AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **41** (2008), Pages 257–261

Improved upper bounds for the k-tuple domination number

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Abstract

We improve the generalized upper bound for the k-tuple domination number given in [A. Gagarin and V.E. Zverovich, A generalized upper bound for the k-tuple domination number, *Discrete Math.* **308** no. 5–6 (2008), 880–885]. Precisely, we show that for any graph G, when $k=3$, or $k=4$ and $d \leq 3.2$,

$$\gamma_{\times k}(G) \leq \frac{\ln(\delta - k+2) + \ln\Big((k-2)d + \sum_{m=2}^{k-2} \frac{(k-m)}{4^{\min\{m,\ k-2-m\}}}\widehat{d}_m + \widehat{d}_{k-1}\Big) + 1}{\delta - k + 2} n,$$

and, when $k=4$ and $d > 3.2$, or $k \geq 5$,

$$\gamma_{\times k}(G) \leq \frac{\ln(\delta - k+2) + \ln\left(\sum_{m=0}^{k-2} \frac{(k-m)}{4^{\min\{m,\ k-2-m\}}}\widehat{d}_m + \widehat{d}_{k-1}\right) + 1}{\delta - k + 2} n,$$

where $\gamma_{\times k}(G)$ is the k-tuple domination number, δ is the minimum degree, d is the average degree, and $\widehat{d}_m$ is the m-degree of G. Moreover, when $k \geq 5$, the latter bound can be improved to

$$\gamma_{\times k}(G) \leq \frac{\ln(\delta - k+2) + \ln\left(\sum_{m=0}^{k-2} \frac{(k-m)}{\mathrm{P}(k-2,m)}\widehat{d}_m + \widehat{d}_{k-1}\right) + 1}{\delta - k + 2} n,$$

where the coefficient $\mathrm{P}(t,m) = \frac{t^t}{m^m (t-m)^{t-m}}$ for $t > m > 0$, $\mathrm{P}(t,0) = \mathrm{P}(t,t) = 1$, with $t = k-2$.

1 Introduction

We consider undirected simple graphs. Given a graph G having the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, d_i denotes the degree of v_i, $i = 1, 2, \ldots, n$, $d = \frac{1}{n}\sum_{i=1}^{n} d_i$ is the average vertex degree of G, and $\delta = \delta(G) = \min_{1\le i\le n}\{d_i\}$ is the minimum vertex degree of G. For $0 \le m \le \delta$, the *m-degree* $\widehat{d}_m$ of G is defined as

$$\widehat{d}_m = \widehat{d}_m(G) = \frac{1}{n}\sum_{i=1}^{n}\binom{d_i}{m}.$$

Notice that $\widehat{d}_0 = 1$ and $\widehat{d}_1 = d$. Denote by $N(x)$ the neighborhood and by $N[x] = N(x) \cup \{x\}$ the closed neighborhood of a vertex $x \in V(G)$. For a set of vertices $X \subseteq V(G)$, denote by $N(X) = \cup_{x\in X} N(x)$ and by $N[X] = N(X) \cup X$.

A set $X \subseteq V(G)$ is called a *dominating set* in G if every vertex in $V(G)\backslash X$ is adjacent to a vertex in X. The minimum cardinality of a dominating set in G is the *domination number* $\gamma(G)$ of G. A set $X \subseteq V(G)$ is called a *k-tuple dominating set* in G if for every vertex $v \in V(G)$, $|N[v] \cap X| \ge k$. The minimum cardinality of a k-tuple dominating set in G is the *k-tuple domination number* $\gamma_{\times k}(G)$ of G.

The k-tuple domination number was apparently introduced by Harary and Haynes in [5]. Since a vertex can be dominated only by itself and its neighbours, we must have $k \le \delta + 1$, and $\gamma_{\times k}(G)$ is defined only when $1 \le k \le \delta + 1$. Clearly, $\gamma(G) = \gamma_{\times 1}(G)$ and $\gamma_{\times k}(G) \le \gamma_{\times k'}(G)$ when $k \le k'$. The 2-tuple domination number $\gamma_{\times 2}(G)$ and the 3-tuple domination number $\gamma_{\times 3}(G)$ are called, respectively, the *double* and *triple domination numbers.*

The following upper bound for the domination number was independently obtained by Alon and Spencer [1], Arnautov [2], and Payan [6]:

Theorem 1 ([1, 2, 6]) *For any graph G,*

$$\gamma(G) \le \frac{\ln(\delta+1)+1}{\delta+1}n.$$

An upper bound for the double domination number was shown by Harant and Henning in [4]:

Theorem 2 ([4]) *For any graph G with $\delta \ge 1$,*

$$\gamma_{\times 2}(G) \le \frac{\ln\delta + \ln(d+1)+1}{\delta}n.$$

An upper bound for the triple domination number was provided by Rautenbach and Volkmann in [7]:

Theorem 3 ([7]) *For any graph G with $\delta \ge 2$,*

$$\gamma_{\times 3}(G) \le \frac{\ln(\delta-1) + \ln(\widehat{d}_2 + d)+1}{\delta-1}n.$$

Gagarin and Zverovich [3] generalized Theorem 3 as follows:

Theorem 4 ([3]) *For any graph G with $3 \le k \le \delta + 1$,*

$$\gamma_{\times k}(G) \le \frac{\ln(\delta-k+2) + \ln\left(\sum_{m=1}^{k-1}(k-m)\widehat{d}_m - d\right)+1}{\delta-k+2}n. \tag{1}$$

In this paper we improve the general upper bound of Theorem 4. More precisely, we show two new upper bounds and decide which one is better depending on the values of parameters k and d. We also mention another improvement of (1) for $k \ge 5$.

2 Improved upper bounds

The following theorem provides improved upper bounds which are similar to the upper bound of Theorem 4.

Theorem 5 *For any graph G with $3 \le k \le \delta+1$, when $k=3$, or $k=4$ and $d \le 3.2$, we have*

$$\gamma_{\times k}(G) \le \frac{\ln(\delta-k+2)+\ln\left((k-2)d+\sum_{m=2}^{k-2}\frac{(k-m)}{4^{\min\{m,\ k-2-m\}}}\widehat{d}_m+\widehat{d}_{k-1}\right)+1}{\delta-k+2}n, \quad (2)$$

and, when $k=4$ and $d>3.2$, or $k\ge 5$, we have

$$\gamma_{\times k}(G) \le \frac{\ln(\delta-k+2)+\ln\left(\sum_{m=0}^{k-2}\frac{(k-m)}{4^{\min\{m,\ k-2-m\}}}\widehat{d}_m+\widehat{d}_{k-1}\right)+1}{\delta-k+2}n. \quad (3)$$

Proof: We use the random construction and follow the proof of Theorem 4 in [3]. Let A be a set formed by an independent choice of vertices of G, where each vertex is selected with probability p, $0 \le p \le 1$. For $m = 0,1,\dots,k-1$, let $B_m = \{v_i \in V(G)\backslash A : |N(v_i)\cap A| = m\}$, and, for $m=0,1,\dots,k-2$, let $A_m = \{v_i \in A : |N(v_i)\cap A| = m\}$.

A set A'_m corresponding to a set A_m is constructed as follows: for each vertex in the set A_m, we take its $k-(m+1)$ neighbours that are not in A. Such neighbours always exist because $\delta \ge k-1$. It is obvious that $|A'_m| \le (k-m-1)|A_m|$. A set B'_m corresponding to a set B_m is constructed by taking $k-(m+1)$ neighbours that are not in A for every vertex that is in B_m. We have $|B'_m| \le (k-m-1)|B_m|$. Then a set D is defined as

$$D = A \cup \left(\bigcup_{m=0}^{k-2} A'_m\right) \cup \left(\bigcup_{m=0}^{k-1} B_m \cup B'_m\right).$$

The set D is a k-tuple dominating set in G. Indeed, if there is a vertex v which is not k-tuple dominated by D, then v is not k-tuple dominated by A. Therefore, v would belong to A_m or B_m for some m, but all such vertices are k-tuple dominated by the set D by construction. The expectation of $|D|$ is

$$\begin{aligned} E(|D|) &\le E(|A| + \sum_{m=0}^{k-2}|A'_m| + \sum_{m=0}^{k-1}|B_m| + \sum_{m=0}^{k-1}|B'_m|) \\ &\le E(|A| + \sum_{m=0}^{k-2}(k-m-1)|A_m| + \sum_{m=0}^{k-1}(k-m)|B_m|) \\ &= E(|A|) + \sum_{m=0}^{k-2}(k-m-1)E(|A_m|) + \sum_{m=0}^{k-1}(k-m)E(|B_m|). \quad (4) \end{aligned}$$

The remaining part of the proof follows the lines of the proof of Theorem 4 and Corollary 1 in [3]. The only difference that provides an improvement is that we use the following upper bound (here $\mu = \delta - k + 2$):

$$\begin{aligned}\sum_{m=2}^{k-2}(k-m)(E(|A_m|)+E(|B_m|)) &\leq \sum_{m=2}^{k-2}(k-m)p^m(1-p)^{(k-2)-m}(1-p)^{\mu}\widehat{d}_m n \\ &\leq e^{-p\mu}n\sum_{m=2}^{k-2}(k-m)p^m(1-p)^{(k-2)-m}\widehat{d}_m \\ &\leq e^{-p\mu}n\sum_{m=2}^{k-2}\frac{(k-m)}{4^{\min\{m,\ k-m-2\}}}\widehat{d}_m, \end{aligned} \tag{5}$$

when the terms $(k-1)E(|A_0|), (k-2)E(|A_1|), kE(|B_0|)$, and $(k-1)E(|B_1|)$ of the sum (4) are considered separately in [3]. This gives the upper bound (2). It is also possible to consider all the terms of the sum (4) together to have

$$\sum_{m=0}^{k-2}(k-m)(E(|A_m|)+E(|B_m|)) \leq e^{-p\mu}n\sum_{m=0}^{k-2}\frac{(k-m)}{4^{\min\{m,\ k-m-2\}}}\widehat{d}_m. \tag{6}$$

This results in the upper bound (3). Other details are omitted and can be easily figured out from the proof in [3].

It remains to decide which upper bound of (2) and (3) is better. Bound (3) is worse than bound (2) in the case $k = 3$. Notice that bound (2) is the same as in Theorem 3 in this case. For $k \geq 4$, bound (2) is better than bound (3) if

$$\frac{4k}{3k-7} \geq d \geq \delta \geq k-1. \tag{7}$$

Resolving (7), we obtain

$$3 \leq d \leq 3.2$$

when $k = 4$. In all the other cases, $k = 4$ and $d > 3.2$, and $k \geq 5$, bound (3) should be used instead of bound (2). ∎

3 Final remarks

Similarly to the improvement technique used in the proof of Theorem 5, it is possible to maximize the product $p^m(1-p)^{(k-2)-m}$ used in (5) and (6) with respect to p for fixed values of parameters k and m, $k-2 \geq m \geq 0$. This provides the following upper bound:

$$\gamma_{\times k}(G) \leq \frac{\ln(\delta-k+2)+\ln\left(\sum_{m=0}^{k-2}\frac{(k-m)}{\mathrm{P}(k-2,m)}\widehat{d}_m+\widehat{d}_{k-1}\right)+1}{\delta-k+2}n, \tag{8}$$

where the coefficient $\mathrm{P}(t,m) = \frac{t^t}{m^m(t-m)^{t-m}}$ for $t > m > 0$, $\mathrm{P}(t,0) = \mathrm{P}(t,t) = 1$, with $t = k-2$. Bound (8) is better than bound (3) for $k \geq 5$. It is easy to see that $\mathrm{P}(t,m) \geq 4^{\min\{m,\ t-m\}}$ and $\mathrm{P}(t,m) \geq \mathrm{C}(t,m)$, where $\mathrm{C}(t,m) = \frac{t!}{m!(t-m)!}$ is the usual binomial coefficient.

The random construction used in this paper and in [3] provides a randomzed algorithm to find a k-tuple dominating set in a given graph G. An interesting direction in this research would be to derandomize this algorithm or to obtain an independent deterministic algorithm to find a k-tuple dominating set satisfying the upper bound (2), (3), or (8), respectively.

Acknowledgment

The author is thankful to the referee for constructive and insightful comments and remarks.

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(Received 27 June 2007; revised 3 Aug 2007)