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Latin k-hypercubes

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Abstract

We study k dimensional Latin hypercubes of order n. We describe the automorphism groups of the hypercubes and define the parity of a hypercube and relate the parity with the determinant of a permutation hypercube. We determine the parity in the orbits of the automorphism group. Based on this definition of parity we make a conjecture similar to the Alon-Tarsi conjecture. We define an orthogonality relation between hypercubes and we show that a set of mutually orthogonal Latin k dimensional hypercubes correspond to MDS codes and to $k+1$ dimensional permutation hypercubes satisfying a combinatorial condition.

1 Introduction

The study of Latin squares began with Euler's landmark paper in 1782 [5]. In this paper Euler describes orthogonal Latin squares in an attempt to solve the 36 officer problem; see [14] for a full description. Since then numerous uses of Latin squares have been found and various generalizations have been made; see [2] and [11] and the references therein.

In this paper we study a generalization of Latin squares to more dimensions beginning with the necessary definitions in Section 1. In Section 2, we study the group of isometries of a Latin hypercube and describe the automorphism group, isotopy class and the main class of a Latin hypercube. In Section 3, we define various parities of a Latin hypercube and make a conjecture similar to the Alon-Tarsi conjecture. In Section 4, we study a particular definition of orthogonality for Latin hypercubes.

We begin with some definitions.

Definition 1.1 *A Latin k-hypercube $L_{j_1,\ldots,j_k}$ of order n is a k dimensional array with n^k elements given as a cube with each side containing n coordinates, where the elements come from the set $\{0,1,2,\ldots,n-1\}$, such that if $j_i = j_i'$ for all $i \neq \alpha$ and $j_\alpha \neq j_{\alpha'}$ then $L_{j_1,j_2,\ldots,j_k} \neq L_{j_1',j_2',\ldots,j_k'}$.*

This definition implies that if $k-1$ coordinates are fixed and the remaining coordinate is allowed to take on all possibilities then these entries are a permutation of the set $\{0,1,2,\ldots,n-1\}$. As an example, if $k=1$ then a Latin k-hypercube is a permutation, which we shall refer to as a Latin line. If $k=2$ we have a Latin square.

The next proposition follows easily from the definition.

Proposition 1.2 *The subhypercube of a Latin k-hypercube of order n formed by fixing $k-s$ coordinates is a Latin s-hypercube of order n.*

Proof. Each line in the hypercube still contains each element exactly once. □

Theorem 1.3 *For $s \leq k$, a Latin k-hypercube of order n contains $\binom{k}{k-s}n^{k-s} = \binom{k}{s}n^{k-s}$ s-hypercubes of order n.*

Proof. There are $\binom{k}{k-s}$ ways of picking $k-s$ coordinates to fix, and at each coordinate there are n choices of which element to fix. □

A subset of the subhypercubes can be seen to partition the hypercube. For example, there are kn^{k-1} Latin lines in a hypercube which split into k sets of n^{k-1}, each of which partition the hypercube. There are $\binom{k}{2}n^{k-2}$ Latin squares which split into $\binom{k}{k-2}$ sets of n^{k-2} each of which partitions the hypercube. These two partitions of the hypercube will be used in defining the i-parity and orthogonality respectively. In general there are $\binom{k}{s}n^{k-s}$ Latin s-hypercubes which split into sets of size n^{k-s} which partition the hypercube.

We can view a Latin k-hypercube of order n as an array in a manner similar to an orthogonal array for Latin squares. Namely, given a Latin k-hypercube of order n, $L_{j_1,\ldots,j_k}$, let M be the matrix with n^k rows and $k+1$ columns where the columns are labeled $x_1, x_2, \ldots, x_k, s$ and $j_1, j_2, \ldots, j_k, j_{k+1}$ is a row in M if and only if $L_{j_1,\ldots,j_k} = j_{k+1}$. We shall refer to this matrix as the orthogonal array of L, denoted by $OA(L)$, and refer to the coordinates throughout the paper as $x_1, x_2, \ldots, x_k, s$, adding additional columns when discussing orthogonality.

We shall describe a few techniques for constructing Latin k-hypercubes. For other constructions see [10].

Given a Latin k-hypercube L of order n define $E^+(L)$ by

$$E^+(L)_{a_1,\ldots,a_k,a_{k+1}} = L_{a_1,\ldots,a_k} + (a_{k+1}) \pmod{n}. \tag{1}$$

Then $E^+(L)$ is a Latin $k+1$ hypercube. If $L=(0,1,2,\ldots,n-1)$ is a Latin line then $(E^+)^{k-1}(L)$ is said to be the circulant Latin k-hypercube of order n.

Given a Latin k-hypercube L of order n define $E^-(L)$ by

$$E^-(L)_{a_1,\ldots,a_k,a_{k+1}} = L_{a_1,\ldots,a_k} - (a_{k+1}) \pmod{n}. \tag{2}$$

Then $E^-(L)$ is a Latin $k+1$ hypercube. If $L=(0,1,2,\ldots,n-1)$ is a Latin line then $(E^-)^{k-1}(L)$ is said to be the reverse circulant Latin k-hypercube of order n.

It is immediate then that Latin k-hypercubes of order n exist for all k and n by simply applying E^+ to the identity permutation $(k-1)$ times.

Any Latin square of order n determines an algebraic structure known as a quasigroup, where the operation is given by $i * j = L_{ij}$. We shall say that a Latin square L represents the operation $*$ in this case. A quasigroup is not necessarily associative nor does it necessarily have an identity but it does have cancelation.

Theorem 1.4 *Let $L_1, L_2, \dots, L_s$ be Latin squares of order n with L_i representing the operation $*_i$. Let $M = L_1 * L_2 * \cdots * L_s$ with $M_{a_1,a_2,\dots,a_s,a_{s+1}} = a_1 *_1 (a_2 *_2 (a_3 *_3 \dots (a_s *_s a_{s+1}))\dots)$. Then M is a Latin $(s+1)$-hypercube of order n.*

Proof. Fix any s dimensions. If $a_1 *_1 (a_2 *_2 (a_3 *_3 \dots (a_i *_i \dots (a_s *_s a_{s+1}))\dots) = a_1 *_1 (a_2 *_2 (a_3 *_3 \dots (a_i' *_i \dots (a_s *_s a_{s+1}))\dots)$, then applying left cancelation $i-1$ times followed by right cancelation gives that $a_i = a_i'$. The result must be a Latin line. Hence M is a Latin $s+1$-hypercube. $\square$

Notice that the Latin squares need not be distinct. Hence we can construct Latin k-hypercubes for all k and n. In fact, we can construct an abundance of Latin k-hypercubes using this technique.

More generally a Latin k-hypercube of order n can be viewed as a generalization of a quasigroup. Namely it is a function $f : \mathbb{Z}_n^k \to \mathbb{Z}_n$ with $f(a_1, \dots, a_k) = a_{k+1}$, where $M_{a_1,\dots,a_k} = a_{k+1}$. Let $M_1, M_2, \dots, M_s$ be Latin k_i hypercubes of order n where M_i is associated with the function f_i and each $k_i \geq 2$. Then let $M = M_1 * M_2 * \cdots * M_s$ given by the function g, where g is defined by:

$$\begin{aligned} & g(a_1^1, \dots, a_{k_1}^1, a_2^2, \dots, a_{k_2}^2, a_2^3, \dots, a_{k_3}^3, \dots, a_2^s, \dots, a_{k_s}^s) \\ = \; & f_s(\dots f_3(f_2(f_1(a_1^1, \dots, a_{k_1}^1), a_2^2, \dots, a_{k_2}^2), a_2^3, \dots, a_{k_2}^3), \dots, a_2^s, \dots, a_{k_2}^s). \end{aligned}$$

Then M is a $(\sum k_i - (s-1))$-hypercube of order n.

2 Group of isometries

We shall define the group of isometries of Latin k-hypercubes.

Let x_i denote the i-th coordinate of a Latin k-hypercube and let s be the symbol. That is, we have $x_1, x_2, \dots, x_n, s$ as the coordinates of $OA(L)$. We shall refer to the lines formed by fixing all but x_i as i-lines. For example, in a Latin square 1-lines are columns and 2-lines are rows.

Definition 2.1 *Two Latin k-hypercubes are isotopic if one can be obtained from the other by permuting the i-lines and the symbols of the hypercube.*

Let S_n denote the symmetric group on n letters. Each Latin k-hypercube is acted on by $\oplus_{i=1}^{k+1} S_n$, where S_n permutes the i-lines and the symbols of the hypercube. Then two Latin k-hypercubes, L and L', are isotopic if there exists $(\sigma_1, \sigma_2, \dots, \sigma_{k+1}) \in \oplus_{i=1}^{k+1} S_n$ with $(\sigma_1, \sigma_2, \dots, \sigma_{k+1})(L) = L'$.

We can also create a new Latin k-hypercube from an existing one by permuting the roles of the coordinates and symbols. That is, acting on the set $\{x_1, x_2, \dots, x_k, s\}$ by S_{k+1} creates a new Latin k-hypercube. An operation of this type is said to be an adjugate. For Latin squares you can take the row adjugate (switching the roles of rows and symbols), the column adjugate (switching the roles of columns and symbols), the transpose (switching the roles of rows and columns), and combinations of them. In this case there are $|S_3| = 6$ possible adjugates. In general there are $|S_{k+1}| = (k+1)!$ possible adjugates.

Definition 2.2 *Two Latin k-hypercubes are equivalent if one can be obtained from the other by taking adjugates of an isotopic Latin k-hypercube.*

For an example of counting the number of equivalence classes see [12]. It is immediate that if M and M' are isotopic then $E^+(M)$ and $E^+(M')$ are isotopic and $E^-(M)$ and $E^-(M')$ are isotopic.

Define the group $G_{n,k}$ by

$$G_{n,k} = \oplus_{i=1}^{k+1} S_n \times_s S_{k+1}, \tag{3}$$

where $\times_s$ denotes a semi-direct product.

The group $G_{n,k}$ acts on the set of all Latin k-hypercubes of order n denoted by $T_{n,k}$. If $(\sigma_1, \sigma_2, \ldots, \sigma_{k+1}, \tau)$ is an element of $G_{n,k}$ then the action is described by the following: σ_i permutes the elements of x_i for $1 \leq i \leq k$, σ_{k+1} permutes the symbols, and τ permutes the x_i and s.

Hence we have two Latin k-hypercubes L and L' are equivalent if there exists

$$(\sigma_1, \sigma_2, \ldots, \sigma_{k+1}, \tau) \in G_{n,k}$$

with $(\sigma_1, \sigma_2, \ldots, \sigma_{k+1}, \tau)(L) = L'$. In this case we shall write $L \sim L'$.

Proposition 2.3 *The relation $\sim$ is an equivalence relation.*

Proof. Let ϵ denote the identity of S_n. Then $(\epsilon, \epsilon, \ldots, \epsilon, \epsilon)(L) = L$ and hence $\sim$ is reflexive. If $L \sim L'$ then $(\sigma_1, \sigma_2, \ldots, \sigma_{k+1}, \tau)(L) = L'$, and then $(\sigma^{-1}_{\tau^{-1}(1)}, \sigma^{-1}_{\tau^{-1}(2)}, \ldots, \sigma^{-1}_{\tau^{-1}(k+1)}, \tau^{-1})(L') = L$. Then the relation $\sim$ is symmetric. If $L \sim L'$ and $L' \sim L''$ then there exist elements of $G_{n,k}$ with $(\sigma_1, \sigma_2, \ldots, \sigma_{k+1}, \tau)(L) = L'$, and $(\lambda_1, \lambda_2, \ldots, \lambda_{k+1}, \mu)(L') = L''$. Then

$$(\lambda_{\tau(1)} \circ \sigma_1, \lambda_{\tau(2)} \circ \sigma_2, \ldots, \lambda_{\tau(k+1)} \circ \sigma_{k+1}, \mu \circ \tau)(L) = L''$$

and the relation is transitive. $\square$

Definition 2.4 *The main class of L is the set of all Latin k-hypercubes of order n that are equivalent to L, i.e.*

$$\begin{aligned} M_L &= \{L' \mid \text{ there exists } (\sigma_1, \sigma_2, \ldots, \sigma_{k+1}, \tau) \in G_{n,k} \text{ where} \\ & \quad (\sigma_1, \sigma_2, \ldots, \sigma_{k+1}, \tau)(L) = L'\}. \end{aligned}$$

The isotopy class of L is the set of all Latin k-hypercubes of order n that are isotopic to L, i.e.

$$\begin{aligned} I_L &= \{L' \mid \text{ there exists } (\sigma_1, \sigma_2, \ldots, \sigma_{k+1}, \epsilon) \in G_{n,k} \text{ where} \\ & \quad (\sigma_1, \sigma_2, \ldots, \sigma_{k+1}, \epsilon)(L) = L'\}, \end{aligned}$$

and ϵ is the identity permutation.

It is clear that for any L, $I_L \subseteq M_L$. A main class of maximal cardinality has $(k+1)!(n!)^{k+1}$ elements and an isotopy class of maximal cardinality has $(n!)^{k+1}$ elements.

We say that a Latin k-hypercube is i,j-symmetric if switching the roles of the i-lines and j-lines results in the same Latin k-hypercube. We also allow i and j to be equal to $k+1$ allowing for the use of symbols as well. More precisely L is i,j-symmetric if τ is the transposition (ij) and $(\epsilon, \epsilon, \dots, \epsilon, \tau)(L) = L$, where ϵ is the identity permutation. As an example, consider the circulant Latin k-hypercube L of order n. Here $L_{a_1,a_2,\dots,a_k} = a_1 + a_2 + \cdots + a_k$. Then for any i,j, with $1 \le i,j \le k$ reversing the roles of i and j does not change the entry in $L_{a_1,a_2,\dots,a_k}$, hence the hypercube is i,j-symmetric for all $1 \le i,j \le k$.

We can generalize a result about Latin squares from [8].

Theorem 2.5 *If I_L has maximal size and it contains an i,j-symmetric Latin k-hypercube then it contains precisely $(n!)^k$ such hypercubes.*

Proof. If L is i,j-symmetric then for permutations $\{\sigma_1, \dots, \sigma_{k+1}\}$ with $\sigma_i = \sigma_j$ and the other σ_ℓ are arbitrary, we have that $(\sigma_1, \sigma_2, \dots, \sigma_k, \sigma_{k+1})(L)$ is i,j-symmetric. Hence there are at least $(n!)^k$ such hypercubes. We know that each produces a different Latin hypercube since the isotopy class is of maximal size.

Next assume that $(\sigma_1, \sigma_2, \dots, \sigma_k, \sigma_{k+1})(L)$ is i,j-symmetric then we have

$$(\sigma_1, \sigma_2, \dots, \sigma_k, \sigma_{k+1}, \tau)(L) = (\sigma_1, \sigma_2, \dots, \sigma_k, \sigma_{k+1}, \epsilon)(L),$$

which implies that $\sigma_i = \sigma_j$ so there are at most $(n!)^k$ i,j-symmetric Latin k-hypercubes in the isotopy class. $\square$

The autoparatopism group of a Latin hypercube is

$$Aut(L) = \{(\sigma_1, \sigma_2, \dots, \sigma_{k+1}, \tau) \in G_{n,k} \mid (\sigma_1, \sigma_2, \dots, \sigma_{k+1}, \tau)(L) = L\}.$$

We avoid the use of the word automorphism to avoid confusion with its use with relation to the quasigroup operation. It is clear that $Aut(L)$ is a subgroup of $G_{n,k}$. The following proposition follows from the orbit-stabilizer theorem.

Proposition 2.6 *For any Latin square L,*

$$|M_L| = \frac{|G_{n,k}|}{|Aut(L)|} = \frac{(n!)^{k+1}(k+1)!}{|Aut(L)|}. \tag{4}$$

3 Parities of Latin k-hypercubes

In this section we shall introduce the notion of parity of a Latin k-hypercube.

If L is a Latin k-hypercube of order n, an i line consists of

$$L_{\alpha_1,\alpha_2,\dots,\alpha_{i-1},0,\alpha_{i+1},\dots,\alpha_k}, L_{\alpha_1,\alpha_2,\dots,\alpha_{i-1},1,\alpha_{i+1},\dots,\alpha_k}, \dots, L_{\alpha_1,\alpha_2,\dots,\alpha_{i-1},n-1,\alpha_{i+1},\dots,\alpha_k}.$$

That is fix $k-1$ dimensions and let the i-th coordinate vary over $0,1,2,\ldots,n-1$. Hence for each i there are n^{k-1} distinct i-lines. Then there are kn^{k-1} different lines in the k-hypercube.

Each i-line can be viewed as a permutation of $0,1,2,\ldots,n-1$. An i-line

$$(\sigma(0),\sigma(1),\ldots,\sigma(n-1))$$

is odd if the permutation σ is odd and even if the permutation σ is even. This is equivalent to saying that the i-line is odd if the number of inversions is odd and the i-line is even if the number of inversions is even, where an inversion is a pair $(\sigma(i),\sigma(j))$ such that $\sigma(i)<\sigma(j)$ and $i>j$.

Define $\Psi : S_n \to \{-1,1\}$ by

$$\Psi(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even,} \\ -1 & \text{if } \sigma \text{ is odd.} \end{cases} \tag{5}$$

Let $\mathcal{A}_i(L)$ be the set of i lines of a Latin k-hypercube of order n. Then we define the i-parity of L to be

$$\mathrm{Par}_i(L) = \prod_{\sigma\in\mathcal{A}_i(L)} \Psi(\sigma), \tag{6}$$

and the parity of L to be

$$\mathrm{Par}(L) = \prod_{i=1}^{k}(\mathrm{Par}_i(L)). \tag{7}$$

A Latin k-hypercube is said to be i-odd if $\mathrm{Par}_i(L)=-1$ and i-even if $\mathrm{Par}_i(L)=1$. Similarly, it is said to be odd if $\mathrm{Par}(L)=-1$ and even if $\mathrm{Par}(L)=1$.

Let L be a Latin k-hypercube of order n. If τ is an i-line of L, then $\tau,\sigma\tau,\sigma^2\tau,\ldots,\sigma^{n-1}\tau$ are now i-lines of $E^+(L)$ and $\tau,(\sigma^{-1})\tau,(\sigma^{-1})^2\tau,\ldots,(\sigma^{-1})^{n-1}\tau$ are now i-lines of $E^-(L)$, where $\sigma=(0,1,2,\ldots,n-1)$. Note that σ and σ^{-1} are even when n is odd and odd when n is even. This gives that for n odd each sign of a permutation of L appears oddly many times as a sign of a permutation in $E^+(L)$ and $E^-(L)$ and hence the parity does not change. If n is even then each sign of a permutation appears evenly many times and hence the i-parity becomes 1. Hence, we have if n is even then $\mathrm{Par}_i(E^+(L))=\mathrm{Par}_i(E^-(L))=1$ and if n is odd then $\mathrm{Par}_i(E^+(L))=\mathrm{Par}_i(E^-(L))=\mathrm{Par}_i(L)$.

Let $T_{n,k}$ be the set of all Latin k-hypercubes of order n. For each i we partition $T_{n,k}$ into $O^i_{n,k}$ and $E^i_{n,k}$, where $O^i_{n,k}$ is the set of all i-odd Latin k-hypercubes of order n and $E^i_{n,k}$ is the set of all i-even Latin k-hypercubes of order n. Likewise, $O_{n,k}$ is the set of all odd Latin k-hypercubes of order n and $E_{n,k}$ is the set of all even Latin k-hypercubes of order n.

Theorem 3.1 *If n is odd, then $|E^i_{n,k}|=|O^i_{n,k}|$ and $|E_{n,k}|=|O_{n,k}|$.*

Proof. Let $\Phi_i : T_{n,k}\to T_{n,k}$ where $\Phi_i(L)=L'$ such that if $(a_1,a_2,a_3\ldots,a_n)$ is an i-line of L then the corresponding i-line of L' is $(a_2,a_1,a_3,\ldots,a_n)$. It is clear that L' is a Latin k-hypercube of order n and that $\Psi(a_2, a_1, a_3, \ldots, a_n) = (-1)\Psi(a_1, a_2, a_3, \ldots, a_n)$. This gives

$$\mathrm{Par}_i(L') = (-1)^{n^{k-1}} \mathrm{Par}_i(L) = -(\mathrm{Par}_i(L)).$$

Thus Φ is an injection from $E^i_{n,k}$ to $O^i_{n,k}$ and from $O^i_{n,k}$ to $E^i_{n,k}$. This yields $|E^i_{n,k}| = |O^i_{n,k}|$.

Notice that $\mathrm{Par}_j(\Phi_i(L)) = \mathrm{Par}_j(L)$ when $j \neq i$. Then the map Φ is a bijection between $E_{n,k}$ and $O_{n,k}$ as well, which implies that $|E_{n,k}| = |O_{n,k}|$. □

Corollary 3.2 *If n is odd, then for all Latin k-hypercubes L we have* $|E_{n,k} \cap I_L| = |O_{n,k} \cap I_L|$.

Proof. We note that $L' = \Phi(L) \in I_L$ so the proof of Theorem 3.1 applies here as well. □

These results imply that when n is odd, then each isotopy and main class splits into pieces of equal size of even and odd Latin hypercubes for each i and that the entire space splits into pieces of equal size of even and odd Latin hypercubes as well. When n is even the map Φ_i preserves parity.

Proposition 3.3 *Let $\Omega = (\epsilon, \epsilon, \ldots, \epsilon, \tau) \in G_{n,k}$, where ϵ is the identity of S_n and $\tau(k+1) = k+1$. Then* $\mathrm{Par}_i(L) = \mathrm{Par}_{\tau(i)}(\Omega(L))$ *and* $\mathrm{Par}(L) = \mathrm{Par}(\Omega(L))$.

Proof. We know that the (n^{k-1}) i-lines of L are exactly the (n^{k-1}) $\tau(i)$-lines of $\Omega(L)$. Thus $\mathrm{Par}_i(L) = \mathrm{Par}_{\tau(i)}(\Omega(L))$.

Then we have $\mathrm{Par}(L) = \prod(\mathrm{Par}_i(L)) = \prod(\mathrm{Par}_{\tau(i)}(\Omega(L))) = \mathrm{Par}(\Omega(L))$. □

In [1], Alon and Tarsi conjectured that for Latin squares of even order the number of even Latin squares does not equal the number of odd Latin squares. This conjecture is shown to be equivalent to a variety of conjectures, see [7] for a complete description. Proofs of this conjecture when the order is $p+1$ and $n = 2^r p$ where p is a prime can be found in [3] and [4] respectively. We generalize this conjecture to the following.

Conjecture 3.4 *If n is even then*

$$|E^i_{n,k}| \neq |O^i_{n,k}| \qquad \textit{and} \qquad |E_{n,k}| \neq |O_{n,k}|.$$

When $n = 2$, $|L_{2,k}| = 2$ for all k, since any coordinate determines the rest. The two hypercubes are even and so the conjecture is true when $n = 2$.

Define L^i to be the Latin k-hypercube formed when the symbols are switched with the i-lines of L.

Theorem 3.5 *If L is a Latin k-hypercube then* $\mathrm{Par}_i(L) = \mathrm{Par}_i(L^i)$.

Proof. If $L_{(\alpha_1,\dots,0,\dots,\alpha_k)}, L_{(\alpha_1,\dots,1,\dots,\alpha_k)}, \dots L_{(\alpha_1,\dots,n-1,\dots,\alpha_k)} = (s_0, s_1, \dots, s_{n-1})$ is an i-line in L then $L_{(\alpha_1,\dots,s_0,\dots,\alpha_k)}, L_{(\alpha_1,\dots,s_1,\dots,\alpha_k)}, \dots L_{(\alpha_1,\dots,s_{n-1},\dots,\alpha_k)} = (0, 1, \dots, n-1)$ is the corresponding i-line in L^i. In the first the permutation is given by $i \to s_i$ and the second the permutation is given by $s_i \to i$. Then the result follows from the fact that $\Psi(\sigma) = \Psi(\sigma^{-1})$. □

The previous result for Latin squares can be found in ([9],page 34). In their terminology, as an example, the row adjugate has the same column parity as the original square, which is slightly different terminology than ours.

Theorem 3.6 *Let L be the circulant Latin k-hypercube of order n. For all $k > 2$ and for all n we have* $\mathrm{Par}_i(L) = 1$.

Proof. When n is even and $k > 2$ each of $\{(0, 1, 2, \dots, n-1), (1, 2, \dots, n-1, 0), (2, 3, \dots, n-1, 0, 1), \dots, (n-1, 0, 1, \dots, n-2)\}$ appears evenly many times in the i-th direction. If n is odd then each i-line has even parity and so the parity is even. □

For odd n, the i-parity of the reverse circulant hypercube depends on the direction. For example, if $n = 3$ and $k = 2$ the reverse circulant Latin square is

$$\begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}.$$

The row parity is 1 and the column parity is -1. Hence for the reverse circulant hypercube we can obtain either 1 or -1 depending on i.

4 Permutation hypercubes

Let $L = (L_{a_1,a_2,\dots,a_k})$ be a Latin k-hypercube of order n then define a $(k+1)$ dimensional permutation hypercube with entries from $\{0, 1\}$ by

$$P_{a_1,a_2,\dots,a_k,a_{k+1}} = 1 \text{ if and only if } L_{a_1,a_2,\dots,a_k} = a_{k+1}. \tag{8}$$

This connection was first noticed by Gupta in [6].

There are n^k entries in L, so in P there are n^{k+1} entries, n^k of which are 1.

Theorem 4.1 *In the permutation hypercube P, on any i-line there is exactly one coordinate with a 1 in it.*

Proof. Assume there exists an i-line with 2 coordinates with a 1 in them, that is,

$$P_{a_1,a_2,\dots,\alpha,\dots,a_k,a_{k+1}} = P_{a_1,a_2,\dots,\beta,\dots,a_k,a_{k+1}} = 1.$$

Then we have $L_{a_1,a_2,\dots,\alpha,\dots,a_k} = a_{k+1}$ and $L_{a_1,a_2,\dots,\beta,\dots,a_k} = a_{k+1}$, which means a_{k+1} appears twice in an i-line of L which is a contradiction. Also a_{k+1} must appear once for some α in $L_{a_1,a_2,\dots,\alpha,\dots,a_k}$ since it is a Latin k-hypercube. □

Let L be a Latin k-hypercube of order n and P the corresponding permutation hypercube. Let $L_{a_1,a_2,\ldots,x,\ldots,a_k}$ with $x = 0,1,2,\ldots,n-1$ be an i-line. This line viewed as a permutation has corresponding permutation matrix $P_{a_1,a_2,\ldots,x,\ldots,a_k,y}$ where $x, y = 0,1,\ldots,n-1$. If ℓ is a line of L let P_ℓ be its corresponding permutation matrix. We make the following definition:

$$\mathrm{Det}_i(P) = \prod_{\ell \text{ an i−line}} \mathrm{Det}(P_\ell) \tag{9}$$

where $\mathrm{Det}(P_\ell)$ is the usual determinant, and

$$\mathrm{Det}(P) = \prod_{\ell} \mathrm{Det}(P_\ell). \tag{10}$$

Theorem 4.2 *Let L be a Latin k-hypercube and P its corresponding permutation hypercube; then* $\mathrm{Det}_i(P) = \mathrm{Par}_i(L)$ *and* $\mathrm{Det}(P) = \mathrm{Par}(L)$.

Proof. If $\{\ell_j\}$ are the i-lines then $\Psi(\ell) = \mathrm{Det}(P_\ell)$ so

$$\mathrm{Par}_i(L) = \prod \Psi(\ell_j) = \prod \mathrm{Det}(P_{\ell_j}) = \mathrm{Det}_i(P).$$

Also

$$\mathrm{Par}(L) = \prod_{i=1}^{k} \mathrm{Par}_i(L) = \prod_{i=1}^{k} \mathrm{Det}_i(P) = \mathrm{Det}(P).$$

□

The following theorem is immediate from the construction.

Theorem 4.3 *Let $(\sigma_1,\ldots,\sigma_{k+1},\tau) \in G_{n,k}$. If L is a Latin k-hypercube of order n with corresponding permutation hypercube P, then the Latin k-hypercube $(\sigma_1,\ldots,\sigma_{k+1},\tau)L$ has permutation hypercube P' formed by letting σ_i act on the i-th coordinate of P and τ on the set of coordinates.*

Let L be a Latin k-hypercube of order n and P its corresponding permutation hypercube. There are $2(k+1)$ directions in which P can project, namely each of the coordinate directions in either the positive or negative direction. That is

$$\Pi_i^+(P) = M_{a_1,a_2,\ldots,a_{i-1},a_{i+1},\ldots,a_{k+1}}, \tag{11}$$

where

$$M_{a_1,a_2,\ldots,a_{i-1},a_{i+1},\ldots,a_{k+1}} = a_i \iff P_{a_1,a_2,\ldots,a_{k+1}} = 1,$$

and

$$\Pi_i^-(P) = M_{a_1,a_2,\ldots,a_{i-1},a_{i+1},\ldots,a_{k+1}}, \tag{12}$$

where

$$M_{a_1,a_2,\ldots,a_{i-1},a_{i+1},\ldots,a_{k+1}} = (n-1) - a_i \iff P_{a_1,a_2,\ldots,a_{k+1}} = 1.$$

The following is immediate.

Theorem 4.4 *Let L be a Latin k-hypercube of order n and P its corresponding hypercube. Then $\Pi_i^+(P)$ and $\Pi_i^-(P)$ are Latin k-hypercubes in the main class of L.*

5 Orthogonality Relations

Two Latin squares, L and M are said to be orthogonal if the set $\{(L_{i,j}, M_{i,j})\}$ has n^2 distinct elements. We shall investigate an extension of this definition to hypercubes. Numerous different definitions have been made as an extension to this definition of orthogonal Latin squares introduced by Euler [5]. For a description of various notions of orthogonality see [17] or [15] and the references therein. Our interest will be in one of the most restrictive, namely the following.

Definition 5.1 *Two Latin k-hypercubes are said to be orthogonal if each corresponding pair of Latin subsquares of order n are orthogonal. A set of s Latin k-hypercubes of order n are said to be mutually orthogonal if each pair are orthogonal. In this case we say we have a set of s mutually orthogonal Latin k-hypercubes (MOLkC).*

We shall relate these to an important class of codes, which will require a few additional definitions.

A code of length ℓ over $\mathbb{Z}_n$ is simply a subset of $\mathbb{Z}_n^\ell$. The distance between two vectors is the number of coordinates where they differ. The minimum distance of a code is the smallest distance between any two vectors. Let d be the minimum distance of a code of length ℓ over $\mathbb{Z}_n$ then the Singleton bound gives $d \leq \ell - log_n(M) + 1$ where there are M vectors in the code. Any code meeting this bound is said to be a maximum distance separable code (MDS). MDS codes are extremely important in coding theory and their existence relates to numerous combinatorial and geometric questions. A code with length ℓ, M vectors and minimum distance d is said to be an $[\ell, M, d]$ code. A linear code is a code that is also a submodule of $\mathbb{Z}_n^\ell$. Of course, $\mathbb{Z}_n$ is only a field when n is a prime, but a substantial literature exists for codes over rings. We refer to [18] for example.

We shall now prove some results relating codes and Latin hypercubes. The first few, namely Theorem 5.2, Lemma 5.3 and Theorem 5.4 are certainly known but we shall include proofs for completeness. For a proof of equivalent results see Chapter 11 of [13]. For an argument using this relationship to count MDS codes see [16].

Theorem 5.2 *There exists a set of s mutually orthogonal Latin k hypercubes of order n if and only if there exists a $[k+s, n^k, s+1]$ MDS code over $\mathbb{Z}_n$.*

Proof. Assume $L^1, L^2, \ldots, L^s$ are mutually orthogonal Latin k-hypercubes of order n. Construct the orthogonal array $OA(L^1, L^2, \ldots, L^s)$ where there are n^k rows indexed by $\mathbb{Z}_n^k$ where for $(a_1, a_2, \ldots, a_k) \in \mathbb{Z}_n^k$ we have

$$OA(L^1, L^2, \ldots, L^s)_{(a_1,a_2,\ldots,a_k)} = (a_1, a_2, \ldots, a_k, L^1_{a_1,a_2,\ldots,a_k}, L^2_{a_1,a_2,\ldots,a_k}, \ldots, L^s_{a_1,a_2,\ldots,a_k}). \quad (13)$$

This array has n^k rows and $k+s$ columns, hence we can view this as a code of length $k+s$ with n^k elements.

If two vectors v and w agree in $k-1$ of the first k coordinates then two elements in the column corresponding to L^i in v and w differ since they are in the same Latin line. Hence the distance is $s+1$.

We can view any k columns as the coordinates. Specifically, if we fix $k-2$ of the first coordinates then any pair of columns for L^i and L^j, $i \neq j$, have every pair appearing once since L^i and L^j are orthogonal. Hence L^i and L^j can be used with these coordinates to make k coordinates using the other two coordinates as Latin k-hypercubes. This extends to fixing $k-r$ coordinates of the first k since each pair are orthogonal. So the above proof where v and w agree in the first $k-1$ places applies to any $k-1$ coordinates in which they agree.

Given a $[k+s, n^k, s+1]$ MDS code over $\mathbb{Z}_n$, the first k coordinates must have all possible distinct k-tuples since they cannot agree in k places otherwise their distance would be s which is less than $s+1$. Then for column $k+i$ let

$$L^i_{a_1,a_2,\ldots,a_k} = \alpha$$

where the first k coordinates in the vectors are $a_1, a_2, \ldots, a_k$ and the vector has α in the $(k+i)$-th coordinate.

Assume $k-2$ of the first coordinates are the same then if $L^i_\alpha = L^j_\alpha$ and $L^i_\beta = L^j_\beta$ for $\alpha \neq \beta$, $i \neq j$ then the distance between those vectors is less than or equal to s. This implies that L^i is orthogonal to L^j. □

Lemma 5.3 *There do not exist n mutually orthogonal Latin k-hypercubes of order n.*

Proof. Assume there are n MOLkCs of order n. Fix any $k-2$ dimensions in every hypercube. The corresponding Latin squares are n MOLS of order n which are well known not to exist. □

The well known Singleton bound gives that an $[\ell, n^k, d]$ MDS code over $\mathbb{Z}_n$ satisfies $\ell \leq n+k-1$. In our case this means that $k+s \leq n+k-1 = k+(n-1)$. We know $s \leq n-1$ since there cannot be n-MOLkCs of order n by Lemma 5.3. This means that any MDS code has parameters consistent with Theorem 5.2 which shows that any MDS code corresponds to a set of MOLkCs of order n.

Let q be a prime power, $a_i \in \mathbb{F}_q$, $a_i \neq 0$, $a_i \neq a_j$ for $i \neq j$, $1 < k < q$ and consider the matrix:

$$\mathcal{M} = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 & 1 & 0 \\ a_1 & a_2 & a_3 & \ldots & a_{q-1} & 0 & 0 \\ a_1^2 & a_2^2 & a_3^2 & \ldots & a_{q-1}^2 & 0 & 0 \\ . & & & & & & \\ . & & & & & & \\ a_1^{q-k} & a_2^{q-k} & a_3^{q-k} & \ldots & a_{q-1}^{q-k} & 0 & 1 \end{pmatrix}. \tag{14}$$

The code that has $\mathcal{M}$ as a parity check matrix, that is the set of vectors that are orthogonal to each row of the matrix is a $[q+1, k, q-k+2]$ MDS code, see section 11 of [13] for details. This code is known as a RS (Reed-Solomon) code. This gives the following theorem.

Theorem 5.4 *Let q be a prime power. For $1 < k \leq q+1$ there exists $q-k+1$ MOLkCs of order q.*

Proof. A $[q+1, k, q-k+2]$ MDS code is guaranteed by the above discussion. Then write a_i as i and by Theorem 5.2 we have k MOLkCs of order q. Notice that k must be greater than 1 since orthogonality requires that we have at least Latin squares. □

We shall now show that the relation of orthogonality can be expressed in terms of permutation hypercubes satisfying a specific property.

Theorem 5.5 *Let $L_1, L_2, \ldots, L_s$ be Latin k-hypercubes of order n and $P_1, P_2, \ldots, P_s$ be their corresponding $k+1$ dimensional permutation hypercubes. Then $\{L_i\}$ are mutually orthogonal if and only if given P_i, P_j, $i \neq j$ and letting $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ be the permutation matrices formed when intersecting both permutation hypercubes with n parallel planes we have that A_i and B_j have at most one line in common.*

Proof. We can assume that A_i corresponds to the i-th symbol and B_j corresponds to the j-th symbol in the corresponding Latin lines in L_i and L_j respectively. Assume that A_i and B_j have 2 lines in common, then the ordered pair (i, j) appears twice in the corresponding Latin squares formed from these Latin lines, that is in the two Latin squares corresponding to the permutation hypercube made up of these n permutation matrices. This is a contradiction because the squares must be orthogonal if L_i and L_j are orthogonal.

Next assume that the any two of these matrices share at most one line. This implies that every pair (i, j) appears at least once (and then by counting exactly once) in the overlap of the corresponding Latin squares. Hence the corresponding Latin squares are orthogonal, which shows that the Latin k-hypercubes are orthogonal. □

The results can be collected into the following.

Theorem 5.6 *There is a one to one correspondence between MDS codes over $\mathbb{Z}_n$, mutually orthogonal Latin k-hypercubes of order n, and sets of s permutation hypercubes of dimension $(k+1)$ satisfying the property in Theorem 5.5.*

A Latin k-hypercube L of order n is said to be linear if

$$L_{a_1,a_2,\ldots,a_k} + L_{b_1,b_2,\ldots,b_k} = L_{a_1+b_1,a_2+b_2,\ldots,a_k+b_k} \pmod{n}.$$

If M and M' are linear k and k' hypercubes with functions f and g then $M'' = M * M'$ is linear. We have

$$\begin{aligned} M''_{a_1,\ldots,a_s,b_2,\ldots,b_t} &+ M''_{a'_1,\ldots,a'_s,b'_2,\ldots,b'_t} \\ &= g(f(a_1,\ldots,a_s),b_2,\ldots,b_t) + g(f(a'_1,\ldots,a'_s),b'_2,\ldots,b'_t) \\ &= g(f(a_1,\ldots,a_s)+f(a'_1,\ldots,a'_s),b_2+b'_2,\ldots,b_t+b'_t) \\ &= M''_{a_1+a'_1,\ldots,a_s+a'_s,b_2+b'_2,\ldots,b_t+b'_t}. \end{aligned}$$

It follows by induction that if $M_1, \ldots M_h$ are linear hypercubes of order n then

$$M_1 * M_2 * \cdots * M_h$$

is a linear hypercube of order n.

It is clear that a linear k-hypercube corresponds to a linear $[k+1, k, 2]$ MDS code.

Theorem 5.7 *There is exactly one isotopy class containing linear Latin k-hypercubes.*

Proof. A linear Latin k-hypercube is determined by the coordinates

$$L_{1,0,0,\ldots,0}, L_{0,1,0,\ldots,0}, \ldots, L_{0,0,\ldots,0,1}.$$

Then $L_{a_1,a_2,\ldots,a_k} = a_1 L_{1,0,0,\ldots,0} + \cdots + a_k L_{0,0,\ldots,0,1}$. Each of $L_{1,0,0,\ldots,0}, L_{0,1,0,\ldots,0}, \ldots,$ $L_{0,0,\ldots,0,1}$ must be non-zero since $L_{0,0,0,\ldots,0} = 0$ and the symbol in these coordinates must be different from the symbol in $L_{0,0,0,\ldots,0}$ since each of these symbols is in a different i-line with $L_{0,0,0,\ldots,0}$. The symbols in these coordinates must also be a unit of $\mathbb{Z}_n$ otherwise the elements of the i-line would not be distinct. If $L_{0,0,\ldots,1,\ldots,0} = \alpha_i$ where 1 is in the i-th coordinate, then the hypercube L' with $L_{0,0,\ldots,1,\ldots,0} = \alpha_i'$ is formed by permuting the i-th coordinate of L by multiplying the coordinates by a unit. Hence any two linear Latin k-hypercubes are isotopic. □

Corollary 5.8 *If C is a linear $[\ell, k, \ell-k+1]$ MDS code then the corresponding $\ell-k$ MOLkCs are isotopic.*

Proof. Let L be one of the Latin k-hypercubes corresponding to a coordinate of the code, then if $L_{a_1,a_2,\ldots,a_k} = \alpha$ and $L_{b_1,b_2,\ldots,b_k} = \beta$ then the coordinate beginning with $(a_1 + b_1, \ldots, a_k + b_k)$ must have $\alpha + \beta$ in the coordinate corresponding to L so the hypercube is linear. Then by the previous theorem all of the k-hypercubes are isotopic. □

Corollary 5.9 *If C is a linear $[n+1, 2, n-1]$ MDS code, with n prime, the corresponding $n-1$ MOLS complete to a desarguesian plane.*

Proof. The Latin squares must all be linear since the code is linear and a linear Latin square corresponds to the standard construction of desarguesian planes. □

If L is linear then not every Latin k-hypercube in M_L is linear. For a linear hypercube L we have $L_{0,0,\ldots,0} = 0$ then simply permuting the symbols 0 and 1 results in a k-hypercube in M_L that is not linear.

Theorem 5.10 *Let P be the permutation hypercube corresponding to a k-hypercube L. The k-hypercube L is linear if and only if $P_{a_1,\ldots,a_k,a_{k+1}} = 1$ and $P_{b_1,\ldots,b_k,b_{k+1}} = 1$ then*

$$P_{a_1+b_1,\ldots,a_k+b_k,a_{k+1}+b_{k+1}} = 1.$$

Proof. We have $P_{a_1,\ldots,a_k,a_{k+1}} = 1$ if and only if $L_{a_1,\ldots,a_k} = a_{k+1}$ and $P_{b_1,\ldots,b_k,b_{k+1}} = 1$ if and only if $L_{b_1,\ldots,b_k} = b_{k+1}$. This gives that $P_{a_1+b_1,\ldots,a_{k+1}+b_{k+1}} = 1$ if and only if $L_{a_1+b_1,\ldots,a_k+b_k} = a_{k+1} + b_{k+1}$ if and only if L is linear. □

A well known conjecture about MDS codes is that a $[\ell, k, \ell-k+1]$ MDS code over $\mathbb{Z}_n$ must satisfy $\ell \leq n+1$. For linear codes this bound follows from the maximum size of an arc in projective space. This leads us to conjecture that the maximum number of Mutually Orthogonal Latin k-hypercubes of order n is bounded above by $n+1-k$. We have already shown that this number is naturally bounded by $n-1$ which corresponds to the value when $k=2$. More concretely, we make the following conjecture.

Conjecture 5.11 *Let r be the maximum number of k-MOLkCs of order n then the max number of $(k+1)$-MOLkCs of order n is bounded above by $r-1$.*

Theorem 5.12 *There exist k-MOLkCs of order n if and only if there exists a k-hypercube of order n such that the numbers $0, 1, 2, \ldots, n^k - 1$ can be placed in the hypercube so that in each i-line the digits in any place of each k-digit number written in base n are a permutation of $0, 1, \ldots, n-1$.*

Proof. Let $L_1, \ldots, L_k$ be a set of k-MOLkCs of order n. Define a k dimensional hypercube of order n as follows:

$$\begin{aligned} M_{i_1,i_2,\ldots,i_k} &= (L_1)_{i_1,i_2,\ldots,i_k} + (L_2)_{i_1,i_2,\ldots,i_k} n + \cdots + (L_k)_{i_1,i_2,\ldots,i_k} n^{k-1} \\ &= \sum_{j=1}^{k} (L_j)_{i_1,i_2,\ldots,i_k} n^{j-1}. \end{aligned}$$

It is clear that in any i-line each of the symbols occurs in each place since it is an i-line of a Latin k-hypercube.

Consider the MDS $[2k, k, k+1]$ code constructed from $L^1, L^2, \ldots, L^k$ via Theorem 5.2. Then if $M_{i_1,i_2,\ldots,i_k} = M_{i'_1,i'_2,\ldots,i'_k}$, with at least one $i_j \neq i'_j$ then the corresponding vectors in C would have distance less than $k+1$ which is a contradiction. Hence each element from 0 to $n^k - 1$ appears exactly once.

The other direction is similar; simply let $(L_s)_{i_1,i_2,\ldots,i_k}$ be the s-th place of the element in $M_{i_1,i_2,\ldots,i_k}$ when the elements are written in base n. $\square$

Theorem 5.13 *Let M be a k-hypercube of order n formed in Theorem 5.12; then the sum of every i-line is $(\frac{n(n^k-1)}{2})$.*

Proof. We know that each number in $\{0, 1, 2, \ldots, n-1\}$ occurs once in each digit place. Thus, if you sum the numbers in each digit's place in an i-line we have $(0+1+2+\cdots+n-1)(1+n+n^2+\cdots+n^{k-1}) = (\frac{(n-1)n}{2})(\frac{n^k-1}{n-1}) = (\frac{n(n^k-1)}{2})$. $\square$

Hence, the matrix M is a magic hypercube in the sense that each i-line gives the same sum, namely $n(n^k-1)/2$. Of course, any k-hypercube with the property that all i-lines sum to a constant and each element appearing once from 0 to $n^k - 1$ must have this sum. Notice that we have relaxed the usual conditions given for magic squares that the diagonals have the same sums as the lines.

Corollary 5.14 *The following are equivalent:*

- *A* $[2k, k, k+1]$ *MDS code.*
- *A set of* k *MOLkCs of order* n*.*
- *A magic* k*-hypercube of order* n *where the digits place in each element in an* i*-line is a permutation of* $0, 1, \ldots, n-1$*.*
- *A set of* k *permutation hypercubes of size* $k+1$ *of order* n *satisfying the condition in Theorem 5.5.*

Moreover, if we have a $[2k, k, k+1]$ linear MDS code then the corresponding magic hypercube is constructed from MOLkCs all of which are equivalent.

Given the standard construction of MDS codes given above we have the following.

Corollary 5.15 *Let* q *be odd and then the* $[q+1, \frac{q+1}{2}, \frac{q+3}{2}]$ *RS code gives a magic hypercube.*

Proof. This follows from Theorem 5.4. □

For example, the $[4, 2, 3]$ ternary code gives the magic square:

$$\begin{pmatrix} 0 & 5 & 7 \\ 4 & 6 & 2 \\ 8 & 1 & 3 \end{pmatrix}. \tag{15}$$

The $[6, 3, 4]$ MDS code over $\mathbb{F}_5$ gives the following Magic hypercube of order 5, formed by stacking the following squares:

$$\begin{pmatrix} 0 & 83 & 36 & 119 & 72 \\ 64 & 17 & 95 & 28 & 106 \\ 123 & 51 & 9 & 87 & 40 \\ 32 & 110 & 68 & 21 & 79 \\ 91 & 49 & 102 & 55 & 13 \end{pmatrix}, \begin{pmatrix} 43 & 121 & 54 & 7 & 85 \\ 77 & 30 & 113 & 66 & 24 \\ 11 & 94 & 47 & 100 & 58 \\ 70 & 3 & 81 & 39 & 117 \\ 109 & 62 & 15 & 98 & 26 \end{pmatrix}$$

$$\begin{pmatrix} 56 & 14 & 92 & 45 & 103 \\ 115 & 73 & 1 & 84 & 37 \\ 29 & 107 & 60 & 18 & 96 \\ 88 & 41 & 124 & 52 & 5 \\ 22 & 75 & 33 & 111 & 69 \end{pmatrix}, \begin{pmatrix} 99 & 27 & 105 & 63 & 16 \\ 8 & 86 & 44 & 122 & 50 \\ 67 & 20 & 78 & 31 & 114 \\ 101 & 59 & 12 & 90 & 48 \\ 35 & 118 & 71 & 4 & 82 \end{pmatrix}$$

$$\begin{pmatrix} 112 & 65 & 23 & 76 & 34 \\ 46 & 104 & 57 & 10 & 93 \\ 80 & 38 & 116 & 74 & 2 \\ 19 & 97 & 25 & 108 & 61 \\ 53 & 6 & 89 & 42 & 120 \end{pmatrix}$$

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