ is

$$\Delta(Q) = \sum_{i=1}^{3} Q(e_i)Q(e_{3+i}).$$

Let $L = \{\lambda^2 + \lambda : \lambda \in GF(4)\}$, the image of the homomorphism $\lambda \mapsto \lambda^2 + \lambda$ of the additive group $GF(4)^+$of $GF(4)$ whose kernel is $GF(2)$. Thus L is a subgroup of $GF(4)^+$ isomorphic to $GF(4)^+/GF(2)^+$. Write $\bar{\Delta}(Q)$ for the class of $\Delta(Q)$ modulo L.

Dye shows in [9], [10] that $\bar{\Delta}(Q)$ is independent of the choice of the symplectic base, and two quadratic froms associated with B, say Q_1 and Q_2, are equivalent if and only if $\bar{\Delta}(Q_1) = \bar{\Delta}(Q_2)$. In particular, Q has non-maximal Witt index if and only if $\Delta(Q)$ does not belong to L.The group $G_2(4)$ acts transitively on both $\mathcal{H}_G$ and $\mathcal{E}_G$ [6].

From [5], the group $H = PSL_2(13)$ is always a subgroup of $G_2(4^i)$, $i \geq 1$ and it is a maximal subgroup if and only if $i = 1$. The group H is generated by the matrices:

$$g1 = \begin{pmatrix} \omega & \omega^2 & \omega^2 & 1 & 0 & \omega \\ \omega & 0 & \omega^2 & \omega^2 & 0 & 1 \\ 0 & 1 & \omega^2 & 0 & \omega & 1 \\ 1 & \omega^2 & \omega & 1 & 0 & \omega \\ 0 & 1 & 1 & 0 & \omega & 0 \\ 0 & \omega & \omega^2 & 0 & \omega^2 & 1 \end{pmatrix},$$

and

$$g2 = \begin{pmatrix} \omega & 1 & 1 & 0 & 0 & 0 \\ \omega^2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \omega^2 & 0 & 0 & 0 \\ 1 & 1 & \omega^2 & \omega & \omega^2 & \omega^2 \\ 1 & \omega & \omega & 0 & \omega^2 & \omega \\ 0 & \omega^2 & \omega & 0 & 1 & 0 \end{pmatrix}.$$

where $\omega^2 = \omega + 1$. Direct computations show that H acts transitively on the set $\mathcal{Y}$ of 14 quadrics of $\mathcal{E}_G$ with equations $x_1x_4 + x_2x_5 + x_3x_6 + \sum_{i=1}^{6} \alpha_i x_i^2$, where $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) \in$:

$$\{(1, 0, \omega^2, \omega, 0, \omega), (\omega, \omega, 0, \omega, \omega^2), (\omega, 1, \omega^2, \omega^2, \omega, \omega),$$

$$(\omega^2, 1, 0, \omega^2, 0, 1), (\omega^2, \omega, 1, 1, \omega, \omega), (1, 0, \omega, 1, 0, 1),$$

$$\omega^2, \omega, \omega^2, 1, 0, \omega), (\omega^2, 0, 1, 1, \omega, 1), (\omega^2, 0, \omega, \omega, \omega^2, \omega),$$

$$(0, 1, 1, \omega^2, 1, 1), (0, 1, \omega, 0, \omega, \omega^2), (0, \omega^2, 1, \omega, \omega, 1),$$

$$(0, \omega^2, \omega^2, \omega, 1, 1), (\omega^2, 1, 0, \omega^2, \omega^2, \omega).$$

Proposition 3.1 *The group $PSL_2(13)$ as a subgroup of $G_2(4)$ is the full stabilizer of a set $\mathcal{Y}$ of 14 quadrics in $\mathcal{E}_G$. This action of H is equivalent to the primitive action of $PSL_2(13)$ on the bases of $PG(2, 13)$ when $PSL_2(13)$ is considered as stabilizer of a conic in $PG(2, 13)$.*

Remark 3.2 It should be noted that the group $PSL_2(13)$ as a subgroup of $G_2(4)$ is also the stabilizer of a certain distance 2-ovoid of the split Cayley Hexagon $H(4)$, see [7].

4 The embedding $PSL_2(7) < A_7 < PSU_3(25)$

Let $PG(2,25)$ denote the Desarguesian projective plane over the finite field $GF(25)$. A *Hermitian curve* $\mathcal{H}(2,25)$ in $PG(2,25)$ is defined as the set of all isotropic points of a non-degenerate unitary polarity $\perp$ of $PG(2,25)$. The number of points of $\mathcal{H}(2,25)$ is 126. If P is a point in $PG(2,25)$, then the polar line $P^\perp$ of P meets $\mathcal{H}(2,25)$ in 1 or 6 points, according as P lies on $\mathcal{H}(2,25)$ or does not. Lines of the first type are called *tangents*, those of the second type are called *secants* of $\mathcal{H}(2,25)$. There is just one tangent at every point $P \in \mathcal{H}(2,25)$, whereas the remaining 25 lines through P are secants. If $P \notin \mathcal{H}(2,25)$, then through P there are 6 tangents (meeting $\mathcal{H}(2,25)$ in the points of $P^\perp \cap \mathcal{H}(2,25)$) and 20 secants.

Let $PSU_3(25)$ denote the special projective unitary group associated with $\perp$. The group $PSU_3(q^2)$ acts doubly transitively on the set of points of $\mathcal{H}(2,25)$.

A *spread* of $\mathcal{H}(2,25)$ is a set of 21 pairwise disjoint secants that partition the points of $\mathcal{H}(2,25)$.

An obvious method for constructing a spread of $\mathcal{H}(2,25)$ requires one to start with a non-isotropic point $P \in PG(2,25)$. The set of secants to $\mathcal{H}(2,25)$ through P, together with $P^\perp$, form a spread of $\mathcal{H}(2,25)$. This spread is called *regular*.

Let $\mathcal{H}(2,25)$ be the Hermitian curve with equation $X_1^6 + X_2^6 + X_3^6 = 0$, where X_1, X_2, X_3 are projective homogeneous coordinates in $PG(2,25)$. The group K generated by the matrices

$$h_1 = \begin{pmatrix} \omega^4 & 1 & 3 \\ \omega^8 & \omega^{10} & \omega^{16} \\ \omega^2 & \omega^{16} & \omega^4 \end{pmatrix},$$

and

$$h_2 = \begin{pmatrix} \omega^{11} & \omega^{17} & \omega^5 \\ \omega^7 & \omega^9 & \omega^5 \\ 4 & \omega^{19} & \omega^{11} \end{pmatrix},$$

where $\omega \in GF(25)$ and $\omega^2 = \omega + 2$ is isomorphic to $PSL_2(7)$ and leaves $\mathcal{H}(2,25)$ invariant. The orbit of the line $r : \omega^{14}X_1 + \omega^5 X_2 - X_3 = 0$ under K gives rise to a spread of $\mathcal{H}(2,25)$. This spread was firstly noted by T. Penttila [14]. The group K is easily seen to be maximal in A_7 which in turn is maximal in $PSU_3(25)$. For more geometry on the groups embedding $A_7 < PSU_3(25)$, see [11].

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