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Perfect distance forests

Bill Calhoun   John Polhill

Department of Mathematics, Computer Science, and Statistics
Bloomsburg University
Bloomsburg, PA 17815
U.S.A.

Abstract

In 1975, Leech introduced the problem of finding trees whose edges can be labeled with positive integers so that the sums of the edge labels between vertices is $\{1, 2, \cdots, \binom{n}{2}\}$, where n is the number of vertices. We refer to such trees as perfect distance trees. In a previous paper, this problem was generalized to graphs in general and in particular, forests. Here, we show that $tK_{1,2}$ can be labeled as a perfect distance forest if and only if $t \cong 0, 1 (\text{mod } 4)$ and that $tK_{1,3}$ can always be labeled as a perfect distance forest. Examples of labelings for tP_3 and $sK_{1,3} \cup tP_3$ as perfect distance forests are given. The most important result on perfect distance graphs, Taylor's Theorem, is used to rule out the existence of many forests with same size components. Finally, we examine the case of perfect distance forests with 2 components. For $n \leq 16$, we determine all pairs n_1, n_2 such that there is a perfect distance forest with components of size n_1 and n_2.

1 Introduction

A *weighted tree* is a tree in which each edge is labeled with a positive integer, called the *weight* of the edge. The *distance* between two vertices in a weighted tree is the sum of the weights on the edges of the unique path connecting the pair. Since each pair of vertices determines a distance, there are a total of $\binom{n}{2}$ distances in a tree (or any connected graph) with n vertices. If all of these distances are distinct, we call the tree a *distinct distance tree.* Define the function $M(n)$ to be the smallest integer so that there exists a distinct distance tree with n vertices and maximum distance $M(n)$. We refer to a distinct distance tree on n vertices as a *perfect distance tree* if the set of distances $\{1, 2, \cdots, \binom{n}{2}\}$ can be achieved. To generalize this notion further, we define a *minimal distinct distance tree* to be a distinct distance tree with maximal distance $M(n)$. In [7], Leech performed a hand search and found that the five perfect distance trees shown in Figure 1 are the only ones on 6 or fewer vertices. They remain the only known perfect distance trees.

Perfect distance trees and minimal distinct distance trees have applications in electrical networks, where the weights are interpreted as electrical resistances [1].

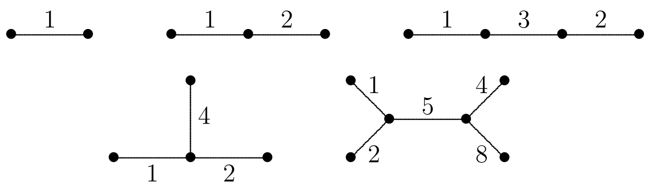


Figure 1: All Known Perfect Distance Trees

When such a network forms a perfect distance tree, the network might be used as an efficient multipurpose resistor, since only $n-1$ resistors are needed to construct the network, and any of $\binom{n}{2}$ different resistances can be obtained. Using a minimal distinct distance tree minimizes the number of gaps corresponding to resistances that cannot be obtained.

A major result on perfect distance trees is known as Taylor's Condition, which states that in order for a tree with n vertices to be labeled as a perfect distance tree $n = k^2$ or $n = k^2 + 2$ for some integer k, see [5] or [8]. In a previous paper [2], we showed among other things that there are no other perfect distance trees with fewer than 18 vertices. Since there seem to be so few, it is natural to consider generalizations. One such generalization is to relax the requirement that the graph need to be a tree. The graphs in this paper will not necessarily be connected, but can have neither loops nor multiple edges.

Let $G = (V, E)$ be a graph on n vertices and k components. We denote the sizes of the components of G by n_i for $i = 1 \ldots k$. An *edge-labeling* of G is a map $\lambda : E \to Z^+$. The (weighted) *distance,* $d(u, v)$*,* between vertices u and v in an edge-labeled graph (G, λ) is defined only if u and v are in the same component of G. In that case $d(u, v)$ is the minimum weight of a path from u to v in G. We define $\text{maxdist}(G, \lambda)$ to be the maximum of the distances between pairs of vertices in (G, λ).

A *distinct distance labeling* of a graph G is an edge-labeling such that if $u_1 \neq v_1$ and $u_2 \neq v_2$ are vertices of G, $\{u_1, v_1\} \neq \{u_2, v_2\}$, and $d(u_1, v_1)$ and $d(u_2, v_2)$ are defined, then $d(u_1, v_1) \neq d(u_2, v_2)$. We say that (G, λ) is a *distinct distance graph* if λ is a distinct distance labeling. A *perfect distance graph* is a distinct distance graph (G, λ) whose maximum distance is the number of pairs of distinct vertices in G such that the two vertices are in the same component of G. That is, $\text{maxdist}(G, \lambda) = \Sigma_{i=1}^{k} \binom{n_i}{2}$, and the distances in (G, λ) are the consecutive integers $\{1, 2, 3, \ldots, \text{maxdist}(G, \lambda)\}$. For any graph G, $M(G)$ is the minimum of $\text{maxdist}(G, \lambda)$ over all λ such that (G, λ) is a distinct distance graph.

Finally, a *Taylor coloring* of an edge-labeled graph (G, λ) is a function $t : V \to \{0, 1\}$, where V is the set of vertices of G, such that if x and y are in the same component of G then $d(x, y) \equiv |t(x) - t(y)| \pmod 2$. We omit the proofs of the following, since they are in the previous paper.

Proposition 1.1. *Every edge-labeled forest* (F, λ) *has a Taylor coloring.*

If t is a Taylor coloring of an edge-labeled graph (G, λ) with n vertices and k

components, then, for $i = 1, \ldots, k$ and $j = 0, 1$, we let

$$n_{i,j} = |\{x : x \text{ is in the } i\text{th component and } t(x) = j\}|.$$

The next theorem generalizes Taylor's Condition to weighted graphs. The proof generalizes Taylor's proof for trees to arbitrary graphs that may have multiple components.

Theorem 1.2 (Generalized Taylor's Condition). *If (G, λ) is an n-vertex perfect distance graph with k components and a Taylor coloring, then there are nonnegative integers $a_1, a_2, \ldots, a_k$, where $a_i \equiv n_i \pmod 2$ and $a_i \le n_i$ for $i = 1, 2, \ldots, k$, such that*

$$a_1^2 + a_2^2 + \cdots + a_k^2 + 2p = n,$$

where $p = maxdist(G, \lambda) \bmod 2$.

We will also use the fact that $a_i = |n_{i,0} - n_{i,1}|$ in Theorem 1.2 as was shown in [2].

2 Perfect Distance Forests - Same Size Components

In the previous paper [2] we had the following result:

Corollary 2.1 (Theorem 1.2). *If $tK_{1,2}$ can be labeled as a perfect distance forest, then $t \equiv 0, 1 \pmod 4$.*

Proof. Each component i in $tK_{1,2}$ is a copy of $K_{1,2}$. Consider a coloring of component i. If the labels are all even, then $a_i = |n_{i,0} - n_{i,1}| = 3$. If the integer labels on this component are either both odd or one odd and one even, then it follows that $|a_i| = 1$. Therefore, ${a_i}^2 = 9$ or ${a_i}^2 = 1$.

Let k represent the number of components that are labeled with two even labels, so there are $t - k$ components labeled otherwise. Inserting this information into the equation from Theorem 1.2 yields $9k + 1(t - k) + 2p = 3t$, which can be simplified to $k = \frac{t-p}{4}$. If t is even, we will have that $p = 0$ and since k must be an integer, $t \equiv 0 \pmod 4$. If t is odd then $p = 1$, and again since k is an integer $t \equiv 1 \pmod 4$. ☐

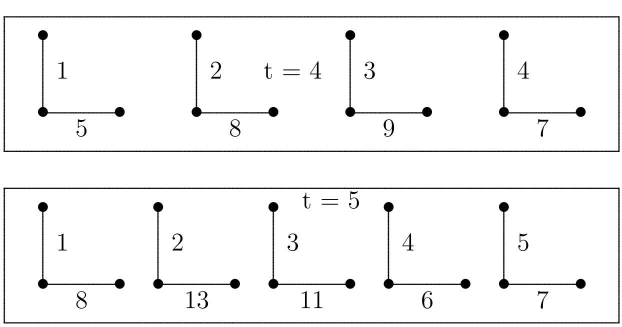


Figure 2: Examples of $tK_{1,2}$

We now show that for the cases not ruled out by the previous result, $tK_{1,2}$ can be labeled as a perfect distance forest. The above labelings in Figure 2 together with the perfect distance tree ($t = 1$) show that this is true for $t < 8$. We give constructions in three separate cases for $t \geq 8$: $t \cong 0 \pmod 4$ and $t \cong 1, 5 \pmod 8$. In each case, we can label one edge of component i with label i. From there, the labelings diverge for the three different cases. We do not give the proof of Theorem 2.2, since it is a straightforward verification that the given labels on $tK_{1,2}$ are perfect distance forests.

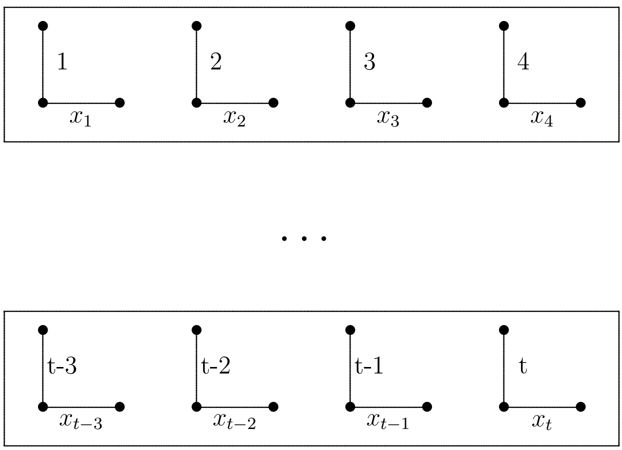


Figure 3: Labeling the Perfect Distance Forest $tK_{1,2}$

$t \cong 0 \pmod 4$:

$$x_i = \begin{cases} \frac{-i+3t}{2} + 1 & \text{if } i \text{ is even} \\ \frac{3t}{2} + 1 & i = t-1 \\ \frac{11t}{4} & i = 1 \\ \frac{5t}{2} - 1 & i = \frac{t}{2} + 1 \\ \frac{-i+5t+1}{2} & i \text{ odd and } \frac{t}{2} + 3 \leq i \leq t-3 \\ \frac{-i+5t-1}{2} & i \text{ odd and } 3 \leq i \leq \frac{t}{2} - 1 \end{cases},$$

$t \cong 1 \pmod 8$:

$$x_i = \begin{cases} \frac{-i+3t+1}{2} & \text{if } i \text{ is even} \\ \frac{3t+1}{2} & i = t \\ \frac{11t-3}{4} & i = 1 \\ 2t+1 & i = \frac{t-3}{2} \\ \frac{-i+5t}{2} + 1 & i \text{ odd and } \frac{t+1}{2} \leq i \leq t-2 \\ \frac{-i+5t}{2} & i \text{ odd and } 3 \leq i \leq \frac{t-7}{2} \end{cases},$$

$t \cong 5 \pmod 8$:

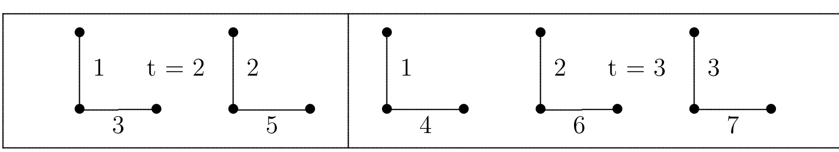
$$x_i = \begin{cases} \frac{-i+3t+1}{2} & \text{if } i \text{ is even} \\ \frac{3t+1}{2} & i = t \\ \frac{9t+3}{4} & i = 1 \\ \frac{5t+3}{2} & i = \frac{t-3}{2} \\ \frac{-i+5t}{2} & i \text{ odd and } \frac{t+1}{2} \leq i \leq t-2 \\ \frac{-i+5t}{2}+1 & i \text{ odd and } 3 \leq i \leq \frac{t-7}{2} \end{cases}.$$

Theorem 2.2. *$tK_{1,2}$ can be labeled as a perfect distance forest if and only if $t \equiv 0, 1$ (mod 4).*

For $t \equiv 2, 3 \pmod 4$ we can label $tK_{1,2}$ so that it is a minimal distinct distance forest with greatest distance $3t + 1$.

Theorem 2.3. *For $t \equiv 2, 3$ (mod 4) there is a labeling of $tK_{1,2}$ such that $tK_{1,2}$ is a minimal distinct distance forest with set of distances $\{1, 2, \ldots, 3t-1\} \cup \{3t+1\}$ and such that component i has an edge labeled i and component t has an edge labeled $2t+1$.*

Proof. We give the cases $t = 2, 3$ in a separate figure. Otherwise we have the following labeling, where component i has edges labeled i and x_i.



$t \cong 2 \pmod 4$, $t \geq 6$:

$$x_i = \begin{cases} 2t+1 & \text{if } i = t \\ \frac{-i+3t}{2} & \text{if } i \neq t \text{ and even} \\ \frac{3t}{2} & i = t-1 \\ \frac{11t-2}{4} & i = 1 \\ 2t & i = \frac{t}{2} \\ \frac{-i+5t+1}{2} & i \text{ odd and } \frac{t}{2}+2 \leq i \leq t-3 \\ \frac{-i+5t-1}{2} & i \text{ odd and } 3 \leq i \leq \frac{t}{2}-2 \end{cases},$$

$t \cong 3 \pmod 4$, $t \geq 7$:

$$x_i = \begin{cases} 2t+1 & \text{if } i = t \\ \frac{-i+3t+1}{2} & \text{if } i \text{ is even} \\ \frac{3t+1}{2} & i = t-2 \\ \frac{11t-5}{4} & i = 1 \\ \frac{5t-1}{2} & i = \frac{t-1}{2} \\ \frac{-i+5t}{2} & i \text{ odd and } \frac{t+3}{2} \leq i \leq t-4 \\ \frac{-i+5t-2}{2} & i \text{ odd and } 3 \leq i \leq \frac{t-5}{2} \end{cases}$$

□

Using the above labelings of $tK_{1,2}$, we can show that $tK_{1,3}$ always can be labeled as a perfect distance forest. We give the idea behind this recursion using the example of $4K_{1,2} \to 4K_{1,3}$ in Figure 4.

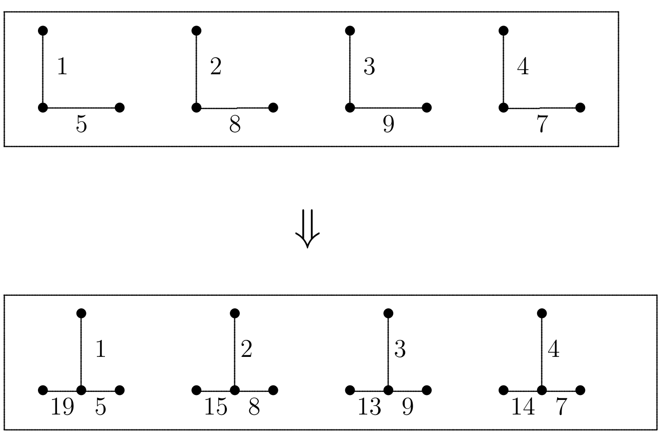


Figure 4: Deriving the PDF $4K_{1,3}$ from the PDF $4K_{1,2}$

Theorem 2.4. *Suppose that $tK_{1,2}$ can be labeled as a perfect distance forest with labels $1, 2, \ldots t$ on separate components. Then $tK_{1,3}$ can be labeled as a perfect distance forest.*

Proof. To label $tK_{1,3}$, we begin by labeling 2 of the 3 edges of each component so that if we ignore the third edge, we have a perfect distance forest with $tK_{1,2}$. This gives us weighted sums of $1, 2, \ldots, 3t$. On component i, we have edges labeled i and x_i. Let the third edge be labeled with $y_i = 6t + 1 - i - x_i$. Note that $\{y_1, \ldots, y_t\} = \{6t+1-(x_1+1), \ldots, 6t+1-(x_t+t)\}$, $\{y_1+1, \ldots, y_t+t\} = \{6t+1-x_1, \ldots, 6t+1-x_t\}$, and $\{y_1 + x_1, \ldots, y_t + x_t\} = \{6t + 1 - 1, \ldots, 6t + 1 - t\}$. Since $tK_{1,2}$ is labeled as a perfect distance forest, $\{y_1, \ldots, y_t\} \cup \{y_1 + 1, \ldots, y_t + t\} \cup \{y_1 + x_1, \ldots, y_t + x_t\} = \{3t+1, \ldots, 6t\}$. This shows that we have labeled $tK_{1,3}$ as a perfect distance forest. □

Theorem 2.5. *Suppose that $tK_{1,2}$ can be labeled as a minimal distinct distance forest with the labels $1, 2, \ldots, t$ on different components, the label $2t+1$ on the same component as t, and the set of distances is $\{1, 2, \ldots, 3t-1\} \cup \{3t+1\}$. Then $tK_{1,3}$ can be labeled as a perfect distance forest.*

Proof. We start with our labeling of $tK_{1,2}$ giving us 2 of the 3 edge labels in each component, call them i and x_i. Let the third edge be labeled $y_i = 6t+1-i-x_i$. We can show as in the previous proof that the distances y_i, y_i+i, and y_i+x_i over all components i will be the set of distances $\{3t, 3t+2, 3t+3, \ldots, 6t\}$. This proves that we have a labeling of $tK_{1,3}$ as a perfect distance forest. □

Corollary 2.6. *$tK_{1,3}$ can be labeled as a perfect distance forest for all t.*

Proof. For the case where $t \equiv 0, 1 \pmod 4$ we combine Theorems 2.2 and 2.4 to give that $tK_{1,3}$ can be labeled as a perfect distance forest, and for the case where $t \equiv 2, 3 \pmod 4$ we combine Theorems 2.3 and 2.5. □

We also consider perfect distance forests where each component is a copy of P_3 the path with 3 edges (and 4 vertices). Our results, obtained by computer search, are recorded in the following proposition. We conjecture that tP_3 can be labeled as a perfect distance forest for all $t \geq 4$. We provide examples in Figure 5, and note that these examples are not unique.

Proposition 2.7. *The forests tP_3 can be labeled as perfect distance forests for $t \in \{1, 4, 5, 6, 7\}$ but not for $t \in \{2, 3\}$.*

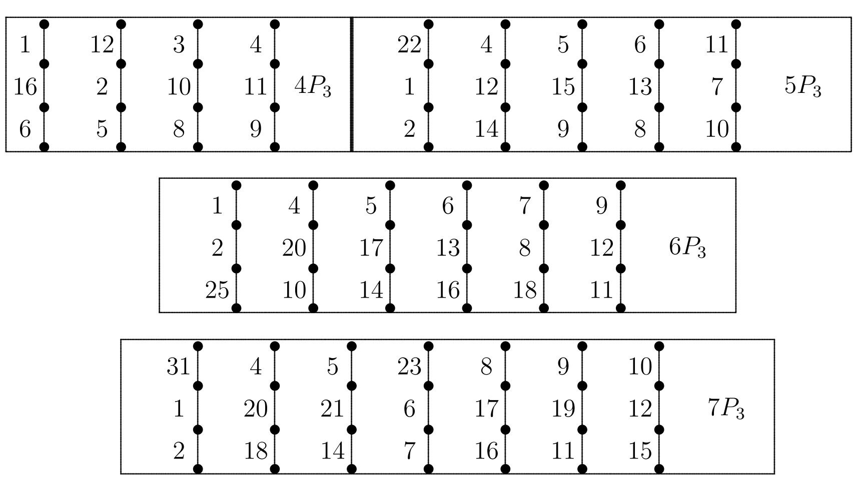
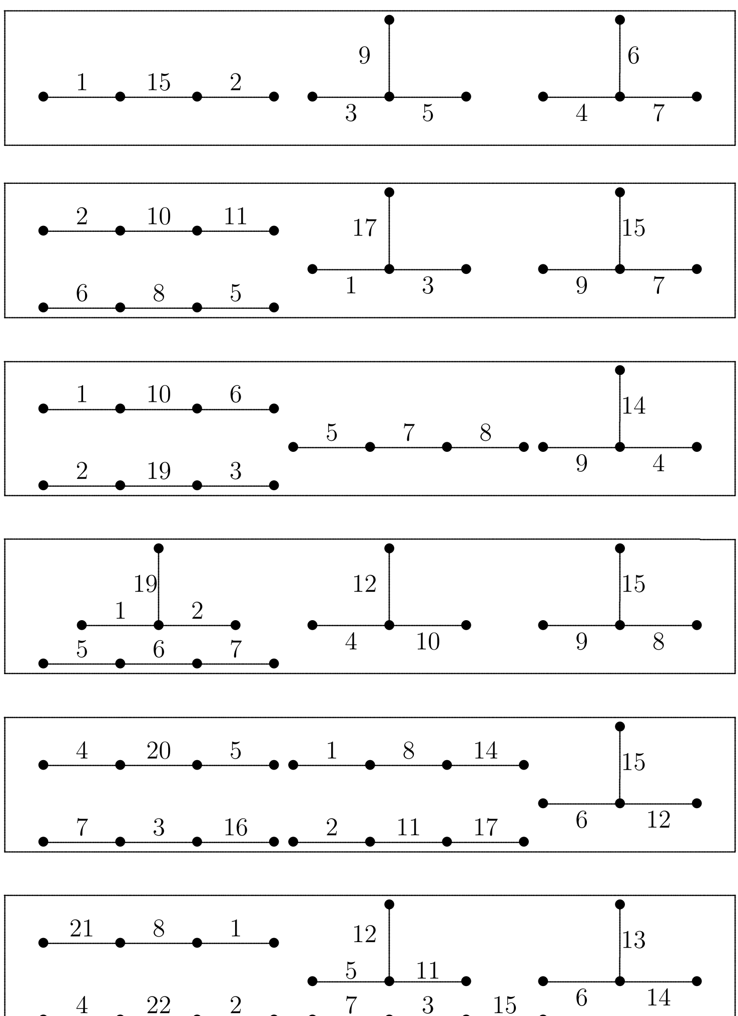


Figure 5: PDF Examples of tP_3

In general, when all components have 4 vertices, the forest is of the form $sK_{1,3} \cup tP_3$. Our computer search has found many labelings of such forests as perfect distance forests. In fact, we conjecture that all such forests with $s+t \geq 4$ can be labeled as perfect distance forests. The following proposition summarizes the results of our computer search. The figure below does not include examples where $s = 0$ or $t = 0$ since those cases have already been addressed. It would seem that as the sum $s + t$ increases, so does the likelihood of being able to label $sK_{1,3} \cup tP_3$ as a perfect distance forest, so we also omit the examples where $s + t = 6$. The examples given are not unique for a given s, t.

Proposition 2.8. *The forests $sK_{1,3} \cup tP_3$ can be labeled as perfect distance forests for all cases in which $s+t \leq 6$ except for the following choices of (s, t):* $(1, 1), (0, 2), (1, 2)$, $(0, 3)$.



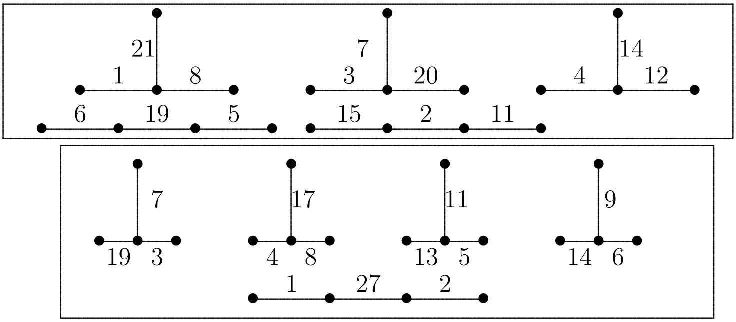


Figure 6: PDF Examples of $sK_{1,3} \cup tP_3$, $s, t \neq 0$ and $s + t \leq 5$

When we move up to components that are larger than 4 vertices, Theorem 1.2 allows us to eliminate many cases. We summarize with the following theorem.

Corollary 2.9. *Suppose that G is a Taylor-colorable graph with t components, each of which has k vertices. If G can be labeled as a perfect distance graph then we have the following:*

(a) $k \equiv 3 \pmod 8$ *implies* $t \equiv 0, 1 \pmod 4$*,*

(b) $k \equiv 5 \pmod 8$ *implies* $t \equiv 0, 2 \pmod 4$*,*

(c) $k \equiv 7 \pmod 8$ *implies* $t \equiv 0, 3 \pmod 4$*,*

(d) $(k, t) \notin \{(6, 2), (8, 1), (10, 1), (10, 3), (12, 1), (12, 2), (14, 1), (14, 2), (17, 1), (18, 2), (19, 1), (20, 3)\}$.

Proof. We use Theorem 1.2 to prove parts (a), (b) and (c). Since k is odd, there are odd positive integers $a_1, a_2, \ldots, a_k$, where $a_i \leq k$ for $i = 1, 2, \ldots, k$, such that

$$a_1^2 + a_2^2 + \cdots + a_k^2 + 2p = n = kt,$$

where $p = t\frac{k(k-1)}{2} \pmod 2 = t\frac{k-1}{2} \bmod 2$. For each odd positive integer j less than or equal to k, let b_j be the size of the set $\{i : a_i = j^2\}$. Then $b_1 + 9b_3 + \cdots k^2 b_k + 2p = kt$ and $b_1 + b_3 + \cdots b_k = t$. Subtracting $k(b_1 + b_3 + \cdots b_k) = kt$ we obtain

$$(1 - k)b_1 + (9 - k)b_3 + \cdots + (k^2 - k)b_k + 2p = 0.$$

Notice that if a is odd, then $(a + 2)^2 = a^2 + 4(a + 1) \equiv a^2 \pmod 8$. Hence

$$(1 - k) \equiv (9 - k) \equiv \cdots \equiv (k^2 - k) \pmod 8$$

and

$$(1 - k)(b_1 + b_3 + \cdots + b_k) + 2p = (1 - k)t + 2p \equiv 0 \pmod 8.$$

Equivalently, $\frac{k-1}{2}t \equiv p \pmod 4$, and substituting for p we obtain

$$t\frac{k-1}{2} \bmod 4 = t\frac{k-1}{2} \bmod 2.$$

It follows that $t\frac{k-1}{2} \bmod 4 \in \{0,1\}$ which can only happen in the following cases:

$$k \equiv 1 \pmod 8,$$

$$k \equiv 3 \pmod 8 \text{ and } t \bmod 4 \in \{0,1\},$$

$$k \equiv 5 \pmod 8 \text{ and } t \bmod 4 \in \{0,2\},$$

and

$$k \equiv 7 \pmod 8 \text{ and } t \bmod 4 \in \{0,3\}.$$

Parts (a), (b) and (c) follow. Each case in part (d) can be proved similarly by application of Theorem 1.2. We omit the details. □

The cases listed in the previous corollary represent all choices (k,t) with $k \leq 20$ that violate the necessary condition of Theorem 1.2.

$n = n_1 + n_2$	n_1	n_2	Existence	Justification
6	3	3	no	Corollary 2.1
7	5	2	yes	[2]
7	4	3	yes	[2]
8	6	2	yes	[2]
8	5	3	no	Computer Search
8	4	4	yes	Corollary 2.6
9	7	2	yes	[2]
10	7	3	yes	[2]
10	6	4	yes	Figure 7
11	6	5	yes	Figure 7
11	7	4	yes	Figure 7
11	8	3	no	Computer Search
11	9	2	no	Computer Search
12	6	6	no	Theorem 1.2
12	7	5	yes	Figure 7
12	8	4	no	Theorem 1.2
12	9	3	no	Computer Search
12	10	2	no	Theorem 1.2
13	9	4	yes	Figure 7
13	7	6	yes	Figure 7
13	11	2	no	Computer Search
13	10	3	no	Computer Search
13	8	5	no	Computer Search
14	all	all	no	Theorem 1.2
$15-16$	all	all	no	Theorem 1.2 + Computer Search

Table 1: 2-component PDFs of size (n_1, n_2)

3 Perfect Distance Forests with Two Components

Another interesting type of perfect distance forest is one having two components. We have found the examples given in Figure 7. A computer program determined that graphs A, C, D, E, and F are the only perfect distance forests with the given sizes for the two components. For instance, A is the only perfect distance forest with components of 6 and 5 vertices. Graph B is one of 5 different examples of perfect distance forests with components of size 6 and 4. Again we can use Theorem 1.2 to rule out perfect distance forests with certain choices for the sizes of the two components. Our computer program also ruled out some other cases. We summarize our results for 2-component perfect distance forests with fewer than 17 total vertices in Table 1. This table includes all of the known 2-component perfect distance forests on fewer than 17 vertices with the exception of the examples that can be obtained from adding a single disconnected edge to the perfect distance trees of Figure 1. We also provide some examples that are ruled out in various ways. Incidentally, there are no 2-component perfect distance forests with total number of vertices between 14 and 16.

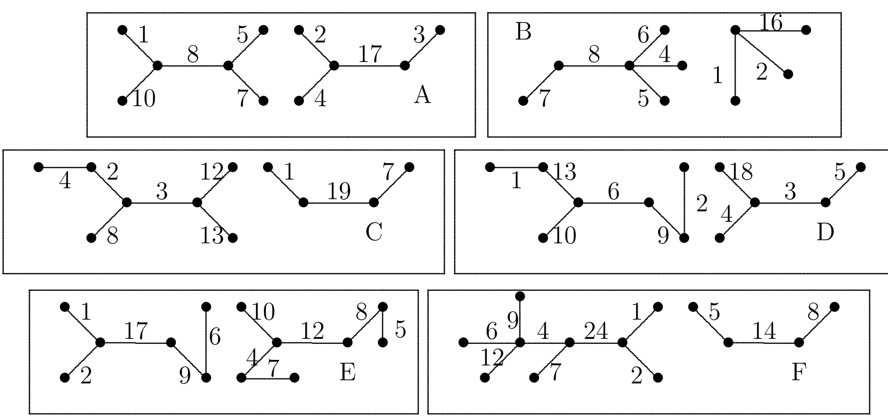


Figure 7: Examples of 2-component perfect distance forests

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