AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **42** (2008), Pages 3–19

Nested balanced ternary designs and Bhaskar Rao designs

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Abstract

In this paper, we consider balanced ternary designs, BTDs, in which every block contains one element singly and the rest doubly. We call these packed BTDs, and we investigate three aspects of these designs: existence, nestings and signings. Construction methods generate classes of packed BTDs that are nested with balanced (BIBD) or partially balanced (PBIBD) incomplete block designs. Some of these classes are signed to produce c-Bhaskar Rao BTDs, most often with $c = 0$. Packed BTDs with block size three and five are studied in detail. The spectrum of possible indices for packed BTDs is determined. In particular, we prove every triple system with index $3t$ is nested within a packed Bhaskar Rao balanced ternary design with $K = 5$ and index $8t$. We give several new families of PBIBDs for block size 3, and show that each is nested within a BTD with block size 5 and index 4 or 6. We show that the necessary conditions are sufficient for the existence of this type of BTD, whether or not it has a design nested within.

1 Introduction

A balanced ternary design, or BTD, with parameters (V, B, R, K, Λ) is a collection of B blocks on V elements such that (1) each element occurs R times in the design, (2) each pair of distinct elements occurs Λ times in the design, and (3) each block contains K elements, where an element may occur 0, 1, or 2 times in a block (i.e., multiset).

BTDs are regular in the sense that each element occurs singly in ρ_1 blocks and doubly in ρ_2 blocks [6]. Because of this regularity, BTD parameters are most often given as $(V; B; \rho_1, \rho_2, R; K; \Lambda)$.

Counting arguments can be used to establish the necessity of the following well-known relationships among the parameters of any BTD:

* The authors wish to express their thanks to The Citadel Foundation for its support.

$$VR = BK,$$
$$\Lambda(V-1) = \rho_1(K-1) + 2\rho_2(K-2), \text{ and}$$
$$R = \rho_1 + 2\rho_2.$$

An example of a BTD with parameters $(4; 12; 3, 6, 15; 5; 16)$ appears in Figure 1. In the figure, each column represents a block of the design.

1	1	1	2	2	2	1	1	1	1	1	1
1	1	1	2	2	3	1	1	1	2	2	3
2	2	3	3	3	3	2	3	2	2	2	3
2	3	4	3	4	4	2	3	4	3	4	4
3	3	4	4	4	4	4	4	4	3	4	4

Figure 1: A BTD(4;12;3,6,15;5;16).

An alternative way of representing a BTD$(V; B; \rho_1, \rho_2, R; K; \Lambda)$ is with a V-by-B matrix, M, whose $(i, j)^{th}$ entry denotes the number of times element i appears in block j. M is referred to as the incidence matrix of the BTD. The incidence matrix for the design in Figure 1 is given in Figure 2. It is straightforward to see that the inner product of any two distinct rows of an incidence matrix of a BTD$(V; B; \rho_1, \rho_2, R; K; \Lambda)$ is Λ, the sum of the entries in any row R, and the sum of the entries in any column K.

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}
1	2	2	2	0	0	0	2	2	2	1	1	1
2	2	1	0	2	2	1	2	0	1	2	2	0
3	1	2	1	2	1	2	0	2	0	2	0	2
4	0	0	2	1	2	2	1	1	2	0	2	2

Figure 2: The incidence matrix for the BTD shown in Figure 1.

Sometimes it is possible to "sign" some of the entries in the rows of the incidence matrix of a BTD and still maintain constant row inner products. For example, changing all the ones to negative ones in the incidence matrix shown in Figure 2 produces a matrix where the inner product of any two rows is zero. Not every BTD incidence matrix can be signed. However, when one can be, the signed incidence matrix is called a Bhaskar Rao balanced ternary design or BRBTD.

Formally, a c-BRBTD$(V; B; \rho_1, \rho_2, R; K; \Lambda)$ is a V-by-B matrix $\widehat{M}$ with entries from the set $\{0, \pm 1, \pm 2\}$ such that (1) the inner product of any two distinct rows of $\widehat{M}$ is c, and (2) the matrix M obtained from $\widehat{M}$ by changing each negative entry of $\widehat{M}$ to its corresponding absolute value yields the incidence matrix of a BTD$(V; B; \rho_1, \rho_2, R; K; \Lambda)$.

A second way to modify the incidence matrix of a BTD is to change all the twos to ones. For example, changing the twos to ones in the matrix of Figure 2 produces the incidence matrix of a balanced incomplete block design, or BIBD, with parameters $(4, 12, 9, 3, 6)$. Not every BTD incidence matrix can be "reduced" to a

BIBD. However, when one can be, the BTD is nested as we will show in Corollary 2 given in Section 2.3.

A BIBD is a BTD where no element appears doubly in a block. Because ρ_2 is zero in a BIBD, the parameters can be given as (v, b, r, k, λ) and the necessary parametric relationships as $vr = bk$ and $\lambda(v-1) = r(k-1)$. These necessary conditions imply that the parameters are always fully specified by v, k, and λ. Thus, throughout the remainder of the paper when listing parameters of a BIBD we only list the triple (v, k, λ).

The purpose of this paper is to describe a class of BTDs which can be used to construct BRBTDs and which are often nested. In Section 2, we define the class and show some general results. In Sections 3 and 4, respectively, we analyze designs with block size three and five. For each case we delineate the spectrum, construct related Bhaskar Rao designs, and identify nested designs.

For the interested reader, more details about balanced ternary designs appear in [2, 3, 9, 15, 17], about Bhaskar Rao designs in [4, 12, 14, 17, 18], and about nested designs in [10, 11, 13, 16].

2 Packed and Fully Packed BTDs

In this section, we introduce the concept of packed and fully packed BTDs and BRBTDs, derive necessary parametric relationships for them, and construct some infinite classes of them. We close the section with a discussion of nested designs and design reductions.

A block of odd size $K = 2N + 1$ in a BTD is said to be *packed* if it contains one element that appears singly and N elements that appear doubly.

A BTD with odd block size $K = 2N + 1$ is said to be packed if every block in the design is packed. For example, the BTD$(3; 3; 1, 1, 3; 3; 2)$ with blocks $\{1, 1, 2\}$, $\{2, 2, 3\}$, and $\{3, 3, 1\}$ is a packed BTD.

A packed BTD with odd block size $K = 2N + 1$ is said to be fully packed if it is simple (i.e., has no repeated blocks) and contains every possible packed block. Adding the blocks $\{1, 2, 2\}$, $\{2, 3, 3\}$, and $\{3, 1, 1\}$ to the above packed BTD gives a fully packed BTD$(3; 6; 2, 2, 6; 3; 4)$.

A c-BRBTD is called packed [fully packed] if the underlying BTD is packed [fully packed]. The 0-BRBTD$(3; 3; 1, 2, 5; 5; 8)$ shown in Figure 3 is fully packed.

	b_1	b_2	b_3
1	2	2	-1
2	2	-1	2
3	-1	2	2

Figure 3: A fully packed 0-BRBTD(3;3;1,2,5;5,8).

BTDs and BRBTDs can be packed without being fully packed. The 0-BRBTD $(5; 10; 2, 4, 10; 5; 8)$ of Figure 4 is one such BRBTD.

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_8	b_{10}
1	2	2	0	0	−1	0	0	2	2	−1
2	2	0	2	−1	0	0	2	0	−1	2
3	0	2	−1	2	0	2	0	−1	0	2
4	0	−1	2	0	2	2	−1	0	2	0
5	−1	0	0	2	2	−1	2	2	0	0

Figure 4: A packed 0-BRBTD(5;10;2,4,10;5;8) that is not fully packed.

The BTD associated with the 0-BRBTD$(5; 10; 2, 4, 10; 5; 8)$ given in Figure 4 is the union of two BTD$(5; 5; 1, 2, 5; 5; 4)$s, namely $\{b_1, b_2, b_3, b_4, b_5\}$ and $\{b_6, b_7, b_8, b_9, b_{10}\}$. However, the signing of the "whole" BTD is not a signing for either of the "half" BTDs.

2.1 Parametric relationships for packed BTDs

We begin by establishing necessary conditions for the parameters of packed BTDs. These conditions are then used to set bounds on the spectrum of fully packed and simple packed BTDs.

Lemma 1 *If a BTD$(V; B; \rho_1, \rho_2, R; 2N + 1; \Lambda)$ is packed, then Λ is even.*

Proof: If D is a packed BTD, exactly one element per block appears singly. Thus whenever distinct v_1 and v_2 appear in the same block, one or both occurs with multiplicity two. The result follows. □

Lemma 2 *If a BTD$(V; B; \rho_1, \rho_2, R; 2N + 1; \Lambda)$ is packed, then*

(a) $B = \rho_1 V$,
(b) $\rho_2 = N\rho_1$,
(c) $R = K\rho_1$, *and*
(d) $\Lambda(V - 1) = 4\rho_1 N^2$.

Proof: Assume $D = \text{BTD}(V; B; \rho_1, \rho_2, R; 2N + 1; \Lambda)$ is a packed BTD. Since every block of D contains exactly one singleton and since every element is a singleton exactly ρ_1 times, there must be $\rho_1 V$ blocks in the design. Similarly counting doubletons we get $NB = V\rho_2$, which together with (a) implies $\rho_2 = N\rho_1$. The remaining two conditions follow from (a), (b), the necessary parametric conditions for BTDs stated in Section 1, and the fact that $K = 2N + 1$. □

Theorem 1 *If a BTD$(V; B; \rho_1, \rho_2, R; 2N + 1; \Lambda)$ is packed, then it is fully specified by the three parameters $V, K,$ and Λ.*

Proof: If a BTD$(V; B; \rho_1, \rho_2, R; 2N + 1; \Lambda)$ is packed, the equation of part (d) of Lemma 2 can be used to define ρ_1 in terms of $V, K,$ and Λ. This formulation can then be substituted into (a) through (c) of that lemma, and the result follows. □

Because of the above result, throughout the remainder of the paper, we use the abbreviated notation BTD(V, K, Λ) or $BTD(V, 2N+1, \Lambda)$ when referring to a packed BTD$(V; B; \rho_1, \rho_2, R; K = 2N+1; \Lambda)$.

Theorem 2 *A fully packed* $BTD(V, 2N+1, \Lambda)$ *exists if and only if* $V \geq N+1$ *and* $\Lambda = 4\binom{V-1}{N}N^2/(V-1)$. *When a fully packed BTD does exist for a triple* (V, K, Λ), *it is unique.*

PROOF: *Since a fully packed design contains every possible packed block exactly once, these designs are straightforward to construct and clearly unique for a given set of parameters. On the other hand, if* $D = BTD(V, 2N+1, \Lambda)$ *is fully packed with element set* E, *then each element* v *must appear singly in a block with all possible* N*-tuples of* E-$\{v\}$. *Thus,* $\rho_1 = \binom{V-1}{N}$. *This, used in conjunction with Lemma 2 (d), implies* $\Lambda = 4\binom{V-1}{N}N^2/(V-1)$. □

Corollary 1 *If* $D = BTD(V, 2N+1, \Lambda)$ *is both simple and packed, then* $\Lambda \leq 4\binom{V-1}{N}N^2/(V-1)$.

2.2 Constructing packed BTDs

Below we construct two infinite classes of packed designs. The method used in these constructions builds packed BTDs from already existing BIBDs. The spectrum of the BTDs produced by the second method is not as far reaching as that from the first method; however, the parameters of the BTDs are generally smaller. The designs from both constructions can be signed to produce BRBTDs.

Not all construction methods for building packed designs are dependent on having a starter design. For example see Theorem 6.

Theorem 3 *Let* $V \geq 4$. *If there exists a BIBD*$(V, 3, \lambda)$, *then there exists a packed BTD*$(V, 7, 6\lambda(V-3))$. *If the BIBD is not complete, then this BTD will not be fully packed. The incidence matrix of the constructed BTD can be signed to create a c-BRBTD with* $c = 2\lambda(V-3) = \Lambda/3$.

PROOF: Assume that $V \geq 4$, and that D_1 is a BIBD with parameters $(V, b, r, 3, \lambda)$ and elements $\{1, 2, \ldots, V\}$. For each block $\{v_1, v_2, v_3\}$ in D_1 and each element v_4 in $\{1, 2, \ldots, V\}\backslash\{v_1, v_2, v_3\}$, define a new block $\{v_1, v_1, v_2, v_2, v_3, v_3, v_4\}$. Call the collection of all such new blocks D_2. By construction, D_2 contains $b(V-3)$ size seven blocks over V elements. Since each element appears in r blocks of D_1, it will appear doubly in $r(V-3)$ blocks of D_2 and singly in $(b-r)$ blocks. Also, since each pair of distinct elements appear together in λ blocks of D_1, and each without the other in $r-\lambda$ blocks, each distinct pair will appear both doubly in $\lambda(V-3)$ blocks of D_2 and one singly and the other doubly in $2(r-\lambda)$ blocks. Thus, D_2 is a BTD where each pair of distinct elements appears in $4\lambda(V-3) + 2 \cdot 2(r-\lambda)$ blocks. The necessary conditions on $BIBD$ parameters imply that $r = \lambda(V-1)/2$. This can be used to rewrite the index (pair repetition number) given above as $6\lambda(V-3)$. If D_1 is not complete, D_2 will not be fully packed.

Assume M is the incidence matrix for D_2. Change each one in M to a negative one. Recall that since each pair of distinct elements v_1 and v_2 of D_1 appear together in λ blocks of D_1 and each without the other in $r - \lambda$ blocks, each pair will appear both doubly in $\lambda(V-3)$ blocks of D_2 and one singly and the other doubly in $2(r-\lambda)$ blocks. Thus, in the signed matrix the inner product of row v_1 and row v_2 will be $4\lambda(V-3) - 2 \cdot 2(r-\lambda) = 2\lambda(V-3)$. □

Before leaving Theorem 3 we mention that it is well known that the necessary conditions for a BIBD$(v, 3, \lambda)$ are sufficient [10].

A BIBD is said to be resolvable if the design blocks can be partitioned into classes, called parallel classes, such that each element of the design appears exactly once in the blocks of a class.

A BIBD is said to be near-resolvable if the design blocks can be partitioned into classes (near-parallel classes) such that each class is missing exactly one element and every element of the design is absent from exactly one class.

If a BIBD(v, K, λ) is near-resolvable, then each element must appear once in $v-1$ classes so $r = v - 1$, which forces $\lambda = K - 1$. Since each class is missing a single element and since each block has size K, $v \equiv 1 (\mathrm{mod}\ K)$.

The necessary conditions for the existence of a near resolvable design are known to be sufficient for $K \leq 7$. For $K \leq 5$, see Table I.6.26 in [1]. For $K = 6$ see [2]. In [3], Abel et al. reduced the exception list for $K = 7$ to five open cases. These five cases have since been solved by Abel [4].

Theorem 4 *If there exists a near-resolvable BIBD*$(V, K, K-1)$*, then there exists a packed BTD*$(V, 2K+1, 4K)$*. The incidence matrix of the constructed BTD can be signed to create a* $4(K-2)$*-BRBTD.*

PROOF: Assume D_1 is a near-resolvable BIBD$(V, K, K-1)$, where C_{v_i} is the near-parallel class missing element v_i. For each element x and each block $\{v_1, \ldots, v_K\}$ in C_x, define a new block $\{v_1, v_1, \ldots, v_K, v_K, x\}$. Call the collection of all such new blocks D_2. By construction, D_2 contains b size $2K+1$ blocks over V elements. Since each element appears in $V-1$ blocks of D_1 and is missing from exactly one class, each element will appear doubly in $V-1$ blocks of D_2 and singly in $b/V = (V-1)/K$ blocks. Also, since each pair of distinct elements appear together in $K-1$ blocks of D_1, and each is missing from exactly one near-resolvable class, each distinct pair will appear both doubly in $K-1$ blocks of D_2 and one singly and the other doubly in two blocks. Thus, D_2 is a BTD where each pair of distinct elements appears $4(K-1) + 4 = 4K$ times.

Assume M is the incidence matrix for D_2. Change each one in M to a negative one. Recall that since each pair of distinct elements will appear together in $K-1$ blocks of D_1, and each is missing from exactly one near-resolvable class, each distinct pair will appear both doubly in $K-1$ blocks of D_2 and one singly and the other doubly in two blocks. Thus, in the signed matrix the inner product of row v_1 and row v_2 will be $4(K-1) - 4 = 4(K-2)$. □

2.3 Nested packed BTDs and BTD reductions

We next turn our attention to nested designs and reductions, and their relevance to packed BTDs.

If each block b_i of a design D_1 can be partitioned into sub (multi) sets c_{ij}, $j = 1, 2, \dots, m$ such that $\{c_{ij}\}_{i=1}^{i=B}$ is a design (with the assigned decomposition of blocks) for each j, then D_1 is said to be a nested design ([14], [15], [17]).

Theorem 5 *If $D =$ BTD(V, K, Λ) is a fully packed BTD, then D is nested.*

PROOF: Let $K = 2N + 1$. For each block $b_i = \{v_1, v_1, v_2, v_2, \dots, v_N, v_N, v_{N+1}\}$ in D, let $c_{i1} = \{v_1, v_2, \dots, v_{N+1}\}$ and $c_{i2} = \{v_1, v_2, \dots, v_N\}$. Both $D_1 = \{c_{i1}\}_{i=1}^{i=B}$ and $D_2 = \{c_{i2}\}_{i=1}^{i=B}$ are designs. Here D_1 is a multiple of the complete (containing every possible $N+1$-tuple) BIBD$(V, N+1, \lambda_1)$, and D_2 is also a multiple of a complete BIBD(V, N, λ_2) design where the blocks have size N. □

This definition of a nested *design* does not specify the design type of the sub-designs. In fact, it does not even demand that the sub-designs all be of the same type. For example, when $K = 3$, the design D_1 in the proof of Theorem 5 is a BIBD$(V, 2, 2)$, while D_2 is a one-design, i.e., a set of elements.

Fully packed BTDs are not the only packed BTDs that are nested. The subdivision of blocks used in Theorem 5 can be applied to the BTDs of Theorems 3 and 4 to show that they too are nested.

Nested BTDs have occurred in the literature previously. Theorem 6 is one such example.

Theorem 6 *[21], [17]. If x be a primitive root of GF(V), where $V = 2t+1$ is a prime or prime power and t is odd, then the initial block $\{0, x^0, x^0, x^2, x^2, \dots, x^{2t-2}, x^{2t-2}\}$ gives, when developed additively over GF(V), a nested packed BTD which reduces to two BIBDs with initial sub-blocks $\{0, x^0, x^2, \dots, x^{2t-2}\}$ and $\{x^0, x^2, \dots, x^{2t-2}\}$.*

Somewhat differently, we say D_2 is a *reduction* of a packed BTD D_1, if D_2 is a design and $\{v_1, v_2, \dots, v_{N+1}\}$ is a block of D_2 if and only if $\{\mathrm{v}_1,\mathrm{v}_1,\ \mathrm{v}_2,\mathrm{v}_2,\ \dots,\ \mathrm{v}_N,\mathrm{v}_N,\ \mathrm{v}_{N+1}\}$ is a block of D_1. For example, the BIBD$(4, 3, 6)$ with blocks $\{1, 2, 3\}$, $\{1, 2, 3\}$, $\{1, 3, 4\}$, $\{2, 3, 4\}$, $\{2, 3, 4\}$, $\{2, 3, 4\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, $\{1, 2, 4\}$, $\{1, 2, 3\}$, $\{1, 2, 4\}$ and $\{1, 3, 4\}$ is a reduction of the packed BTD$(4; 12; 3, 6, 15; 5; 16)$ shown in Figure 1 of Section 1.

If D_1 is a nested design with reduction D_2, it is possible that D_2 is one of the subdesigns of D_1 that makes up the nesting. We illustrate this using Theorem 6. Consider the field of seven elements $\{0, 1, \dots, 6\}$ with operations mod 7. The element 3 is a generator of the cyclic group of non-zero elements ($3^1 = 3$, $3^2 = 2$, $3^3 = 6$, $3^4 = 4$, $3^5 = 5$, $3^6 = 1$). Setting $x = 3$, we get $\{x^0,\ x^2,\ x^4\} = \{1, 2, 4\}$. Thus the starter block $\{0, 1, 1, 2, 2, 4, 4\}$ additively generates a packed BTD$(7, 7, 6)$. By splitting the starter block into $\{0, 1, 2, 4\}$ and $\{1, 2, 4\}$ one obtains starter blocks for a BIBD$(7, 4, 2)$ and a BIBD$(7, 3, 1)$ respectively, thus showing that the BTD is nested. Clearly the BIBD$(7, 4, 2)$ is a reduction of the BTD$(7, 7, 6)$.

Below we give necessary and sufficient conditions for a packed BTD to reduce to a BIBD.

For every distinct pair of elements $v_1$and v_2 in a packed BTD, define $d_2(v_1, v_2)$ to be the number of blocks which contain one of $v_1$or v_2 singly and the other doubly, and define $d_4(v_1, v_2)$ to be the number of blocks which contain each of v_1 and v_2 doubly.

Theorem 7 *A packed BTD$(V, 2N+1, \Lambda)$ reduces to a BIBD$(V, N+1, \lambda)$, if and only if d_2 and d_4 are constants independent of the choice of the elements v_1 and v_2.*

Proof: Assume the packed BTD$(V, 2N+1, \Lambda)$ reduces to a BIBD$(V, N+1, \lambda)$. Then if v_1, v_2 and v_3, v_4, are two pairs of distinct elements of the design, the equations $\Lambda = 2d_2(v_1, v_2)+4d_4(v_1, v_2) = 2d_2(v_3, v_4)+4d_4(v_3, v_4)$ and $\lambda = d_2(v_1, v_2)+d_4(v_1, v_2) = d_2(v_3, v_4) + d_4(v_3, v_4)$ yield $(\Lambda - 2\lambda)/2 = d_4(v_1, v_2) = d_4(v_3, v_4)$ and $(4\lambda - \Lambda)/2 = d_2(v_1, v_2) = d_2(v_3, v_4)$ and the result follows.

Conversely, let D_1 be a packed BTD$(V, 2N+1, \Lambda)$ with d_2 and d_4 constants independent of any v_1 and v_2. Further, let D_2 be the collection of blocks obtained from D_1 by changing each doubleton in a D_1 block to a singlton. Since D_1 is packed, each block in D_2 will be of size $N+1$. The number of blocks a pair of elements appears in does not change in going from D_1 to D_2 blocks, only the multiplicity changes, i.e. all multiplicities are reduced to one. Thus, each pair of elements will appear together in $\lambda = d_2 + d_4$ blocks. Finally since ρ_1 and ρ_2 are constants for D_1, each element will appear in $r = \rho_1 + \rho_2$ reduced blocks. We have shown D_2 is a block system with a constant replication number for each element and a constant index for each pair of elements. Thus, it follows that D_2 is a BIBD$(V, N+1, \lambda)$. □

Corollary 2 *If a packed BTD$(V, 2N+1, \Lambda)$ D_1 reduces to a BIBD$(V, N+1, \lambda)$, D_2, then D_1 is nested. One nesting consists of D_2 and a second BIBD.*

Proof: Assume D_1 is a packed BTD$(V, 2N+1, \Lambda)$ that reduces to D_2 a BIBD$(V, N+1, \lambda)$. For each block $b_i = \{v_1, v_1, v_2, v_2, \ldots, v_N, v_N, v_{N+1}\}$ in D_1, let $c_{i1} = \{v_1, v_2, \ldots, v_{N+1}\}$ and $c_{i2} = \{v_1, v_2, \ldots, v_N\}$. By assumption $D_2 = \{c_{i1}\}_{i=1}^{i=B}$ is a BIBD$(V,\ N+1, \lambda)$. Thus Theorem 7 tells us that d_2 and d_4 are constants such that $\Lambda = 2d_2 + 4d_4$ and $\lambda = d_2 + d_4$. Let $D_3 = \{c_{i2}\}_{i=1}^{i=B}$. Since each element appears in ρ_2 blocks of D_1 as a doubleton, each element will appear in ρ_2 blocks of D_3. Also since each pair of distinct elements appear together in $d_2 + d_4$ D_1 blocks and d_2 D_2 blocks, it follows that they appear together in d_4 D_3 blocks. We have shown D_3 is an equireplicate block system, with block size N, and constant index for all pairs of elements. Thus, D_3 will be a BIBD(V, N, d_4). □

3 Analysis of packed BTDs with block size 3

Each block in a packed BTD$(V, 3, \Lambda)$ has the form $\{v_1, v_1, v_2\}$ for some distinct pair of elements v_1 and v_2. Because of this correspondence between blocks and pairs of distinct elements, it is straightforward to describe completely the fully packed

BTD$(V,3,\Lambda)$s and their spectrum, as well as the spectrum of the simple packed BTD$(V,3,\Lambda)$s that are not fully packed. The designs are interesting in that the entire spectrum of such designs is nested and can be signed.

Theorem 8 *If a packed BTD*$(V,3,\Lambda)$ *is simple, then either* $\Lambda = 4$ *and the design is fully packed, or* $\Lambda = 2$ *and the design is not fully packed.*

PROOF: No block in a simple BTD can be repeated. Thus, for simple BTDs with block size three and distinct elements v_1 and v_2, either the BTD contains both of the blocks $\{v_1, v_1, v_2\}$ and $\{v_1, v_2, v_2\}$ or exactly one of them. If both blocks are included in the BTD, it follows that $\Lambda = 4$ and the BTD is fully packed. If only one is included, it follows that $\Lambda = 2$ and the BTD is packed but not fully packed. □

Theorem 9 *For every* $V \geq 2$*, there exists a unique fully packed BTD*$(V,3,4)$ *whose incidence matrix can be signed to create a* 0*-BRBTD.*

PROOF: The existence and uniqueness of the designs follows from Theorem 2. Now suppose M is the incidence matrix of a fully packed BTD with element set $\{1, 2, \ldots, V\}$. Change a one in M to a negative one if and only if the one corresponds to v_2 in the block $\{v_1, v_1, v_2\}$ and $v_2 < v_1$. Since the design also contains the block $\{v_2, v_2, v_1\}$, the result follows. □

The blocks of a simple BTD(7, 3, 2) that is packed but not fully packed are shown in Figure 5. The design was created using a cyclic construction method. The blocks shown in column 1 of the example are called the starter blocks of the design. Each row of blocks is then cyclically generated from the starter block of the row by repeatedly adding one (mod 7) to the block elements.

001	112	223	334	445	556	660
002	113	224	335	446	550	661
003	114	225	336	440	551	662

Figure 5: A simple packed BTD(7,3,2) that is not fully packed.

Theorem 10 *For every odd* $V = 2t + 1 \geq 3$*, there exists a simple packed BTD*$(2t + 1, 3, 2)$ *which is not fully packed.*

PROOF: We use the cyclic construction method illustrated above to construct a BTD$(2t+1, 3, 2)$. If $V = 2t+1$, then the starter blocks are $\{0, 0, i\}$, $i \in \{1, \ldots, t\}$.□

Theorem 11 *If* $D =$ *BTD*$(2t, 3, \Lambda)$ *is a simple packed BTD, then* D *is fully packed and* $\Lambda = 4$.

PROOF: Assume the design $D =$ BTD$(2t, 3, \Lambda)$ is packed but not fully packed. Since D is not fully packed, Theorem 8 tells us that $\Lambda = 2$. Further, $B = \binom{V}{2}$ since every pair of distinct elements appear together in exactly one block. Thus, $VR = BK$, which implies $VR = 3B = 3V(V-1)/2$ which yields $2R = 3(V-1)$. The left hand side of this last equation is even while the right hand side is odd. The contradiction implies the BTD must have been fully packed. □

Reducing all twos to ones in the incidence matrix of any simple packed BTD($V, 3,$ 2) that is not fully packed produces a matrix that represents the set of all pairs of distinct elements in the design. Similarly, reducing all twos to ones in the incidence matrix of a fully packed BTD($V, 3, 4$) represents two copies of this same set. These facts can be formally stated as:

Theorem 12 *If a simple BTD$(V, 3, 2)$ is packed but not fully packed, then the BTD is nested. The BTD reduces to a one-factor (block size 2) and the remainders, i.e. the block elements not appearing in the reduction, form a list of the elements of the design (a one-design). If the BTD is fully packed, it reduces to two copies of the one-factor. In particular, for every admissible Λ, there exists a packed BTD$(V, 3, \Lambda)$ that is nested.*

4 Analysis of packed BTDs with block size 5

In this section, we analyze packed BTD($V, 5, \Lambda$)s. We begin our analysis by signing the fully packed BTD($V, 5, \Lambda$)s. The remainder of the section focuses on the spectrum, signings and nestings of packed BTD($V, 5, \Lambda$)s that are not fully packed.

Theorem 13 *For every $V \geq 3$ and $\Lambda = 8(V-2)$, there exists a unique fully packed BTD$(V, 5, \Lambda)$ whose incidence matrix can be signed to create a 0-BRBTD.*

Proof: The existence and uniqueness of the designs follows from Theorem 2. Let M be the incidence matrix of a fully packed BTD($V, 5, \Lambda$) and let $\widehat{M}$ be the matrix obtained by changing every one in M to a negative one. If v_1 and v_2 are any two distinct elements of the design, they appear together in $V-2$ blocks of the form $\{v_1, v_1, v_2, v_2, v_i\}$, $V-2$ blocks of the form $\{v_1, v_1, v_2, v_i, v_i\}$, and $V-2$ blocks of the form $\{v_1, v_2, v_2, v_i, v_i\}$. Thus, the inner product of row v_1 and v_2 in $\widehat{M}$ will be $4(V-2) - 2(V-2) - 2(V-2)$ or zero. □

We are now ready to prove the main result of the section. This requires a structure lemma of H. Agrawal's (stated as Lemma 3) on binary equi-replicate designs. A binary equi-replicate design is a collection of b size k blocks (i.e. sets) over a v-set of elements such that each element appears in r blocks.

Lemma 3 [5] *The elements of every binary equi-replicate design with $bk = vr$ and $b = mv$, can be arranged in a k-by-b array such that each column represents a block of the design and each row contains m copies of every element.*

Theorem 14 *Every BIBD$(V, 3, 3s)$ is the reduction of a packed BTD$(V, 5, 8s)$. Moreover, the incidence matrix of the BTD can be signed to create a 0-BRBTD.*

Proof: Assume D_1 is a BIBD($V, B, R, 3, 3s$), (recall this implies by the necessary conditions of BIBDs that $R = 3s(V-1)/2$ and $B = sV(V-1)/2$). For each block $b_{1i} = \{v_1, v_2, v_3\}$ in D_1, set $b_{2i} = \{v_1v_2,\ v_1v_3,\ v_2v_3\}$, the set of unordered pairs of b_{1i}. Call the collection of all such blocks D_2. Now D_2 contains B size three blocks over $V_2 = \binom{V}{2}$ elements, namely the unordered pairs of D_1, each of which appears $3s$ times in D_2. Since the replication number of D_2 is the index $\lambda = 3s$ of D_1, it follows that D_2 is an equi-replicate design with $m = s$. Thus, we can use Agrawal's lemma to get a 3-by-b array, M, in which every pair of D_1 elements appears s times in each row, and each column represents a block of D_2.

For each column i of M, there exist three distinct elements v_1, v_2 and v_3 of D_1 such that v_1v_2 is the element in position $(1, i)$ of M, v_1v_3 is in position $(2, i)$, and v_2v_3 is in position $(3, i)$. Define $b_{3i} = \{v_1,v_2,v_2,v_3,v_3\}$. Call the collection of all such blocks D_3. D_3 contains B size five blocks over the V elements of D_1. When pair v_1v_2 appears in column i and in either row one or two of M, b_{3i} will contain one of them singly and the other doubly. When the pair appears in row three, b_{3i} will contain both doubly. Thus, pair v_1v_2 appears together $\Lambda = 2 \cdot 2s + 4s = 8s$ times in D_3. The only time v_1 appears doubly in b_{3j} is if there exists x such that v_1x is the element in position $(3, i)$ of M. Since v_1 is in $V - 1$ pairs of elements and since each pair appears s times per row of M, we can conclude ρ_2 is constant and has value $s(V-1)$. Let $\rho_1(x)$ represent the number of times x appears singly in a block of D_3. If x appears singly in block b_{3i}, then x appears in 4 pairs in b_{3i}. If x appears doubly in block b_{3i}, then x appears in 6 pairs in b_{3i}. Hence, each element x of D_3 appears in $4\rho_1(x) + 6\rho_2$ element pairs of D_3. But from above we know each element of D_3 appears in $8s(V-1)$ pairs of D_3. Thus, $4\rho_1(x) + 6\rho_2 = 8s(V - 1)$, which implies ρ_1 is constant, $s(V - 1)/2$, regardless of x. It now follows that D_3 is a packed BTD$(V, 5, 8s)$.

In the incidence matrix M_3 of D_3, change each one to a negative one. From the construction of the blocks, one can see that the inner product of any two rows of $\widehat{M_3}$ is zero. Thus, $\widehat{M_3}$ is a 0-BRBTD. If the 0-BRBTD is reduced, with the minus sign being retained, the result is a c-Bhaskar Rao BIBD$(v, 3, 3s)$, i.e., a c-BRD$(v, 3, 3s)$ with $c = -s$. □

Theorem 14 uses a BIBD to build an equi-replicate design. Then adhering to the method prescribed by Agrawal in [5], an array on the equi-replicate design elements is generated. Lastly, the original BIBD blocks and information from the Agrawal array are used to construct packed BTD blocks. We illustrate Theorem 14 by building a packed BTD$(5, 5, 8)$ from a BIBD$(5, 3, 3)$ on elements $\{0, 1, 2, 3, 4\}$. The ten blocks of the BIBD$(5, 3, 3)$ are given in row 2 of Figure 6. Note in order to save space, block $\{a, b, c\}$ has been written as abc. The three 2-sets produced by each BIBD block are used to form the blocks of an equi-replicate design, and an Agrawal array is generated on the equi-replicate design elements. One such Agrawal array is shown in rows 4–6 of Figure 6. It is important to note that Agrawal's construction guarantees an array such that each equi-replicate element appears exactly once in each row of the array and such that each column represents a block of the equi-replicate design. Because of this guarantee, we can use any row of the Agrawal array to tell us which two elements of the BIBD blocks should be doubletons in the BTD blocks; we use row six. The blocks of the BTD are shown in row 8 of Figure 6.

It is straightforward to build the incidence matrix of the BTD shown in row 8 of Figure 6. Changing each 1 in the matrix to a -1 produces a 0-BRBTD.

The blocks of the BIBD									
013	034	014	023	012	134	123	124	024	234
The Agrawal array for the equi-replicate design									
{0,1}	{0,3}	{1,4}	{0,2}	{1,2}	{3,4}	{1,3}	{2,4}	{0,4}	{2,3}
{0,3}	{0,4)	{0,1}	{2,3}	{0,2}	{1,3}	{1,2}	{1,4}	{2,4}	{3,4}
{1,3}	{3,4}	{0,4}	{0,3}	{0,1}	{1,4}	{2,3}	{1,2}	{0,2}	{2,4}
The blocks of the BTD									
11330	33440	00441	00332	00112	11443	22331	11224	00224	22443

Figure 6: An illustration of the sequence of structures Lemma 6 uses to construct packed BTDs.

In the above theorem, the construction implies that the original BIBD is a reduction of the constructed BTD; the construction also implies that the BTD will be simple whenever the BIBD is. This allows us to describe the Λ-spectrum of such designs.

Lemma 4 *If a packed BTD$(V,5,\Lambda)$ reduces to a BIBD$(V,3,\lambda)$, then $\Lambda = 8d_4$ and $\lambda = 3d_4$, where d_4 is as in Theorem 7.*

PROOF: From Lemma 2 we know $\Lambda = 4\rho_1 \cdot 4/(V-1)$ and $\rho_2 = 2\rho_1$. From the necessary conditions for BIBDs we know $\lambda = 2r/(V-1)$. Thus, $6\rho_1/(V-1) = d_2 + d_4$ and $16\rho_1/(V-1) = 2d_2 + 4d_4$. Solving these as a system of equations yields $d_2 = 2d_4$ and the result follows. $\square$

Lemma 5 *If a BTD$(V,5,\Lambda)$ is packed, then $\Lambda \geq 4$. Further, if the BTD reduces to a BIBD$(V,3,\lambda)$, then $\Lambda \equiv 0 (\bmod 8)$, $\lambda \equiv 0 (\bmod 3)$, and $\lambda = 3\Lambda/8$.*

PROOF: Every block of a packed BTD$(V,5,\Lambda)$ has form $\{v_1, v_1, v_2, v_2, v_3\}$ for distinct v_1, v_2 and v_3. Since pair v_1, v_2 appears four times in the block, it follows that $\Lambda \geq 4$. Lemma 4 tells us $\lambda = 3d_4$ and $\Lambda = 8d_4$. Thus, $\Lambda \equiv 0 (\bmod 8)$, $\lambda \equiv 0 (\bmod 3)$ and $\lambda = 3\Lambda/8$. $\square$

Theorem 15 *(a) If V is odd, then the $\Lambda -$ spectrum of the simple packed BTD $(V,5,\Lambda)$s that reduce to a simple BIBD$(V,3,\lambda)$ is the set $\{\Lambda = 8s : 8 \leq 8s \leq 8(V-2)/3\}$.*

(b) If V is even, then the $\Lambda -$ spectrum of the simple packed BTD$(V,5,\Lambda)$s that reduce to a simple BIBD$(V,3,\lambda)$ is the set $\{\Lambda = 16s : 16 \leq 16s \leq 8(V-2)/3\}$.

PROOF: It is well-known that simple BIBD$(V,3,\lambda)$s exist if and only if (1) λ and V satisfy the necessary conditions for a BIBD and (2) $\lambda \leq V-2$ (see p.85 of [11]). When V is odd this implies simple BIBD$(V,3,3s)$s exist for $3s \leq V-2$. When V is even this implies simple BIBD$(V,3,3s)$s exist for s even and $3s \leq V-2$. This together with Theorem 14, and Lemma 5 yields the result. $\square$

Corollary 3 *If V is even, then the Λ-spectrum of the simple packed $BTD(V, 5, \Lambda)s$ is the same as the spectrum of the simple packed $BTD(V, 5, \Lambda)s$ that reduces to a simple $BIBD(V, 3, \lambda)$, namely the set $\{\Lambda = 16s : 16 \leq 16s \leq 8(V-2)/3\}$.*

PROOF: *If V is even, then the index Λ must be a multiple of 16 since Lemma 2 (d) tells us that $\Lambda(V-1) = 16\rho_1$ for any packed $BTD(V, 5, \Lambda)$. But the Λ-spectrum of the simple nested packed BTDs already covers all such values.* □

Corollary 4 *The necessary conditions are sufficient for the existence of a simple packed 0-$BRBTD(V, 5, \Lambda)$ which reduces to a $BIBD(V, 3, \lambda)$.*

Although Theorem 14 constructed simple packed BTD$(V, 5, \Lambda)$s that reduced to simple BIBD$(V, 3, \lambda)$s, there is no guarantee that every simple packed BTD$(V, 5, \Lambda)$ does so; in fact there is no guarantee that they even reduce to *BIBD*s. Below we show two constructions that illustrate these facts. The first construction is adapted from page 135 of [6].

Theorem 16 *If a $BIBD(V, 3, 1)$ exists, then a simple packed $BTD(V, 5, 8)$ exists that reduces to a $BIBD(V, 3, 3)$ that is not simple. Further, the $BTD(V, 5, 8)$ can be signed to produce a 0-$BRBTD(V, 5, 8)$.*

PROOF: Let D_1 be a simple BIBD$(V, 3, 1)$ with incidence matrix M_1, and let $\widehat{M_2}$ be the 0-BRBTD$(3, 5, 8)$ shown in Figure 3. (Note M_2 is a simple BTD that reduces to a BIBD that is not simple.) We build a new matrix $\widehat{M_3}$ by replacing each element of M_1 with a triple of elements. In particular, replace the three ones in each column of M_1 with the three different rows of $\widehat{M_2}$ and replace each zero in M with a triple of three zeros. Since M_1 is the incidence matrix of a simple BIBD$(V, 3, 1)$ and $\widehat{M_2}$ a simple 0-BRBTD$(3, 5, 8)$, $\widehat{M_3}$ will be a simple 0-BRBTD$(V, 5, 8)$. However, each of the columns of M_3 generated from a single column of M_1 will reduce to the same block. □

Note a BIBD$(V, 3, 1)$ exists if and only if $V \equiv 1, 3 (\text{mod}\, 6)$ [11].

Let D_1 be the packed BTD$(5, 5, 4)$ with blocks 11225, 44551, 11334, 33552, and 22443. The blocks of D_1 reduce to the blocks 125, 451, 134, 352, and 243. Since pair 12 appears once in the reduced blocks while pair 15 appears twice, D_1 is an example of a packed BTD$(5, 5, 4)$ that does not reduce to a BIBD. We use D_1 to build a class of BTD$(V, 5, 4)$s that do not reduce to BIBDs.

Theorem 17 *If a $BIBD(V, 5, \lambda)$ exists, then a packed $BTD(V, 5, 4\lambda)$ which does not reduce to a BIBD exists. If $v \equiv 1, 5 (\text{mod}\, 20)$, then λ can be taken to be one.*

PROOF: Let D_1be the BTD$(5, 5, 4)$ described above and let D_2 be a BIBD$(V, 5, \lambda)$ that contains distinct elements v_1, v_2, and v_3. For every block b in D_2 generate a copy of D_1 using the elements of b. In the case where one or more of v_1, v_2, or v_3 is in b, identify v_1 with 1, v_2 with 2, and v_3 with 3. The collection of generated blocks will be a packed BTD$(V, 5, 4\lambda)$. When the blocks are reduced, v_2 and v_3 will appear together in the reduced blocks twice as many times as v_1 and v_2. □

We continue our analysis of the five case by examining packed BTDs that do not reduce to BIBDs. Here a different type of design, a partially balanced incomplete block design, arises. Let X be a v-set with a symmetric m-association scheme defined on it, then a partially balanced incomplete block design with m associate classes (PBIBD(M)) having parameters $(v; b; r; \lambda_1, \lambda_1, \ldots, \lambda_m)$ is a design based on the v-set X with b blocks each of size k such that (1) every element occurs in r blocks, and (2) each pair of i-th associates occur together in λ_i blocks; see [23].

If v is an odd integer and D is an additively generated collection of b blocks of size k over Z_v with s starter blocks, then D is a PBIBD($(V-1)/2$) with $r = 3s$. The $(v-1)/2$ associate classes are defined by the non-zero differences less than $(v-1)/2$ (i.e., x and y are $i-th$ associates iff $x - y = \pm i(\bmod V), 1 \leq i \leq (V-1)/2$), and each λ_i is defined by the number of times i appears as a difference in the starter blocks. For example, in the additively generated design over Z_9, with starter blocks $\{0, 1, 4\}$ and $\{0, 2, 3\}$ there are 4 associate classes and $\lambda_1 = \lambda_3 = 2$ and $\lambda_2 = \lambda_4 = 1$.

As seen above the indices corresponding to the various associate classes in a PBIBD(m) do not have to be unique. We are interested in PBIBDs where there are exactly two unique λ_is. The above example is one such situation. In the example each element x appears twice with $x+1$, $x+3$, $x+6$, and $x+8$ (the addition is mod 9) and once with $x+2$, $x+4$, $x+5$, $x+7$.

If we extend the starter blocks from this example to $\{0, 0, 1, 4, 4\}$ and $\{0, 0, 3, 2, 2\}$, it is straightforward to check that they additively generate a packed BTD$(9, 5, 4)$ which reduces to the example PBIBD.

The following two theorems use the idea illustrated above. That is, in the proof of each theorem a PBIBD with two unique λ_is is generated and then extended to a BTD that reduces back to the PBIBD. In the two proofs we also make use of the fact that a design additively generated from starter blocks, where each element appears zero, one or two times, will be a BTD iff each non-zero difference appears the same number of times.

Theorem 18 *For every $V \equiv 1(\bmod 4)$, there exists a cyclic nested packed BTD(V, 5, 4) which reduces to a cyclic PBIBD.*

Proof: Let $V = 4t+1$. Additively generate the blocks of a PBIBD over Z_V using the t starter blocks $\{0, 1, 2t\}, \{0, 2, 2t-1\}, \ldots, \{0, t, t+1\}$. Calculating and counting the differences of the elements in the starter blocks shows that each of $1, 3, \ldots, 2t-1$ and their negatives *mod* v will be a difference two times, while each of $2, 4, \ldots, 2t$ and their negatives *mod* v will be a difference one time. Now extend each starter block to five elements by repeating zero and the non-zero even element in the block. The construction of the extended starter blocks increases the number of times the differences $1, 3, \ldots, 2t-1$ appear by one, and the number of times the differences $2, 4, \ldots, 2t$ appear by two. Thus the design additively generated by the extended starter blocks is a BTD$(V, 5, 4)$. Clearly it reduces to the original PBIBD.

If we delete the element 0 from each of the starter blocks of the PBIBD used above, and additively generate these new blocks over Z_V, we get an equi-replicate design with each block having size two and $r = 2t$ ($r = 2t$ since there are t starter blocks with two elements per block). Further each element x appears once with element y if and only if $y = x \pm 2$, $i = 1, 2, \ldots, t$. Thus this design is a PBIBD and it follows that the BTD is nested. □

Theorem 19 *For every* $V \equiv 1 (\text{mod}\, 8)$, *there exists a cyclic nested packed BTD* $(V, 5, 6)$ *which reduces to a cyclic PBIBD.*

Proof: Let $V = 8t + 1$. Additively generate the blocks of a PBIBD over Z_V using the $3t$ starter blocks: $\{0, i, t + 4i - 2\}$, $\{0, -i, t + 2i - 1\}$ and $\{0, i, t + 4i\}$, $i = 1, 2, \ldots, t$. Calculating and counting the differences of the elements in the starter blocks confirms that each of 1 through t is a difference three times, while each of $t+1$ through $4t$ is a difference two times (so the two indices are 3 and 2). Now extend each starter block by repeating the two non-zero elements in the block. The construction of the extended starter blocks increases the number of times the differences $1, 2, \ldots, t$ appear by three, and the number of times the differences $t + 1, t + 2, \ldots, 4t$ appear by four. Thus the design additively generated by the extended starter blocks is a BTD$(V, 5, 6)$. Clearly it reduces to the original PBIBD.

If we delete the element 0 from each of the $3t$ starter blocks of the PBIBD used above, and additively generate these new blocks over Z_V, we get an equi-replicate design with each block having size two and $r = 6t$ ($r = 6t$ since there are $3t$ starter blocks with two elements per block). Further each element x fails to appear with y if $y = x \pm 1$, otherwise x appears exactly once with y. Thus this design is a PBIBD and it follows that the BTD is nested. □

Note the PBIBD families used in the proofs of the last two theorems do not appear in Clatworthy [9] since the PBIBD association scheme, in each of our examples, has $(v-1)/2$ associate classes; whereas the Clatworthy designs have two associate classes

We close with a general result about the existence of packed BTD.

Lemma 6 *If a packed BTD*$(V, 5, \Lambda)$ *exists, then* $\Lambda = 2s \geq 4$. *Further, (1)* $V = 2t$ *implies* $\Lambda = 16s$, *(2)* $V = 4t + 3$ *implies* $\Lambda = 8s$, *and (3)* $V = 8t + 5$ *implies* $\Lambda = 4s$.

Proof: *Lemmas 1 and 5 imply* $\Lambda = 2s \geq 4$. *Since* $K = 5$, *Lemma 2 (d) tells us that* $\Lambda(V - 1) = 16\rho_1$. *Thus, when* V *is even,* $V - 1$ *is odd which implies* $\Lambda = 16s$. *When* $V = 4t + 3$, $V - 1$ *is twice an odd number which implies* $\Lambda = 8s$. *When* $V = 8t + 5, V - 1$ *is four times an odd number which implies* $\Lambda = 4s$. □

Theorem 20 *The necessary conditions are sufficient for the existence of a packed (and/or nested) BTD*$(V, 5, \Lambda)$.

Proof: We follow the cases outlined in Lemma 6. (1) Let $V = 2t$. Then since there is a BIBD$(2t, 3, 6)$, applying the construction of Lemma 14 produces a BTD$(2t, 5, 16)$. (2) Similarly if $V = 4t + 3$, since there is a BIBD$(V, 3, 3)$ applying the construction of Lemma 14 produces a BTD$(4t + 3, 5, 8)$. (3) If $V = 8t + 5$, the construction in Theorem 18 produces a BTD$(8t+5, 5, 4)$. Multiples of these indices may be obtained from multiple copies of these designs. If $V = 8t+1$, every even value of Λ greater than two may be obtained from a linear combination of 4 and 6 using the constructions in Theorems 18 and 19. □

Acknowledgment

The authors would like to thank the referee for the helpful suggestions.

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(Received 9 May 2006; revised 12 Dec 2007)