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A neighborhood and degree condition for pancyclicity and vertex pancyclicity

Ryota Matsubara Masao Tsugaki

Department of Mathematical Information Science
Tokyo University of Science
1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601
Japan
qq8c6dt9n@able.ocn.ne.jp tsugaki@hotmail.com

Tomoki Yamashita

Department of Mathematics
School of Dentistry, Asahi University
1851 Hozumi, Gifu 501–0296
Japan
tomoki@dent.asahi-u.ac.jp

Abstract

Let G be a 2-connected graph of order n. For any $u \in V(G)$ and $l \in \{m, m+1, \ldots, n\}$, if G has a cycle of length l, then G is called $[m,n]$-pancyclic, and if G has a cycle of length l which contains u, then G is called $[m,n]$-vertex pancyclic. Let $\delta(G)$ be a minimum degree of G and let $N_G(x)$ be the neighborhood of a vertex x in G. In [*Australas. J. Combin.* 12 (1995), 81–91] Liu, Lou and Zhao proved that if $|N_G(u) \cup N_G(v)| + \delta(G) \geq n+1$ for any nonadjacent vertices u, v of G, then G is $[3,n]$-vertex pancyclic. In this paper, we prove if $n \geq 6$ and $|N_G(u) \cup N_G(v)| + d_G(w) \geq n$ for every triple independent vertices u, v, w of G, then (i) G is $[3,n]$-pancyclic or isomorphic to the complete bipartite graph $K_{n/2,n/2}$, and (ii) G is $[5,n]$-vertex pancyclic or isomorphic to the complete bipartite graph $K_{n/2,n/2}$.

1 Introduction

In this paper, we consider only finite graphs without loops or multiple edges. For standard graph-theoretic terminology not explained in this paper, we refer the reader to [3]. We denote by $N_G(x)$ the neighborhood of a vertex x in a graph G. For a subgraph H of G and a vertex $x \in V(G)$, we also denote $N_H(x) := N_G(x) \cap V(H)$

and $d_H(x) := |N_H(x)|$. For $X \subset V(G)$, $N_G(X)$ denote the set of vertices in $G\backslash X$ which are adjacent to some vertex in X. If there is no fear of confusion, then we often identify a subgraph H of a graph G with its vertex set $V(H)$. Let G be a graph of order n and a, b be two integers with $3 \leq a \leq b \leq n$. Then G is said to be $[a, b]$-pancyclic if for every $l \in \{a, a+1, \ldots, b\}$, it contains a cycle of length l. Especially, $[3, n]$-pancyclic is called pancyclic.

The study of pancyclic graphs was initiated by Bondy [1] in 1971. In [1], he proved that if G is a graph of order n with $d_G(x) + d_G(y) \geq n$ for each pair of nonadjacent vertices x, y of G, then G is either pancyclic or isomorphic to the complete bipartite graph $K_{n/2,n/2}$. In particular, a pancyclic graph is hamiltonian. Therefore this shows that Ore's condition for a graph to be hamiltonian also implies that it is pancyclic except for some exceptional graphs. He proposed in [2] his famous "metaconjecture". It says that almost all nontrivial sufficient conditions for a graph to be hamiltonian also imply that it is pancyclic except for maybe a simple family of exceptional graphs.

The following proposition concerning hamiltonian graphs is obtained easily.

Proposition 1. *Let G be a 2-connected graph of order n. Suppose that $|N_G(x) \cup N_G(y)| + d_G(z) \geq n$ for every triple independent vertices x, y, z of G. Then G is hamiltonian.*

The condition of Proposition 1 is weaker than Ore's condition. In this paper, motivated by Proposition 1 and Bondy's metaconjecture, we prove the following.

Theorem 2. *Let G be a 2-connected graph of order $n \geq 6$. Suppose that $|N_G(x) \cup N_G(y)| + d_G(z) \geq n$ for every triple independent vertices x, y, z of G. Then G is pancyclic or isomorphic to the complete bipartite graph $K_{n/2,n/2}$.*

Let G be a 2-connected graph of order n. For any $u \in V(G)$ and $l \in \{m, m+1, \cdots, n\}$, if G has a cycle of length l which contains u, then G is called $[m, n]$-vertex pancyclic. Especially, $[3, n]$-vertex pancyclic is called vertex pancyclic. Let $\kappa(G)$ and $\delta(G)$ be the connectivity and the minimum degree of a graph G. In [4], Faudree, Gould, Jacobson and Lesniak conjectured that if a connected graph G of order n with $\delta(G) \geq \kappa(G) + 1$ satisfies $|N_G(u) \cup N_G(v)| \geq n - \kappa(G)$ for any nonadjacent vertices u, v of G, then G is vertex pancyclic. Song, in [6], refomulated this conjecture. He conjectured that if a 2-connected graph G of order n satisfies $|N_G(u) \cup N_G(v)| \geq n - \delta(G) + 1$ for any nonadjacent vertices u, v of G, then G is vertex pancyclic. Obviously, Song's conjecture implies the conjecture by Faudree et al. In [5], Liu, Lou and Zhao settled Song's conjecture.

Theorem 3 (Liu, Lou and Zhao [5])**.** *Let G be a 2-connected graph of order n. Suppose that $|N_G(u) \cup N_G(v)| + \delta(G) \geq n + 1$ for any nonadjacent vertices u, v of G. Then G is vertex pancyclic.*

By observing Theorems 2 and 3, one might expect the condition of Theorem 2 can yield vertex pancyclic. However, there exist graphs which satisfy the conditions of Theorem 2 but do not have a cycle of length 3 and 4 containing some $u \in V(G)$.

We define a graph G_0 of order $3m+3$ as follows: let $H_i (1 \le i \le 3)$ be complete graphs of order $m \ge 3$, and let

$$\begin{aligned} V(G_0) &= \bigcup_{1\le i\le 3} V(H_i) \cup \{a_1, a_2\} \cup \{u\}, \\ E(G_0) &= \bigcup_{1\le i\le 3} (E(H_i) \cup \{vw \colon v \in V(H_i), w \in V(H_{i+1})\}) \\ &\quad \cup \{a_1 v \colon v \in V(H_1)\} \cup \{a_2 w \colon w \in V(H_2)\} \cup \{a_1 u, a_2 u\}, \end{aligned}$$

where $H_4 = H_1$. Then G_0 is not isomorphic to a complete bipartite graph and does not have a cycle of length 3 and 4 containing u, and $|N_{G_0}(a_1) \cup N_{G_0}(w)| + d_{G_0}(a_2) = 4m+1 \ge 3m+4 = |V(G_0)|+1$, where $w \in V(H_3)$.

Therefore, we prove the following theorem.

Theorem 4. *Let G be a 2-connected graph of order $n \ge 6$. Suppose that $|N_G(x) \cup N_G(y)| + d_G(z) \ge n$ for every triple independent vertices x, y, z of G. Then G is $[5, n]$-vertex pancyclic or isomorphic to the complete bipartite graph $K_{n/2,n/2}$.*

By Theorem 4, we can obtain the following corollary. Again, by considering G_0, we cannot replace the conclusion "$[5, n]$-vertex pancyclic" with "$[4, n]$-vertex pancyclic".

Corollary 5. *Let G be a 2-connected graph of order $n \ge 6$. Suppose that $|N_G(x) \cup N_G(y)| + d_G(z) \ge n+1$ for every triple independent vertices x, y, z of G. Then G is $[5, n]$-vertex pancyclic.*

In [7], Wei and Zhu considered similar conditions for $[a, b]$-panconnected graphs. A graph G is called $[a, b]$-panconnected, if for any two distinct vertices u, v, there exists a path joining u and v with l vertices, for each $a \le l \le b$.

Theorem 6 (Wei and Zhu [7])**.** *Let G be a 3-connected graph of order n. Suppose that $|N_G(x) \cup N_G(y)| + d_G(z) \ge n+1$ for every triple independent vertices x, y, z of G. Then G is $[7, n]$-panconnected.*

We write a cycle C with a given orientation by $\overrightarrow{C}$. For $x \in V(C)$, we denote the successor and the predecessor of x on $\overrightarrow{C}$ by $x^{+(C)}$ and $x^{-(C)}$, respectively. If there is no fear of confusion, we write v^+ and v^- in stead of $v^{+(C)}$ and $v^{-(C)}$, respectively. For a cycle $\overrightarrow{C}$ and $X \subset V(C)$, we define $X^+ := \{x^+ \colon x \in X\}$ and $X^- := \{x^- \colon x \in X\}$. For $x, y \in V(C)$, we denote by $C[x, y]$ a path from x to y on $\overrightarrow{C}$. The reverse sequence of $C[x, y]$ is denoted by $C^-[y, x]$. For $x, y \in V(G)$, we let $d_G(x, y)$ denote the length of the shortest path connecting x and y. For a subset S of $V(G)$, we let $G[S]$ denote the subgraph induced by S in G.

2 Proof of Theorem 4

Suppose that G satisfies the assumptions of Theorem 4. Then we can obtain the following fact.

Fact 2.1. *If G is a bipartite graph, then G is balanced and complete.*

Let $u \in V(G)$ and let $\mathcal{C}_m$ be the set of cycles of length m which contains u.

Lemma 2.2. *At least one of the following statements hold:*

(i) $\mathcal{C}_3 \neq \emptyset$ *and* $\mathcal{C}_4 \neq \emptyset$.

(ii) $\mathcal{C}_4 \neq \emptyset$ *and* $\mathcal{C}_5 \neq \emptyset$.

(iii) $\mathcal{C}_5 \neq \emptyset$ *and* $\mathcal{C}_6 \neq \emptyset$.

(iv) $G = K_{n/2,n/2}$.

Proof. Suppose that (i)–(iv) do not hold. Let

$$\begin{aligned} A_1 &= \{v \in V(G) \colon d_G(u,v) = 1\}, \\ A_2 &= \{v \in V(G) \colon d_G(u,v) = 2\} \text{ and} \\ A_3 &= \{v \in V(G) \colon d_G(u,v) \geq 3\}. \end{aligned}$$

Claim 1. $d_G(u) \geq 3$

Proof. Suppose that $d_G(u) = 2$. Let $N_G(u) = \{v_1, v_2\}$. Further assume that there exist $x, y \in A_2 \cup A_3$ such that $xy \notin E(G)$. Then $\{u, x, y\}$ is an independent set, and we obtain

$$|N_G(x) \cup N_G(y)| + d_G(u) \leq |V(G)| - |\{u, x, y\}| + |\{v_1, v_2\}| = n - 1,$$

a contradiction. Therefore $G[A_2 \cup A_3]$ is complete. Since $|A_2 \cup A_3| \geq n - 3 \geq 3$ and G is 2-connected, there exist $w_1, w_2 \in A_2$ such that $v_1w_1, v_2w_2 \in E(G)$. Then $uv_1w_1w_2v_1u \in \mathcal{C}_5$. Let $w_3 \in (A_2 \cup A_3)\backslash\{w_1, w_2\}$. Then $uv_1w_1w_3w_2v_2u \in \mathcal{C}_6$ because $w_1w_3, w_2w_3 \in E(G)$. This contradicts that (iii) does not hold. □

CASE 1. $\mathcal{C}_6 = \emptyset$.

Claim 2. *Let $x, y, z \in A_1$. If $(N_G(x) \cap N_G(y))\backslash\{u\} \neq \emptyset$, then $|(N_G(x) \cup N_G(y)) \cap N_G(z)| \leq 2$.*

Proof. Suppose that there exist $x, y, z \in A_1$ such that $(N_G(x) \cap N_G(y))\backslash\{u\} \neq \emptyset$ and $|((N_G(x) \cup N_G(y)) \cap N_G(z))\backslash\{u\}| \geq 2$. We may assume there exist $a \in (N_G(x) \cap N_G(y))\backslash\{u\}$ and $b \in (N_G(x) \cap N_G(z))\backslash\{u\}$ such that $a \neq b$. Then $uyaxbzu \in \mathcal{C}_6$. This contradicts the assumption of CASE 1. □

Claim 3. *The independence number of A_1 is at most two.*

Proof. Suppose that there exists an independent set $\{x, y, z\} \subseteq A_1$. By Claim 2, we may assume $|(N_G(x) \cup N_G(y)) \cap N_G(z)| \leq 2$. Therefore we have

$$\begin{aligned} |N_G(x) \cup N_G(y)| &\leq |V(G)| - |\{x, y, z\}| - (|N_G(z)| - 2) \\ &= n - d_G(z) - 1, \end{aligned}$$

a contradiction. □

Since (i) does not hold, we obtain the following claim.

Claim 4. *The order of a component of* $G[A_1]$ *is at most two.*

By Claims 1, 3 and 4, $G[A_1]$ consists of two components H_1 and H_2 with $|H_1| = 2$ and $|H_2| \le 2$. Then note that $d_G(u) \le 4$. Since G is 2-connected, $N_G(H_1) \cap A_2 \neq \emptyset$ and $N_G(H_2) \cap A_2 \neq \emptyset$ hold. Let $x \in N_G(H_1) \cap A_2$ and $y \in N_G(H_2) \cap A_2$. By Claims 1 and 3, $\mathcal{C}_3 \neq \emptyset$. Since (i) does not hold, we obtain $x \neq y$ and $|(N_G(x) \cup N_G(y)) \cap A_1| = 2$. Since $\mathcal{C}_6 = \emptyset$, we have $xy \notin E(G)$. Thus, $\{x, y, u\}$ is an independent set, and we obtain

$$\begin{aligned} |N_G(x) \cup N_G(y)| + d_G(u) &\le (2 + |A_2 \cup A_3| - |\{x, y\}|) + 4 \\ &= n - 1, \end{aligned}$$

a contradiction.

CASE 2. $\mathcal{C}_6 \neq \emptyset$.

Then $\mathcal{C}_5 = \emptyset$ holds. Hence we can easily obtain the following claim.

Claim 5. *For* $C = u_1u_2u_3u_4u_5u_6u_1 \in \mathcal{C}_6$ *with* $u_1 = u$*,* $\{u_1, u_3, u_5\}$ *is an independent set.*

Claim 6. *For* $C \in \mathcal{C}_6$*,* $|V(C) \cap A_1| = |V(C) \cap A_2| = 2$.

Proof. Let $C = v_1v_2v_3v_4v_5v_6v_1 \in \mathcal{C}_6$ and $v_1 = u$. By Claim 5, $v_3, v_5 \notin A_1$, that is, $v_3, v_5 \in A_2$. Therefore it suffices to show $v_4 \notin A_1 \cup A_2$. Assume $v_4 \in A_2$. Since $\mathcal{C}_5 = \emptyset$, $N_G(v_4) \cap A_1 = \emptyset$. This contradicts the definition of A_2. Hence assume $v_4 \in A_1$. First, suppose that $|A_1| \le |A_2|$. Since $\mathcal{C}_5 = \emptyset$, $(N_G(v_3) \cup N_G(v_5)) \cap A_2 = \emptyset$. Therefore $\{v_3, v_5, u\}$ is an independent set by Claim 5, and we obtain

$$\begin{aligned} |N_G(v_3) \cup N_G(v_5)| + d_G(u) &\le |A_1| + |A_3| + |A_1| \\ &\le |A_1| + |A_2| + |A_3| \\ &= n - 1, \end{aligned}$$

a contradiction. Next, suppose that $|A_2| \le |A_1| - 1$. Since $uv_2v_3v_4u \in \mathcal{C}_4$ and (i) does not hold, A_1 is an independent set. Especially, we see $N_G(v_i) \cap A_1 = \emptyset$ for $i = 2, 4, 6$ and $\{v_2, v_4, v_6\}$ is an independent set. Therefore we obtain

$$\begin{aligned} |N_G(v_2) \cup N_G(v_6)| + d_G(v_4) &\le 1 + |A_2| + |A_2| + 1 \\ &= |A_2| + |A_2| + 2. \end{aligned}$$

If $|A_2| \le |A_1| - 2$ or $A_3 \neq \emptyset$, then $|N_G(v_2) \cup N_G(v_6)| + d_G(v_4) \le n - 1$, a contradiction. Hence $|A_2| = |A_1| - 1$, $A_3 = \emptyset$ and $N_G(v_2) \cup N_G(v_6) = N_G(v_4) = A_2 \cup \{u\}$. Therefore A_2 is an independent set, because $\mathcal{C}_5 = \emptyset$. By Fact 2.1, G is a balanced complete bipartite graph, which contradicts that (iv) does not hold. $\square$

Let $C = v_1v_2v_3v_4v_5v_6v_1 \in \mathcal{C}_6$ with $v_1 = u$. By Claim 6, we see $v_2, v_6 \in A_1$,$v_3, v_5 \in A_2$ and $v_4 \in A_3$. By Claim 1, $A_1\backslash\{v_2, v_6\} \neq \emptyset$, say $x \in A_1\backslash\{v_2, v_6\}$. Assume $v_3, v_5 \notin N_G(x)$. Then $\{x, v_3, v_5\}$ is an independent set by Claim 5. Since $\mathcal{C}_5 = \emptyset$, $N_G(x) \cap N_G(v_3) \subseteq \{v_2\}$ and $N_G(x) \cap N_G(v_5) \subseteq \{v_6\}$. Thus we have

$$\begin{aligned} |N_G(v_3) \cup N_G(v_5)| &\leq |V(G)| - |\{x, v_3, v_5\}| - |N_G(x)\backslash\{v_2, v_6\}| \\ &= n - d_G(x) - 1, \end{aligned}$$

a contradiction. Hence $v_3 \in N_G(x)$ or $v_5 \in N_G(x)$ holds. By symmetry, we may assume that $v_5 \in N_G(x)$. By Claim 6, $v_6v_3, xv_3 \notin E(G)$. Since $uxv_5v_6 \in \mathcal{C}_4$ and (i) dose not hold, it follows that $v_6 \notin N_G(x)$. Hence $\{v_3, v_6, x\}$ is an independent set. Since $\mathcal{C}_5 = \emptyset$, we see $N_G(v_3) \cap N_G(v_6) = \emptyset$ and $N_G(v_3) \cap N_G(x) = \emptyset$. Thus, we obtain

$$\begin{aligned} |N_G(v_6) \cup N_G(x)| &\leq |V(G)| - |\{v_3, v_6, x\}| - |N_G(v_3)| \\ &= n - d_G(v_3) - 3, \end{aligned}$$

a contradiction. □

Lemma 2.3. *If $\mathcal{C}_m \neq \emptyset$ for $3 \leq m \leq n-2$, then $\mathcal{C}_{m+2} \neq \emptyset$ holds.*

Proof. Suppose that there exists $3 \leq m \leq n-2$ such that $\mathcal{C}_m \neq \emptyset$ and $\mathcal{C}_{m+2} = \emptyset$. Since $\mathcal{C}_{m+2} = \emptyset$, we obtain the following two claims.

Claim 7. *$N_{G\backslash C}(x) \cap N_{G\backslash C}(y^+) = \emptyset$ holds for $C \in \mathcal{C}_m$, $x \in V(G\backslash C)$ and $y \in N_C(x)$.*

Claim 8. *For $C \in \mathcal{C}_{m+1}$, $x \in V(G\backslash C)$ and $y, z \in N_C(x)$, the followings hold.*

(i) *$y^+ \notin N_C(x)$.*

(ii) *$\{x, y^+, z^+\}$ is an independent set.*

(iii) *$N_C(x)^+ \cap (N_C(y^+) \cup N_C(z^+)) = \emptyset$.*

Claim 9. *Suppose that C is a cycle and $\{x, y, z\}$ is an independent set such that $(C1)$ $x \in V(G\backslash C)$ and $y, z \in V(C)$ or $(C2)$ $x \in V(C)$ and $y, z \in V(G\backslash C)$. Then one of the following holds.*

(i) *If $(C1)$ holds, then $|N_C(x)^+ \cap (N_C(y) \cup N_C(z))| \geq 1$. If $(C2)$ holds, then $|N_C(x)^- \cap (N_C(y) \cup N_C(z))| \geq 2$ and $|N_C(x)^+ \cap (N_C(y) \cup N_C(z))| \geq 2$.*

(ii) *$N_{G\backslash C}(x) \cap (N_{G\backslash C}(y) \cup N_{G\backslash C}(z)) \neq \emptyset$.*

Proof. Suppose that neither (i) nor (ii) holds. Since (i) does not hold, we have

$$\begin{aligned} |N_C(y) \cup N_C(z)| + d_C(x) &= |N_C(y) \cup N_C(z)| + |N_C(x)^+| \\ &\leq |V(C)| + |N_C(x)^+ \cap (N_C(y) \cup N_C(z))| \\ &\leq |V(C)| \end{aligned}$$

if $(C1)$ holds; otherwise similarly $|N_C(y) \cup N_C(z)| + d_C(x) \leq |V(C)| + 1$. In either case, we obtain

$$|N_C(y) \cup N_C(z)| + d_C(x) \leq |V(C)| + |V(G \backslash C) \cap \{x, y, z\}| - 1.$$

Since (ii) does not hold, we now obtain

$$\begin{aligned} |N_{G\backslash C}(y) \cup N_{G\backslash C}(z)| + d_{G\backslash C}(x) &= |(N_{G\backslash C}(y) \cup N_{G\backslash C}(z)) \cup N_{G\backslash C}(x)| \\ &\leq |V(G\backslash C)| - |V(G\backslash C) \cap \{x, y, z\}|. \end{aligned}$$

Therefore we deduce $|N_G(y) \cup N_G(z)| + d_G(x) \leq n - 1$, a contradiction. $\square$

CASE 1. $|\{v \in V(G\backslash C) : |N_C(v)| \geq 2\}| \geq 2$ for some $C \in \mathcal{C}_m$.

Let $C \in \mathcal{C}_m$ with $|\{v \in V(G\backslash C) : |N_C(v)| \geq 2\}| \geq 2$, say $x_1, x_2 \in \{v \in V(G\backslash C) : |N_C(v)| \geq 2\}$.

CASE 1.1. $|N_C(x_1) \cup N_C(x_2)| = 2$.

Let $N_C(x_1) = N_C(x_2) = \{v_1, v_2\}$. We may assume that $v_1^+ \neq v_2$. Suppose that $x_1x_2 \notin E(G)$. Then $\{x_1, x_2, v_1^+\}$ is an independent set. By the assumption of CASE 1.1, $N_C(v_1^+)^- \cap (N_C(x_1) \cup N_C(x_2)) \subseteq \{v_2\}$. By Claim 7, $N_{G\backslash C}(v_1^+) \cap (N_{G\backslash C}(x_1) \cup N_{G\backslash C}(x_2)) = \emptyset$. This contradicts Claim 9. Hence $x_1x_2 \in E(G)$ holds. By Claim 7, we obtain $v_2^+ \neq v_1$ and so $v_1^+, v_2^+ \notin N_G(x_1)$. If $v_1^+v_2^+ \in E(G)$, then $C[v_2^+, v_1]x_1x_2C^-[v_2, v_1^+]v_2^+ \in \mathcal{C}_{m+2}$, a contradiction. Thus $\{v_1^+, v_2^+, x_1\}$ is an independent set. Hence, by the assumption of CASE 1.1, $N_C(x_1)^+ \cap (N_C(v_1^+) \cup N_C(v_2^+)) = \emptyset$. By Claim 7, $N_{G\backslash C}(x_1) \cap (N_{G\backslash C}(v_1^+) \cup N_{G\backslash C}(v_2^+)) = \emptyset$. These contradict Claim 9.

CASE 1.2. $|N_C(x_1) \cup N_C(x_2)| \geq 3$.

Let $B_i = \{v \in V(C) : v, v^- \in N_G(x_i)\}$ for $i = 1, 2$.

Claim 10. *For some* $1 \leq i \leq 2$, $B_i = \emptyset$.

Proof. Suppose that $B_1 \neq \emptyset$ and $B_2 \neq \emptyset$. Since $\mathcal{C}_{m+2} = \emptyset$, it follows that $B_1 = B_2$, $|B_1| = |B_2| = 1$ and $x_1x_2 \notin E(G)$. Let $B_1 = \{v_1\}$. Then $\{x_1, x_2, v_1^+\}$ is an independent set. By Claim 7, $N_{G\backslash C}(v_1^+) \cap (N_{G\backslash C}(x_1) \cup N_{G\backslash C}(x_2)) = \emptyset$. By Claim 9, $(N_C(v_1^+)^- \cap (N_C(x_1) \cup N_C(x_2))) \backslash \{v_1^-\} \neq \emptyset$. Without loss of generality, we may assume that $v_2 \in (N_C(v_1^+)^- \cap N_C(x_1)) \backslash \{v_1^-\}$. Note that $v_2 \neq v_1$ because $v_1 \notin N_C(v_1^+)^-$. Then $x_1C^-[v_2, v_1^+]C[v_2^+, v_1^-]x_2v_1x_1 \in \mathcal{C}_{m+2}$, a contradiction. $\square$

Claim 11. *For some* $1 \leq i \leq 2$, *there exist* $v_1, v_2 \in N_C(x_i)$ *such that* (i) $v_1^+, v_2^+ \notin N_C(x_i)$, (ii) $v_1^+v_2^+ \in E(G)$ *and* (iii) $N_G(x_{3-i}) \backslash \{v_1, v_2\} \neq \emptyset$.

Proof. If $B_1 = B_2 = \emptyset$, then $|N_C(x_i)| \geq 3$ for some i by the assumption of CASE 1.2. If $B_i \neq \emptyset$ for some $1 \leq i \leq 2$, then $B_{3-i} = \emptyset$ by Claim 10. Let $v \in B_i$. Then $\{v, v^-\} \not\subseteq N_C(x_{3-i})$ because $B_{3-i} = \emptyset$. Therefore we may assume that there exist $v_1, v_2 \in N_C(x_1)$ such that $\{v_1^+, v_2^+, v_1^-, v_2^-\} \cap N_C(x_1) = \emptyset$ and $N_G(x_2) \backslash \{v_1, v_2\} \neq \emptyset$. If $v_1^+v_2^+ \in E(G)$, then $v_1, v_2 \in N_C(x_1)$ are desired vertices.

Hence we may assume that $\{x_1, v_1^+, v_2^+\}$ is an independent set. By Claims 7 and 9, we have $(N_C(x_1)^+ \cap (N_C(v_1^+) \cup N_C(v_2^+))\backslash\{v_1, v_2\} \neq \emptyset$. By symmetry, we may assume that $v_3^+ \in (N_C(x_1)^+ \cap N_C(v_1^+))\backslash\{v_1, v_2\}$. If $N_C(x_2) \neq \{v_1, v_3\}$, then $v_1, v_3 \in N_C(x_1)$ are desired vertices; otherwise $v_1, v_3 \in N_C(x_2)$ are desired vertices. $\square$

By Claim 11, we may assume that there exist $v_1, v_2 \in N_C(x_1)$ such that $v_1^+, v_2^+ \notin N_C(x_1)$, $v_1^+v_2^+ \in E(G)$ and $N_G(x_2)\backslash\{v_1, v_2\} \neq \emptyset$. Let $C_1 = x_1C^-[v_1, v_2^+]C[v_1^+, v_2]x_1$ and $w_1, w_2 \in N_C(x_2)$ such that $\{w_1, w_2\} \neq \{v_1, v_2\}$. By Claims 8 (ii),(iii) and 9, $N_{G\backslash C_1}(x_2) \cap (N_{G\backslash C_1}(w_1^{+(C_1)}) \cup N_{G\backslash C_1}(w_2^{+(C_1)})) \neq \emptyset$. Without loss of generality, we may assume that $N_{G\backslash C_1}(x_2) \cap N_{G\backslash C_1}(w_1^{+(C_1)}) \neq \emptyset$, say $x_3 \in N_{G\backslash C_1}(x_2) \cap N_{G\backslash C_1}(w_1^{+(C_1)})$. By Claim 7, w_1 and $w_1^{+(C_1)}$ are not consecutive vertices on C. Therefore $w_1 \in \{v_2, v_2^+\}$. By considering $C_2 = x_1C^-[v_2, v_1^+]C[v_2^+, v_1]x_1$, it follows from Claim 11 (i) that $w_2 \in \{v_1, v_1^+\}$. By Claim 8 (i) we have $\{w_1, w_2\} \neq \{v_2^+, v_1^+\}$. Therefore we may assume $\{w_1, w_2\} = \{v_2^+, v_1\}$ by symmetry. Then $x_3C[v_1^+, v_1]x_2x_3 \in \mathcal{C}_{m+2}$, a contradiction.

CASE 2. $|\{v \in V(G\backslash C) : |N_C(v)| \geq 2\}| \leq 1$ for any $C \in \mathcal{C}_m$.

Let $C \in \mathcal{C}_m$ be a cycle and $P = xy_1 \cdots y_kz$ be a path such that $x, z \in V(C)$ and $y_i \in V(G\backslash C)$ for $1 \leq i \leq k$. Since $m \leq n-2$, it follows that $|V(G\backslash C)| \geq 2$. By the assumption of CASE 2, we may assume that $N_C(y_1) = \{x\}$ and $k \geq 2$. Take such a cycle C and a path P as (a-i) $|V(P)|$ is as small as possible, and (a-ii) $|V(C[z, x])|$ is as small as possible subject to (a-i).

CASE 2.1. $k = 2$.

Since $\mathcal{C}_{m+2} = \emptyset$, it follows that $z^+ \neq x$ and $x^+ \neq z$. Since $N_C(y_1) = \{x\}$, we obtain $x^+, z^+ \notin N_C(y_1)$. If $x^+z^+ \in E(G)$, then $y_1y_2C^-[z, x^+]C[z^+, x]y_1 \in \mathcal{C}_{m+2}$, a contradiction. Thus $x^+z^+ \notin E(G)$. Hence $\{y_1, x^+, z^+\}$ is an independent set. Since $N_C(y_1) = \{x\}$, $N_C(y_1)^+ \cap (N_C(x^+) \cup N_C(z^+)) = \emptyset$. By Claim 7, $N_{G\backslash C}(x^+) \cap N_{G\backslash C}(y_1) = \emptyset$. By (a-ii), $N_{G\backslash C}(z^+) \cap N_{G\backslash C}(y_1) = \emptyset$. These contradict Claim 9.

CASE 2.2. $k = 3$.

Suppose that $m \geq 4$. Since $\mathcal{C}_{m+2} = \emptyset$, either $|V(C[x, z])| \geq 4$ or $|V(C[z, x])| \geq 4$ holds. Therefore $|V(C[x, z])| \geq 4$ holds by (a-ii). Then note that $x^{++} \in C[x, z^-]$. By (a-i), $y_1y_3 \notin E(G)$. First, assume $x^+ \neq u$. Since $\mathcal{C}_{m+2} = \emptyset$, $y_3x^{++} \notin E(G)$. Thus $\{y_1, y_3, x^{++}\}$ is an independent set. Since $N_C(y_1) = \{x\}$, $N_C(x^{++})^- \cap N_C(y_1) = \{x\}$. Since $\mathcal{C}_{m+2} = \emptyset$, $N_C(x^{++})^- \cap N_C(y_3) \subseteq \{x\}$. By (a-i) and the assumption of CASE 2, $N_{G\backslash C}(x^{++}) \cap (N_{G\backslash C}(y_1) \cup N_{G\backslash C}(y_3)) = \emptyset$. These contradict Claim 9. Next, assume $x^+ = u$. Then note that $z^{--} \in C[u, z]$. Since $N_C(y_1) = \{x\}$ and $\mathcal{C}_{m+2} = \emptyset$, $N_C(z^{--})^+ \cap N_C(y_1) = \emptyset$. If $v \in (N_C(z^{--})^+ \cap N_C(y_3))\backslash\{z\}$, then $C' = y_3C[v, z^{--}]C^-[v^-, z]y_3 \in \mathcal{C}_m$ and $xy_1y_2y_3$ is a path such that $x, y_3 \in V(C')$ and $y_1, y_2 \in V(G\backslash C')$. This contradicts (a-i). Hence $N_C(z^{--})^+ \cap N_C(y_3) = \{z\}$. This implies $y_3z^{--} \notin E(G)$. Hence $\{y_1, y_3, z^{--}\}$ is an independent set. By (a-i), $N_{G\backslash C}(z^{--}) \cap (N_{G\backslash C}(y_1) \cup N_{G\backslash C}(y_3)) = \emptyset$. These contradict Claim 9.

Suppose that $m = 3$. Since $\mathcal{C}_5 = \emptyset$, we see that $C = uxzu$, $N(u) \cap (N_G(y_1) \cup$

$N_G(y_3)) = \{x, z\}$ and $\{u, y_1, y_3\}$ is an independent set. Then we obtain $|N(y_1) \cup N(y_3)| + d_G(u) = |V(G)| - |\{y_1, y_3, u\}| + |\{x, z\}| \leq n - 1$, a contradiction.

CASE 2.3. $k \geq 4$.

Let $v \in V(C \backslash \{x\})$. By (a-i), we obtain $y_1y_3, vy_3 \notin E(G)$ and $N_C(v)^- \cap N_C(y_3) = \emptyset$. Hence $\{y_1, y_3, v\}$ is an independent set. Since $N_C(y_1) = \{x\}$, $N_C(v)^- \cap N_C(y_1) \subseteq \{x\}$. By (a-i), $N_{G \backslash C}(v) \cap (N_{G \backslash C}(y_1) \cup N_{G \backslash C}(y_3)) = \emptyset$. These contradict Claim 9. □

By Lemmas 2.2 and 2.3, Theorem 4 holds immediately.

3 Proof of Theorem 2

Proof. Suppose that G satisfies the assumption of Theorem 2. By Theorem 4, we have only to show that G has cycles of length 3 and 4 or G is isomorphic to the complete bipartite graph $K_{n/2,n/2}$.

First, we shall show that G has a cycle of length 4. Suppose not. Assume $d_G(v) = 2$ holds for any $v \in V(G)$. If $n = 6$, then $|N_G(x) \cup N_G(y)| + d_G(z) \leq 5$ hold for every triple independent vertices x, y, z of G, a contradiction. If $n \geq 7$, then $|N_G(x) \cup N_G(y)| + d_G(z) \leq 6$ holds for every triple independent vertices x, y, z of G, a contradiction. Therefore there exists $u \in V(G)$ such that $d_G(u) \geq 3$. Let $U_1 = \{v \in V(G) \colon d_G(u, v) = 1\}$, $U_2 = \{v \in V(G) \colon d_G(u, v) = 2\}$ and $U_3 = \{v \in V(G) \colon d_G(u, v) \geq 3\}$. Let $x, y, z \in U_1$. Then $(N_G(x) \cup N_G(y)) \cap N_G(z) = \{u\}$. If $\{x, y, z\}$ is an independent set, then we obtain

$$|N_G(x) \cup N_G(y)| + d_G(z) \leq |V(G)| - |\{x, y, z\}| + |\{u\}| \leq n - 2,$$

a contradiction. Therefore we may assume that $xy \in E(G)$. Note that $((N_G(x) \cup N_G(y)) \cap U_1) \backslash \{x, y\} = \emptyset$. Since G is 2-connected, we may assume that there exists $w \in U_2$ such that $xw \in E(G)$. Since G does not have a cycle of length 4, $\{y, z, w\}$ is an independent set and $(N_G(z) \cup N_G(w)) \cap N_G(y) = \{x, u\}$. Hence we obtain

$$|N_G(z) \cup N_G(w)| + d_G(y) \leq |V(G)| - |\{y, z, w\}| + |\{x, u\}| \leq n - 1,$$

a contradiction. Therefore G has a cycle of length 4.

Next, suppose that G has no cycle of length 3. Let $C = u_1u_2u_3u_4u_1$ be a cycle of length 4. Since $n \geq 6$, we may assume that there exists $w \in V(G)$ such that $wu_1 \in E(G)$. Let $W_1 = \{v \in V(G) \colon d_G(w, v) = 1\}$, $W_2 = \{v \in V(G) \colon d_G(w, v) = 2\}$ and $W_3 = \{v \in V(G) \colon d_G(w, v) \geq 3\}$. Then W_1 and $\{w, u_2, u_4\}$ are independent sets, $u_2, u_4 \in W_2$ and $V(C) \cap W_1 \subset \{u_1, u_3\}$. If $((N_G(u_2) \cup N_G(u_4)) \cap W_1) \backslash \{u_1, u_3\} = \emptyset$, then we obtain

$$|N_G(u_2) \cup N_G(u_4)| + d_G(w) \quad \leq \quad (2 + |W_2 \backslash \{u_2, u_4\}| + |W_3|) + |W_1| \leq n - 1,$$

a contradiction. Therefore we may assume that there exists $u_5 \in (N_G(u_2) \cap W_1) \backslash \{u_1, u_3\}$. Since G does not have a cycle of length 3, we see that $(N_G(u_1) \cup N_G(u_3)) \cap$

$(N_G(u_2) \cup N_G(u_4)) = \emptyset$ and $\{u_1, u_3, u_5\}$ is an independent set. Hence we obtain

$$\begin{aligned} 2n &\leq (|N_G(u_1) \cup N_G(u_3)| + d_G(u_5)) + (|N_G(u_2) \cup N_G(u_4)| + d_G(w)) \\ &= (|N_G(u_1) \cup N_G(u_3)| + |N_G(u_2) \cup N_G(u_4)|) + d_G(u_5) + d_G(w) \\ &\leq (1 + |W_1| + |W_2| + |W_3|) + (1 + |W_2|) + |W_1| \leq 2n - |W_3|. \end{aligned}$$

This yields $W_3 = \emptyset$ and $N_G(u_5) = W_2 \cup \{w\}$. Hence $W_2 \cup \{w\}$ is an independent set, because G has no cycle of length 3. Thus G is a bipartite graph. By Fact 2.1, G is balanced and complete. This completes the proof of Theorem 2. □

4 Further results

We consider the graph G_0, again. Since $d_{G_0}(u) = 2$, one might expect that the conditions of Theorem 4 guarantee the existence of a cycle of length l, $3 \leq l \leq |V(G)|$ containing any vertex $u \in V(G)$ with $d_G(u) \geq 3$. However, there exist examples which satisfy the conditions of Theorem 4 but have no cycle of length 3 or 4 containing some vertex u of degree three. We first construct a graph G_1 which has no cycle of length 3 containing for some $u \in V(G_1)$. We define a graph G_1 of order $m + 4$ as follows: Let H be a complete graph of order $m \geq 2$, and let

$$\begin{aligned} V(G_1) &= V(H) \cup \{a_1, a_2, a_3\} \cup \{u\}, \\ E(G_1) &= E(H) \cup \{a_i v, a_i u \colon 1 \leq i \leq 3,\ v \in V(H)\}. \end{aligned}$$

Then G_1 has no cycle of length 3 containing u, and $|N_{G_1}(a_1) \cup N_{G_1}(a_2)| + d_{G_1}(a_3) = 2m + 2 \geq m + 4 = |V(G_1)|$.

Next, we construct a graph G_2 which does not have a cycle of length 4 containing for some $u \in V(G_2)$. We define a graph G_2 of order $3m + 4$ as follows: let $H_i (1 \leq i \leq 3)$ be complete graphs of order $m \geq 1$, and let

$$\begin{aligned} V(G_2) &= \bigcup_{1 \leq i \leq 3} (V(H_i) \cup \{a_i\}) \cup \{u\}, \\ E(G_2) &= \bigcup_{1 \leq i \leq 3} (E(H_i) \cup \{a_i v, a_i u, vw \colon v \in V(H_i), w \in V(H_{i+1})\}) \cup \{a_2 a_3\}, \end{aligned}$$

where $H_4 = H_1$. Then G_2 does not have a cycle of length 4 containing u, and $|N_{G_2}(a_1) \cup N_{G_2}(w)| + d_{G_2}(a_2) = 4m + 3 \geq 3m + 4 = |V(G_2)|$, where $w \in V(H_3)$.

Therefore we prove the following two theorems.

Theorem 7. *Let G be a 2-connected graph of order $n \geq 6$ and $u \in V(G)$ with $d_G(u) \geq 3$. Suppose that $|N_G(x) \cup N_G(y)| + d_G(z) \geq n$ for every triple independent vertices x, y, z of G. Then G has a cycle containing u of length l, $l = 3, 5, 6, \ldots, n$ or a cycle containing u of length m, $m = 4, 5, 6, \ldots, n$ or is isomorphic to the complete bipartite graph $K_{n/2,n/2}$.*

Theorem 8. *Let G be a 2-connected graph of order $n \geq 6$ and $u \in V(G)$ with $d_G(u) \geq 5$. Suppose that $|N_G(x) \cup N_G(y)| + d_G(z) \geq n$ for every triple independent vertices x, y, z of G. Then G has a cycle containing u of length l, $l = 4, 5, 6, \ldots, n$ or is isomorphic to the complete bipartite graph $K_{n/2,n/2}$.*

Proof. For $u \in V(G)$, let $\mathcal{C}_i$ and A_i be as in the proof of Theorem 4. Suppose that $d_G(u) \geq 3$ and $\mathcal{C}_3 = \mathcal{C}_4 = \emptyset$. Then A_1 is an independent set such that $|A_1| \geq 3$. Let $x, y, z \in A_1$. Since $\mathcal{C}_4 = \emptyset$, $N_G(x) \cap N_G(z) = \{u\}$ and $N_G(y) \cap N_G(z) = \{u\}$. These imply

$$\begin{aligned} |N_G(x) \cup N_G(y)| &\leq |V(G)| - |\{x, y, z\}| - |N_G(z) \backslash \{u\}| \\ &= n - 2 - d_G(z), \end{aligned}$$

a contradiction. Next, suppose that $d_G(u) \geq 5$ and $\mathcal{C}_4 = \emptyset$. Then the independence number of A_1 is at least 3. Therefore we obtain a same contradiction as above. Hence Theorem 4 implies Theorems 7 and 8. □

References

[1] J.A. Bondy, Pancyclic graphs I, *J. Combin Theory* **11** (1971), 80–84.

[2] J.A. Bondy, Pancyclic graphs, Proc. Second Louisiana Conf. Combinatorics, Graph Theory and Computing, 1971, pp. 167–172.

[3] J.A. Bondy, "Basic graph theory—paths and cycles," Handbook of Combinatorics, Vol. I, Elsevier, Amsterdam (1995), 5–110.

[4] R.J. Faudree, R.J. Gould, M.S Jacobson and L. Lesniak, Neighbourhood unions and highly hamilton graphs, *Ars Combin.* **31** (1991), 139–148.

[5] Bolian Liu, Dingjun Lou and Kewen Zhao, A Neighbourhood Union Condition for Pancyclicity, *Australas. J. Combin.* **12** (1995), 81–91.

[6] Zeng Min Song, Conjecture 5.1, Proc. Chinese Symposium on Cycle Problems in Graph Theory, Journal of Nanjing University, **27** (1991), 234.

[7] B. Wei and Y. Zhu, On the panconnectivity of graphs with large degrees and neighborhood unions, *Graphs Combin.* **14** (1998), 263–274.

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